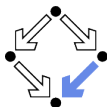
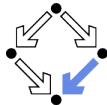


Loose Specifications

Wolfgang Schreiner
Wolfgang.Schreiner@risc.jku.at

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University, Linz, Austria
<http://www.risc.jku.at>





1. General Remarks

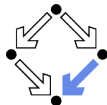
2. Loose Specifications

3. Loose Specifications with Constructors

4. Loose Specifications with Free Constructors

5. Summary

Specifications

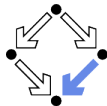


We will introduce various flavors of specifications of ADTs.

- **Specification semantics:** $sp \rightarrow \mathcal{M}(sp)$.
 - Specification sp .
 - Its meaning $\mathcal{M}(sp)$ (an abstract datatype).
- sp is an **adequate specification of an ADT \mathcal{C}** :
 - $\mathcal{C} \subseteq \mathcal{M}(sp)$.
- sp is a **strictly adequate specification of an ADT \mathcal{C}** :
 - $\mathcal{C} = \mathcal{M}(sp)$.
- sp is a **(strictly) adequate specification of an algebra A** :
 - sp is (strictly) adequate specification of the monomorphic ADT $[A]$.
- sp is **polymorphic (monomorphic)**:
 - sp defines a polymorphic (monomorphic) ADT.

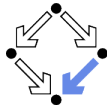
General notions independent of the kind of specification.

Properties of Specifications



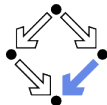
- Is the specification inconsistent?
 - Is the specified ADT empty (i.e. does not contain any algebras)?
- Is the specification monomorphic?
 - Are all algebras of the specified ADT isomorphic?
- Are two specifications equivalent?
 - Do they specify the same ADT?
- Does the specification (strictly) adequately describe a given ADT?
 - Assumes that the ADT is mathematically defined by other means.
 - But specification itself is typically the *only* definition of the ADT.
 - Then no mathematical proof of adequacy is possible.
 - Nevertheless, by “executing the specifications” (mechanically evaluating ground terms), we may investigate the properties of the specified ADT to increase our confidence in its adequacy.

All these questions now have a precise meaning.



-
1. General Remarks
 - 2. Loose Specifications**
 3. Loose Specifications with Constructors
 4. Loose Specifications with Free Constructors
 5. Summary

Loose Specifications

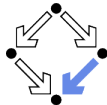


Take logic L .

- **Loose specification** $sp = (\Sigma, \Phi)$ in L :
 - Signature Σ , set of formulas $\Phi \subseteq L(\Sigma)$.
- **Semantics** $\mathcal{M}(sp) = \text{Mod}_{\Sigma}(\Phi)$.
 - All Σ -algebras are candidates for the specified ADT.
 - $\text{Mod}_{\Sigma}(\Phi) = \text{Mod}_{\text{Alg}(\Sigma), \Sigma}(\Phi)$.

A loose specification specifies as the abstract datatype the class of all models of its formula set.

Concrete Syntax



loose spec

sorts *sort* ...

opns *operation* ...

vars *variable*: *sort* ...

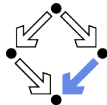
axioms *formula* ...

endspec

- Signature $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$.
- Set of formulas $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$.

We will only use the concrete syntax to define specifications.

Example



loose spec

sorts *el, bool, list*

opns

True :→ *bool*

False :→ *bool*

[] :→ *list*

Add : *el* × *list* → *list*

_ . _ : *list* × *list* → *list*

vars *l, m* : *list*, *e* : *el*

axioms

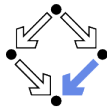
[].*l* = *l*

Add(*e, l*).*m* = *Add*(*e, l.m*)

endspec

Adequate specification of the “classical” list algebra in *EL*.

Strict Adequacy



Not a *strictly* adequate specification of the “classical” list algebra.

- Carrier for *bool* may collapse (“confusion” among *True* and *False*).

PL: **axiom** $\neg(\text{True} = \text{False})$

- Carrier for *list* may collapse (“confusion” among $[]$ and $\text{Add}(e, l)$).

PL: **axiom** $\forall e : \text{el}, l : \text{list} . \neg([] = \text{Add}(e, l))$

- Size of lists may be bound (“confusion” among *Add* terms).

PL: **axiom** $\forall e_1, e_2 : \text{elem}, l_1, l_2 : \text{list} .$

$\text{Add}(e_1, l_1) = \text{Add}(e_2, l_2) \Rightarrow e_1 = e_2 \wedge l_1 = l_2$

- Carriers may contain extra values (“junk”).

- There may a *bool* value different from *True* and *False*.

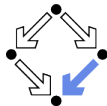
PL: **axiom** $\forall b : \text{bool} . b = \text{True} \vee b = \text{False}$

- There may be *list* values different from those that can be constructed by application of $[]$ and *Add*.

- No axiom can express this in *PL*, a solution will be later presented.

