Loose Specifications

Wolfgang Schreiner
Wolfgang.Schreiner@risc.jku.at

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University, Linz, Austria
http://www.risc.jku.at

Specifications

We will introduce various flavors of specifications of ADTs.

- Specification semantics: $sp \to \mathcal{M}(sp)$.
  - Specification sp.
  - Its meaning $\mathcal{M}(sp)$ (an abstract datatype).
- sp is an adequate specification of an ADT $\mathcal{C}$:
  - $\mathcal{C} \subseteq \mathcal{M}(sp)$.
- sp is a strictly adequate specification of an ADT $\mathcal{C}$:
  - $\mathcal{C} = \mathcal{M}(sp)$.
- sp is a (strictly) adequate specification of an algebra A:
  - sp is (strictly) adequate specification of the monomorphic ADT $[A]$.
- sp is polymorphic (monomorphic):
  - sp defines a polymorphic (monomorphic) ADT.

General notions independent of the kind of specification.

Properties of Specifications

- Is the specification inconsistent?
  - Is the specified ADT empty (i.e. does not contain any algebras)?
- Is the specification monomorphic?
  - Are all algebras of the specified ADT isomorphic?
- Are two specifications equivalent?
  - Do they specify the same ADT?
- Does the specification (strictly) adequately describe a given ADT?
  - Assumes that the ADT is mathematically defined by other means.
    - But specification itself is typically the *only* definition of the ADT.
    - Then no mathematical proof of adequacy is possible.
    - Nevertheless, by "executing the specifications" (mechanically evaluating ground terms), we may investigate the properties of the specified ADT to increase our confidence in its adequacy.

All these questions now have a precise meaning.

1. General Remarks

2. Loose Specifications

3. Loose Specifications with Constructors

4. Loose Specifications with Free Constructors

5. Summary

Loose Specifications

Take logic L.

- Loose specification $sp = (\Sigma, \Phi)$ in L:
  - Signature Σ, set of formulas $\Phi \subseteq L(\Sigma)$.
- Semantics $\mathcal{M}(sp) = Mod_\Sigma(\Phi)$.
  - All Σ-algebras are candidates for the specified ADT.
    - $Mod_\Sigma(\Phi) = Mod_{Alg(\Sigma),\Sigma}(\Phi)$.

A loose specification specifies as the abstract datatype the class of all models of its formula set.

Concrete Syntax

loose spec
    sorts *sort* ...
    opns *operation* ...
    vars *variable*: *sort* ...
    axioms *formula* ...
endspec

- Signature $\Sigma = (\{\mathit{sort}, \ldots\}, \{\mathit{operation}, \ldots\})$.
- Set of formulas $\Phi = \{(\forall \mathit{variable} : \mathit{sort}, \ldots\ .\ \mathit{formula}), \ldots\}$.

We will only use the concrete syntax to define specifications.

Example

loose spec

sorts $el, bool, list$

opns

$True :\to bool$

$False :\to bool$

$[\,] :\to list$

$Add : el \times list \to list$

$\_ . \_ : list \times list \to list$

vars $l, m : list, e : el$

axioms

$[\,].l = l$

$Add(e, l).m = Add(e, l.m)$

endspec

Adequate specification of the "classical" list algebra in *EL*.

Strict Adequacy

Not a *strictly* adequate specification of the "classical" list algebra.

- Carrier for *bool* may collapse ("confusion" among *True* and *False*).

  PL: **axiom** $\neg(True = False)$
- Carrier for *list* may collapse ("confusion" among [ ] and $Add(e, l)$).

  PL: **axiom** $\forall e : el, l : list \,.\, \neg([\,] = Add(e, l))$
- Size of lists may be bound ("confusion" among *Add* terms).

  PL: **axiom** $\forall e_1, e_2 : elem, l_1, l_2 : list \,.$

  $$Add(e_1, l_1) = Add(e_2, l_2) \Rightarrow e_1 = e_2 \wedge l_1 = l_2$$
- Carriers may contain extra values ("junk").
  - There may a *bool* value different from *True* and *False*.

    PL: **axiom** $\forall b : bool \,.\, b = True \vee b = False$
  - There may be *list* values different from those that can be constructed by application of [ ] and *Add*.
    - No axiom can express this in *PL*, a solution will be later presented.

In *PL* (not *EL* or *CEL*), additional axioms may solve *some* problems of "junk" and "confusion".

Example

loose spec

sorts $el, bool, list$

opns

$True :\rightarrow bool$

$False :\rightarrow bool$

$[\,] :\rightarrow list$

$Add : el \times list \rightarrow list$

$\_ . \_ : list \times list \rightarrow list$

vars $l, l_1, l_2 : list, e, e_1, e_2 : el, b : bool$

axioms

$\neg(True = False)$

$b = True \vee b = False$

$\neg([\,] = Add(e, l))$

$Add(e_1, l_1) = Add(e_2, l_2) \;\Rightarrow$

$e_1 = e_2 \wedge l_1 = l_2$

$[\,].l = l$

$Add(e, l_1).l_2 = Add(e, l_1.l_2)$

endspec

More (but still not strictly) adequate specification of the "classical" list algebra in *PL*.

Example

loose spec

sorts $bool, nat$

opns

$True :\to bool$

$False :\to bool$

$0 :\to nat$

$Succ : nat \to nat$

$\_ + \_ : nat \times nat \to nat$

$\_ * \_ : nat \times nat \to nat$

$\_ \leq \_ : nat \times nat \to bool$

vars $m, n : nat, b : bool$

axioms

$\neg(True = False)$

$b = True \vee b = False$

$\neg(0 = Succ(n))$

$Succ(n) = Succ(m) \Rightarrow n = m$

$(0 \leq n) = True$

$(Succ(n) \leq 0) = False$

$(Succ(n) \leq Succ(m)) = (n \leq m)$

$n + 0 = n$

$n + Succ(m) = Succ(n + m)$

$n * 0 = 0$

$n * Succ(m) = n + (n * m)$

endspec

Adequate specification of Peano arithmetic in *PL* (not strictly adequate because *nat* may contain junk).

Proving Strategies for Loose Specifications

Take loose specification $sp = (\Sigma, \Phi)$ in logic L with inference calculus $\vdash$.

- Prove: $\mathcal{M}(sp) \models \varphi$.
  - Every implementation of the specification sp has the property expressed by formula φ.
  - It suffices to prove $\Phi \vdash \varphi$.
    - Formula φ can be derived from the specification axioms Φ.
- Prove: $\mathcal{M}(sp) \subseteq \mathcal{M}(sp')$.
  - Loose specification $sp' = (\Sigma, \Psi)$.
  - Every implementation of the specification sp is also an implementation of the specification sp'.
  - It suffices to prove $\Phi \vdash \Psi$.
    - Every axiom $\psi \in \Psi$ can be derived from the axioms Φ.

Straight-forward reduction of semantic questions to proving.

Expressive Power of Loose Specifications

Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

- Theorem 1: $\mathcal{M}(sp) = Mod_\Sigma(Th_L(\mathcal{M}(sp)))$.
  - $Th_L(C) = \{\phi \in L(\Sigma) \mid \forall A \in C : A \models_\Sigma \phi\}$.
    - The theory of a class of algebras w.r.t. a given logic is the set of all formulas of that logic that are satisfied by every algebra of the class.
  - Thus sp can specify an ADT $\mathcal{C}$ only if $\mathcal{C} = Mod_\Sigma(Th_L(\mathcal{C}))$.
- Example:
  - Signature $\mathrm{NAT} = (\{nat\}, \{0 :\to nat, s : nat \to nat\})$.
  - NAT-algebra $N = (\{\mathbb{N}\}, \{0_\mathbb{N}, (\lambda x\ .\ x+1)\})$.
  - $[N]$ cannot be specified by any specification sp in $EL(NAT)$.
    - Assume specification sp with $\mathcal{M}(sp) = [N]$.
    - $Th_{EL}([N]) = \{0 = 0, s(0) = s(0), s(s(0)) = s(s(0)), \ldots\}$.
    - Take NAT-algebra $A = (\{0,1\}, 0, \lambda x\ .\ 1-x))$
    - Clearly $A \not\simeq N$, thus $A \notin \mathcal{M}(sp)$.
    - But, since $A \models Th_{EL}(\{N\})$, $A \in Mod_\Sigma(Th_{EL}([N]))$, and thus, by Theorem 1, $A \in \mathcal{M}(sp)$.

Algebras can be discriminated only by the expressible formulas.