In *PL* (not *EL* or *CEL*), additional axioms may solve *some* problems of “junk” and “confusion”.

Example



loose spec

sorts $el, bool, list$

opns

$True : \rightarrow bool$

$False : \rightarrow bool$

$[] : \rightarrow list$

$Add : el \times list \rightarrow list$

$.. : list \times list \rightarrow list$

vars $l, l_1, l_2 : list, e, e_1, e_2 : el, b : bool$

axioms

$\neg(True = False)$

$b = True \vee b = False$

$\neg([] = Add(e, l))$

$Add(e_1, l_1) = Add(e_2, l_2) \Rightarrow$
 $e_1 = e_2 \wedge l_1 = l_2$

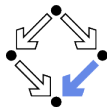
$[].l = l$

$Add(e, l_1).l_2 = Add(e, l_1.l_2)$

endspec

More (but still not strictly) adequate specification of the “classical” list algebra in *PL*.

Example



loose spec

sorts *bool, nat*

opns

True \rightarrow *bool*

False \rightarrow *bool*

0 \rightarrow *nat*

Succ : *nat* \rightarrow *nat*

- $+$ $_{-}$: *nat* \times *nat* \rightarrow *nat*

- $*$ $_{-}$: *nat* \times *nat* \rightarrow *nat*

- \leq $_{-}$: *nat* \times *nat* \rightarrow *bool*

vars *m, n* : *nat*, *b* : *bool*

axioms

$\neg(\text{True} = \text{False})$

$b = \text{True} \vee b = \text{False}$

$\neg(0 = \text{Succ}(n))$

$\text{Succ}(n) = \text{Succ}(m) \Rightarrow n = m$

$(0 \leq n) = \text{True}$

$(\text{Succ}(n) \leq 0) = \text{False}$

$(\text{Succ}(n) \leq \text{Succ}(m)) = (n \leq m)$

$n + 0 = n$

$n + \text{Succ}(m) = \text{Succ}(n + m)$

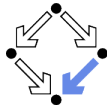
$n * 0 = 0$

$n * \text{Succ}(m) = n + (n * m)$

endspec

Adequate specification of Peano arithmetic in *PL* (not strictly adequate because *nat* may contain junk).

Proving Strategies for Loose Specifications

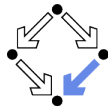


Take loose specification $sp = (\Sigma, \Phi)$ in logic L with inference calculus \vdash .

- Prove: $\mathcal{M}(sp) \models \varphi$.
 - Every implementation of the specification sp has the property expressed by formula φ .
 - It suffices to prove $\Phi \vdash \varphi$.
 - Formula φ can be derived from the specification axioms Φ .
- Prove: $\mathcal{M}(sp) \subseteq \mathcal{M}(sp')$.
 - Loose specification $sp' = (\Sigma, \Psi)$.
 - Every implementation of the specification sp is also an implementation of the specification sp' .
 - It suffices to prove $\Phi \vdash \Psi$.
 - Every axiom $\psi \in \Psi$ can be derived from the axioms Φ .

Straight-forward reduction of semantic questions to proving.

Expressive Power of Loose Specifications

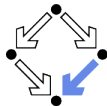


Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

- **Theorem 1:** $\mathcal{M}(sp) = Mod_{\Sigma}(Th_L(\mathcal{M}(sp)))$.
 - $Th_L(\mathcal{C}) = \{\phi \in L(\Sigma) \mid \forall A \in \mathcal{C} : A \models_{\Sigma} \phi\}$.
 - The theory of a class of algebras w.r.t. a given logic is the set of all formulas of that logic that are satisfied by every algebra of the class.
 - Thus sp can specify an ADT \mathcal{C} only if $\mathcal{C} = Mod_{\Sigma}(Th_L(\mathcal{C}))$.
- **Example:**
 - Signature $NAT = (\{nat\}, \{0 : \rightarrow nat, s : nat \rightarrow nat\})$.
 - NAT-algebra $N = (\{\mathbb{N}\}, \{0_{\mathbb{N}}, (\lambda x . x + 1)\})$.
 - $[N]$ cannot be specified by any specification sp in $EL(NAT)$.
 - Assume specification sp with $\mathcal{M}(sp) = [N]$.
 - $Th_{EL}([N]) = \{0 = 0, s(0) = s(0), s(s(0)) = s(s(0)), \dots\}$.
 - Take NAT-algebra $A = (\{0, 1\}, 0, \lambda x . 1 - x)$
 - Clearly $A \not\cong N$, thus $A \notin \mathcal{M}(sp)$.
 - But, since $A \models Th_{EL}(\{N\})$, $A \in Mod_{\Sigma}(Th_{EL}([N]))$, and thus, by Theorem 1, $A \in \mathcal{M}(sp)$.

Algebras can be discriminated only by the expressible formulas.