Expressive Power of Loose Specifications

Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

- Theorem 2: If L has a sound and complete calculus and if Φ is recursively enumerable, then $\mathcal{M}(sp)$ is axiomatizable in L.
  - Set S is recursively enumerable, if there is an algorithm that lists all of its elements (running forever, if necessary).
  - A class $\mathcal{C}$ of Σ-algebras is axiomatizable in L, if $Th_L(\mathcal{C})$ is recursively enumerable.
- An ADT whose theory is not recursively enumerable in the given logic, may not be specifiable by a loose specification.
  - Example: Peano arithmetic (natural numbers with addition and multiplication).
  - The theory of peano arithmetic is not recursively enumerable in first-order predicate logic.
  - Gödel's second incompleteness theorem: Peano arithmetic is not axiomatizable in first-order predicate logic.

Not every ADT can be specified by a loose specification.

Expressive Power of Loose Specifications

Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

- Theorem 3: If L is EL or CEL, then $M(sp)$ also contains algebras whose carriers are singletons (i.e., whose terms are "confused").
  - Consequence: No ADT with non-singleton carriers can be strictly adequately described by a loose specification in EL or CEL.
    - Cannot prevent "collapse" of the carrier.
- Theorem 4: If L is EL, CEL, or PL and $\mathcal{M}(sp)$ contains an algebra with an infinite carrier, then $M(sp)$ also contains algebras whose corresponding carriers contain "junk".
  - Consequence: No ADT with an infinite carrier can be strictly adequately described by a loose specification in EL, CEL, or PL.
    - Cannot rule out "extra" values in addition to the desired ones.

We need some more mechanisms for strictly adequate specifications.

Generated Algebras

Take signature $\Sigma = (S, \Omega)$, Σ-algebra A.

- Define set of operations $\Omega_c \subseteq \Omega$ (the constructors).
  - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is generated by Ω_c:
  - For each sort $s \in S$ and $a \in A(s)$, there exists a ground term $t \in T_{\Sigma_c,s}$ with $a = A(t)$.
    - Carrier a can be described by a term t that involves only constructors.
  - A is generated if it is generated by Ω.
- $Gen(\Sigma, \Omega_c) := \{A \in Alg(\Sigma) \mid A \text{ is generated by } \Omega_c\}$.
  - The set of all Σ-algebras generated by constructors Ω_c.
  - $Gen(\Sigma) := Gen(\Sigma, \Omega)$.

Generated algebra does not contain "junk" in the carriers.

Example

Take signature
$\mathrm{NAT} = (\{nat\}, \Omega = \{0 :\to nat, Succ : nat \to nat, + : nat \times nat \to nat\})$.

- Classical NAT-algebra $A = (\mathbb{N}, 0_{\mathbb{N}}, +_{\mathbb{N}})$.
- Constructors $\Omega_c := \{0 :\to nat, Succ : nat \to nat\}$.
- A is generated by Ω_c:
  - For every $n \in \mathbb{N}, n = A(\underbrace{s(s(s(\ldots(s}_{n \text{ times}}(0))))))$.
- A is also generated by Ω.
  - Any superset of a set of constructors is also a set of constructors.

Usually one looks for the minimal set of constructors.

Algebras Generated in Some Sorts

Take signature $\Sigma = (S, \Omega)$, Σ-algebra A.

- Define set of sorts $S_c \subseteq S$ and set of operations $\Omega_c \subseteq \Omega$ (the constructors) with target sorts in S_c.
  - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is generated by Ω_c in S_c:
  - For each sort $s \in S_c$ and $a \in A(s)$, there exists
    - a set X of variables in Σ with $X_s = \emptyset$ for every s in S_c,
    - an assignment $\alpha : X \to A$,
    - and a term $t \in T_{\Sigma_c(X),s}$

    with $a = A(\alpha)(t)$.
    - Value a can be described by a term t that involves only constructors in the generated sorts and variables in the non-generated sorts.
  - A is generated in Sc if it is generated in S_c by Ω.

Algebra does not contain "junk" in the carrriers of the generated sorts.

Example

- Signature $\mathrm{LIST} = (S, \Omega)$:
  - $S = \{el, list\}$.
  - $\Omega = \{[\,] :\rightarrow list, Add : el \times list \rightarrow list, \_ \cdot \_ : list \times list \rightarrow list$.
- LIST-algebra A:
  - $A(el)$ ... a set of "elements".
  - $A(list)$ ... the set of finite lists of elements.
  - $A([\,])$ ... the empty list.
  - $A(Add)$ adds an element at the front of the list.
  - $A(\cdot)$ concatenates two lists.
- A is generated by $\Omega_c = \{[\,], Add\}$ in $S_c = \{list\}$:
  - Take arbitrary $l = [e_1, e_2, \ldots, e_n] \in A(list)$.
  - Define $X_{el} := \{x_1, x_2, \ldots, x_n\}$.
  - Define $\alpha_{el} := [x_1 \mapsto e_1, x_2 \mapsto e_2, \ldots, x_n \mapsto e_n]$.
  - Then $l = A(\alpha)(Add(x_1, Add(x_2, \ldots, Add(x_n, [\,]))))$.

Proofs by Induction

In generated sorts, the principle of structural induction can be applied.

- Take the LIST-algebra A of the previous example.
  - Notation: c_A for $A(c)$.
  - Knowledge: (1) $\forall l \in list_A : [\,]_A \cdot_A l = l$.
    (2) $\forall e \in el_A, l, r \in list_A :$
    $Add_A(e, l) \cdot_A r = Add_A(e, l \cdot_A r)$.
- Prove: $\forall l \in list_A : l \cdot_A [\,]_A = l$.
- Induction base $l = [\,]_A$:
  - $l \cdot_A [\,]_A = [\,]_A \cdot_A [\,]_A \overset{(1)}{=} [\,]_A = l$.
- Induction step $l = Add_A(e, r)$ (for some $e \in el_A, r \in list_A$).
  - Induction Hypothesis (H): $r \cdot_A [\,]_A = r$.
  - $l \cdot_A [\,]_A = Add_A(e, r) \cdot_A [\,]_A$

    $\overset{(2)}{=} Add_A(e, r \cdot_A [\,]_A)$

    $\overset{(H)}{=} Add_A(e, r) = l$.

Loose Specifications with Constructors

Take logic L.

- Loose specification with constructors $sp = (\Sigma, \Phi, S_c, \Omega_c)$ in L:
  - Signature $\Sigma = (S, \Omega)$, set of formulas $\Phi \subseteq L(\Sigma)$, generated sorts $S_c \subseteq S$, constructors $\Omega_c \subseteq \Omega$ with target sorts in S_c.
- Semantics $\mathcal{M}(sp) = Mod_{\mathcal{U},\Sigma}(\Phi)$ where

  $\mathcal{U} = \{A \in Alg(\Sigma) \mid A \text{ is generated in } S_c \text{ by } \Omega_c\}$.

  - Only generated Σ-algebras are candidates for the specified ADT.

A loose specification with constructors specifies as the ADT the class of all models of its formula set that are generated by the constructors.

Concrete Syntax

loose spec
    sorts [**generated**] *sort* ...
    opns [**constr**] *operation* ...
    vars *variable*: *sort* ...
    axioms *formula* ...
endspec

- Signature $\Sigma = (\{sort, \ldots\}, \{operation, \ldots\})$.
- Set of formulas $\Phi = \{(\forall variable : sort, \ldots\ .\ formula), \ldots\}$.
- Generated sorts $S_c = \{\textbf{generated}\ sort, \ldots\}$.
- Constructors $\Omega_c = \{\textbf{constr}\ operation, \ldots\}$.

We will only use the concrete syntax to define specifications.

Example

loose spec
- **sorts** *el*
  - **generated** *bool*
  - **generated** *list*
- **opns**
  - **constr** $True :\to bool$
  - **constr** $False :\to bool$
  - **constr** $[\,] :\to list$
  - **constr** $Add : el \times list \to list$
  - $\_ . \_ : list \times list \to list$

vars $l, m : list, e, e_1, e_2 : el$

axioms

$\neg(True = False)$

$\neg([\,] = Add(e, l))$

$Add(e_1, l_1) = Add(e_2, l_2) \Rightarrow e_1 = e_2$

$[\,].l = l$

$Add(e, l).m = Add(e, l.m)$

endspec

Strictly adequate specification of the “classical” list algebra in *PL*.

Example

loose spec
- **sorts**
  - **generated** *bool*
  - **generated** *nat*
- **opns**
  - **constr** $True :\to bool$
  - **constr** $False :\to bool$
  - **constr** $0 :\to nat$
  - **constr** $Succ : nat \to nat$
  - $\_ + \_ : nat \times nat \to nat$
  - $\_ * \_ : nat \times nat \to nat$
  - $\_ \leq \_ : nat \times nat \to bool$

vars $m, n : nat$

axioms

$\neg(True = False)$

$\neg(0 = Succ(n))$

$Succ(n) = Succ(m) \Rightarrow n = m$

$(0 \leq n) = True$

$(Succ(n) \leq 0) = False$

$(Succ(n) \leq Succ(m)) = (n \leq m)$

$n + 0 = n$

$n + Succ(m) = Succ(n + m)$

$n * 0 = 0$

$n * Succ(m) = n + (n * m)$

endspec

Strictly adequate specification of Peano arithmetic in *PL*.

Specified ADT is Strictly Adequate

Proof requires two parts.

- Peano arithmetic satisfies the specified axioms.
  - Can be easily checked.
- Specified ADT is monomorphic: $\forall B, C \in \mathcal{M}(sp) : B \simeq C$.
  - There is an isomorphism $h : B \to C$.
    - A bijective homomorphism.
  - Definition of unique term representation for every value.
    - Simplifies the remainder of the proof.
  - Definition of bijective mapping h:
    - By pattern matching on term representation.
  - Proof that h is a homomorphism:
    - By using properties expressed with the help of the term representation.

Term representation essential for this kind of proofs.

Values have Unique Term Representations

Take aribitrary $A \in \mathcal{M}(sp)$.

- $bool_A = \{True_A, False_A\}$ and $True_A \neq False_A$.
  - A is generated by $\{True, False\}$ in $bool$.
  - **axiom** $\neg(True = False)$.
- $nat_A = \{Succ^k(0)_A : k \in \mathbb{N}\}$ and $\forall k \neq l : Succ^k(0)_A \neq Succ^l(0)_A$.
  - A is generated by $\{0, Succ\}$ in nat.
  - Proof by induction on k: $\forall l \neq k : Succ^k(0)_A \neq Succ^l(0)_A$.
    - $k = 0, l \neq 0$: $0_A \neq Succ^l(0)_A$ (by **axiom** $\neg(0 = Succ(n))$).
    - $k \neq 0, l \neq k$: assume $Succ^k(0)_A = Succ^l(0)_A$, show $k = l$.

      Know $l \neq 0$ (by **axiom** $\neg(0 = Succ(n))$).
      Thus $k = k' + 1, l = l' + 1$, it suffices to show $k` = l'$.
      By assumption, $Succ(Succ^{k'}(0))_A = Succ(Succ^{l'}(0))_A$.
      Thus $Succ^{k'}(0)_A = Succ^{l'}(0)_A$ (**axiom** $Succ(n) = Succ(m) \Rightarrow n = m$).
      By induction hypothesis, $k' = l'$.

Values are uniquely described by constructor applications.

Definition of Bijective Mapping

Take arbitrary $B, C \in \mathcal{M}(sp)$.

- h is defined by pattern matching on constructor terms:
  - $h_{bool}(True_B) := True_C$.
  - $h_{bool}(False_B) := False_C$.
  - $h_{nat}(Succ^k(0)_B) = Succ^k(0)_C$, for all $k \geq 0$.
- h is consistently defined:
  - $True_B$ and $False_B$ denote different values.
  - $Succ^k(0)_B$ denote different values for different k.
- h is bijective:
  - $True_C$ and $False_C$ denote different values.
  - $Succ^k(0)_C$ denote different values for different k.

One-to-one correspondence between the carriers of B and C.

Homomorphism Proof

- Clear for constructors *True*, *False*, 0, *Succ*:
  - Definition of h already expresses homomorphism condition.
- Goal: $\forall m, n \in nat_B \,.\, h(op_B(m, n)) = op_C(h(m), h(n))$.

  $op \ldots +, *, \leq$.

  - $\forall k, l \geq 0 \,.\, h(op_B(Succ^k(0)_B, Succ^l(0)_B)) =$
    $op_C(h(Succ^k(0)_B), h(Succ^l(0)_B))$.
    - B and C are generated by $\{0, Succ\}$ in nat.
  - $\forall k, l \geq 0 \,.\, h(op_B(Succ^k(0)_B, Succ^l(0)_B)) =$
    $op_C(Succ^k(0)_C, Succ^l(0)_C)$.
    - By definition of h.
  - $\forall k, l \geq 0 \,.\, h(op(Succ^k(0), Succ^l(0))_B) =$
    $op(Succ^k(0), Succ^l(0))_C$.
    - By definition of term semantics.

Proof goal is expressed with the help of constructor terms.

Homomorphism Proof

The core of the homomorphism proof.

- Goal: $h((Succ^k(0) + Succ^l(0))_B) = (Succ^k(0) + Succ^l(0))_C$.
  - First simplify left and right hand side of the equation.
- Lemma: $\forall A \in \mathcal{M}(sp) : (Succ^k(0) + Succ^l(0))_A = Succ^{k+l}(0)_A$.
  - Induction base $l = 0$: by **axiom** $n + 0 = n$.
  - Induction step $l = l' + 1$:

$$\begin{aligned}
&(Succ^k(0) + Succ^{l'+1}(0))_A \\
&= Succ(Succ^k(0) + Succ^{l'}(0))_A \\
&= Succ(Succ^{k+l'}(0))_A \\
&= Succ^{k+l'+1}(0)_A.
\end{aligned}$$

- Simplified goal: $h(Succ^{k+l}(0)_B) = Succ^{k+l}(0)_C$.
  - By definition of h.

Similar for the homomorphism proofs of the other operations.

1. General Remarks

2. Loose Specifications

3. Loose Specifications with Constructors

4. Loose Specifications with Free Constructors

5. Summary

Freely Generated Algebras

Take signature $\Sigma = (S, \Omega)$, Σ-algebra A.

- Define set of operations $\Omega_c \subseteq \Omega$ (the constructors).
  - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is freely generated by Ω_c:
  - For each sort $s \in S$ and $a \in A(s)$, there exists exactly one ground term $t \in T_{\Sigma_c,s}$ with $a = A(t)$.
    - Value a can be described by a unique term t that involves only constructors.
  - A is freely generated if it is generated by Ω.
- A is freely generated by Ω_c in S_c:
  - Analogous definition as for generated by ... in ... .

Freely generated algebras have unique constructor term representations for the values of the freely generated sorts (no "junk" in carriers and no "confusion" among constructor terms).

Example

- The "classical" BOOL-algebra $(\{\mathit{true}, \mathit{false}\}, \ldots)$:
  - Freely generated by $\{\mathit{True}, \mathit{False}\}$.
  - Not freely generated by $\{\mathit{True}, \mathit{False}, \neg\}$.
- The "one-element" BOOL-algebra $(\{\#\}, \ldots)$.
  - Freely generated by $\{\mathit{True}\}$ and by $\{\mathit{False}\}$.
  - Not freely generated by $\{\mathit{True}, \mathit{False}\}$.
- The "classical" NAT-algebra $(\mathbb{N}, \ldots)$:
  - Freely generated by $\{0, \mathit{Succ}\}$.
  - Not freely generated by $\{0, \mathit{Succ}, +\}$.
- The "classical" INT-algebra $(\mathbb{Z}, \ldots)$:
  - $\mathrm{INT} = (\mathit{int}, \{0 :\to \mathit{int}, \mathrm{Succ} : \mathit{int} \to \mathit{int}, \mathrm{Pred} : \mathit{int} \to \mathit{int}\})$.
  - Not freely generated by (any subset of) operations.

A set of free constructors cannot be extended.

Inductive Function Definitions

Freely generated algebras allow inductive function definitions.

- Signature $\mathrm{LIST} = (S, \Omega)$:
  - $S = \{el, list\}$.
  - $\Omega = \{[\,] :\to list, Add : el \times list \to list, \_ \cdot \_ : list \times list \to list\}$.
- Classical LIST-algebra A as in the previous example.
  - A is freely generated by $\Omega_c = \{[\,], Add\}$ in $S_c = \{list\}$:
- Inductive definition of function $g : A(list) \to \mathbb{N}$.
  - $g([\,]_A) = 0$.
  - $g(Add(x, t)_A) = g(t_A) + 1$ for all $x \in X, t \in T_{\Sigma_c(X), list}$.

Inductive definition by "pattern matching" on constructor terms (independent of the nature of the carrier).

Loose Specifications with Free Constructors

Take logic L.

- Loose specification with free constructors $sp = (\Sigma, \Phi, S_c, \Omega_c)$ in L:
  - Signature $\Sigma = (S, \Omega)$, set of formulas $\Phi \subseteq L(\Sigma)$, freely generated sorts $S_c \subseteq S$, constructors $\Omega_c \subseteq \Omega$ with target sorts in S_c.
- Semantics $\mathcal{M}(sp) = Mod_{\mathcal{U},\Sigma}(\Phi)$ where

  $$\mathcal{U} = \{A \in Alg(\Sigma) \mid A \text{ is freely generated in } S_c \text{ by } \Omega_c\}.$$

  - Only freely generated Σ-algebras are candidates for the specified ADT.

A loose specification with free constructors specifies the class of all models of its formula set that are freely generated by the constructors.

Concrete Syntax

loose spec
  sorts [**freely generated**] *sort* ...
  opns [**constr**] *operation* ...
  vars *variable*: *sort* ...
  axioms *formula* ...
endspec

- Signature $\Sigma = (\{sort, \ldots\}, \{operation, \ldots\})$.
- Set of formulas $\Phi = \{(\forall variable : sort, \ldots\ .\ formula), \ldots\}$.
- Generated sorts $S_c = \{$**freely generated** $sort, \ldots\}$.
- Constructors $\Omega_c = \{$**constr** $operation, \ldots\}$.

Also mixing of generated sorts with freely generated sorts possible.

Example

loose spec

sorts *el*

freely generated *bool*

freely generated *list*

opns

constr $True :\to bool$

constr $False :\to bool$

constr $[\,] :\to list$

constr $Add : el \times list \to list$

$\_ . \_ : list \times list \to list$

vars $l, m : list, e, e_1, e_2 : el$

axioms

$[\,].l = l$

$Add(e, l).m = Add(e, l.m)$

endspec

Strictly adequate specification of the "classical" list algebra in *EL*; the non-constructor operation is inductively defined.

Example

loose spec
  sorts
    freely generated *bool*
    freely generated *nat*
  opns
    constr $True :\to bool$
    constr $False :\to bool$
    constr $0 :\to nat$
    constr $Succ : nat \to nat$
    $\_ + \_ : nat \times nat \to nat$
    $\_ * \_ : nat \times nat \to nat$
    $\_ \leq \_ : nat \times nat \to bool$
  vars $m, n : nat$
  axioms
    $(0 \leq n) = True$
    $(Succ(n) \leq 0) = False$
    $(Succ(n) \leq Succ(m)) = (n \leq m)$
    $n + 0 = n$
    $n + Succ(m) = Succ(n + m)$
    $n * 0 = 0$
    $n * Succ(m) = n + (n * m)$
endspec

Strictly adequate specification of the "classical" list algebra in *EL*; the non-constructor operations are inductively defined.

Summary

A couple of core messages. . .

- A loose specification describes a class of models as an ADT.
  - To check whether a given algebra implements the specification (i.e., whether it is an element of the specified ADT):
    - Check whether the algebra satisfies the specification axioms.
  - There may exist "confusion" among terms.
    - Carriers may collapse to singletons (or be too "small").
    - In *PL*, additional axioms can prevent this.
    - Non-equalities of operation results (injectiveness of operations).
  - Carriers may contain "junk".
    - In *PL*, an additional axiom can prevent this for a finite carrier.
    - Axiom enumerates constants that denote all s of the sort.

Without constructors, loose specifications are generally clumsy because many "boring" axioms are needed.

Summary (Contd)

- Loose specifications with constructors.
  - Every value is denoted by some constructor term.
  - Thus junk is removed from (also infinite) carriers.
  - Induction proofs on term representation of s become possible.
  - Problem: not all carriers allow term representations.
    - ADT "real" (carrier is not countable).
- Loose specifications with free constructors.
  - Every value is denoted by exactly one constructor term.
  - Thus there is no "confusion" among constructor terms and the collapse of carriers is prevented.
  - Inductive function definitions by pattern matching on term representations of s become possible.
  - Problem: not all carriers have unique term representations.
    - ADT "set" (no unique representation at all).
    - ADT "integer" (unique representation is unconvient).

With constructors, loose specifications become easy to use.

Summary (Contd)

So what is the role of loose specifications. . .

- Loose specifications are good for specifying requirements.
  - May specify zero, one, many datatypes (polymorphic ADTs).
  - Thus allow arbitrarily many implementations.
    - A loose specification may not have any model (implementation) at all!
  - Specification axioms can (should) be abstract.
    - Later verification that concrete implementation satisfies the axioms.
- Loose specifications are not good for specifying designs.
  - Not descriptions of concrete algorithms/implementations.
- Loose specifications are generally not executable.
  - No engines to execute loose specifications for rapid prototyping.

Loose specifications are for *reasoning*, not for *executing*; they are the basis of program specification languages such as Larch/C++.