Expressive Power of Loose Specifications

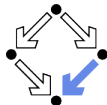


Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

- **Theorem 2:** If L has a sound and complete calculus and if Φ is recursively enumerable, then $\mathcal{M}(sp)$ is axiomatizable in L .
 - Set S is recursively enumerable, if there is an algorithm that lists all of its elements (running forever, if necessary).
 - A class \mathcal{C} of Σ -algebras is axiomatizable in L , if $Th_L(\mathcal{C})$ is recursively enumerable.
- An ADT whose theory is not recursively enumerable in the given logic, may not be specifiable by a loose specification.
 - Example: Peano arithmetic (natural numbers with addition and multiplication).
 - The theory of peano arithmetic is not recursively enumerable in first-order predicate logic.
 - Gödel's second incompleteness theorem: Peano arithmetic is not axiomatizable in first-order predicate logic.

Not every ADT can be specified by a loose specification.

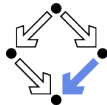
Expressive Power of Loose Specifications



Take loose specification $sp = (\Sigma, \Phi)$ with $\Phi \subseteq L(\Sigma)$.

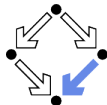
- **Theorem 3:** If L is *EL* or *CEL*, then $M(sp)$ also contains algebras whose carriers are singletons (i.e., whose terms are “confused”).
 - **Consequence:** No ADT with non-singleton carriers can be strictly adequately described by a loose specification in *EL* or *CEL*.
 - Cannot prevent “collapse” of the carrier.
- **Theorem 4:** If L is *EL*, *CEL*, or *PL* and $\mathcal{M}(sp)$ contains an algebra with an infinite carrier, then $M(sp)$ also contains algebras whose corresponding carriers contain “junk”.
 - **Consequence:** No ADT with an infinite carrier can be strictly adequately described by a loose specification in *EL*, *CEL*, or *PL*.
 - Cannot rule out “extra” values in addition to the desired ones.

We need some more mechanisms for strictly adequate specifications.



-
1. General Remarks
 2. Loose Specifications
 - 3. Loose Specifications with Constructors**
 4. Loose Specifications with Free Constructors
 5. Summary

Generated Algebras

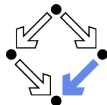


Take signature $\Sigma = (S, \Omega)$, Σ -algebra A .

- Define set of operations $\Omega_c \subseteq \Omega$ (the **constructors**).
 - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is **generated by** Ω_c :
 - For each sort $s \in S$ and $a \in A(s)$, there exists a ground term $t \in T_{\Sigma_c, s}$ with $a = A(t)$.
 - Carrier a can be described by a term t that involves only constructors.
 - A is **generated** if it is generated by Ω .
- $Gen(\Sigma, \Omega_c) := \{A \in Alg(\Sigma) \mid A \text{ is generated by } \Omega_c\}$.
 - The set of all Σ -algebras generated by constructors Ω_c .
 - $Gen(\Sigma) := Gen(\Sigma, \Omega)$.

Generated algebra does not contain “junk” in the carriers.

Example

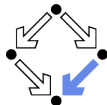


Take signature

$\text{NAT} = (\{\text{nat}\}, \Omega = \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}, + : \text{nat} \times \text{nat} \rightarrow \text{nat}\})$.

- Classical NAT-algebra $A = (\mathbb{N}, 0_{\mathbb{N}}, +_{\mathbb{N}})$.
- Constructors $\Omega_c := \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}\}$.
- A is generated by Ω_c :
 - For every $n \in \mathbb{N}$, $n = A(\underbrace{s(s(s(\dots(s(0))))}_{n \text{ times}}))$.
- A is also generated by Ω .
 - Any superset of a set of constructors is also a set of constructors.

Usually one looks for the minimal set of constructors.



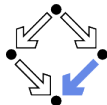
Algebras Generated in Some Sorts

Take signature $\Sigma = (S, \Omega)$, Σ -algebra A .

- Define set of sorts $S_c \subseteq S$ and set of operations $\Omega_c \subseteq \Omega$ (the **constructors**) with target sorts in S_c .
 - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is **generated by** Ω_c in S_c :
 - For each sort $s \in S_c$ and $a \in A(s)$, there exists
 - a set X of variables in Σ with $X_s = \emptyset$ for every s in S_c ,
 - an assignment $\alpha : X \rightarrow A$,
 - and a term $t \in T_{\Sigma_c(X), s}$with $a = A(\alpha)(t)$.
 - Value a can be described by a term t that involves only constructors in the generated sorts and variables in the non-generated sorts.
- A is **generated in** S_c if it is generated in S_c by Ω .

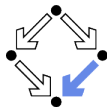
Algebra does not contain “junk” in the carriers of the generated sorts.

Example



- Signature $LIST = (S, \Omega)$:
 - $S = \{el, list\}$.
 - $\Omega = \{[] : \rightarrow list, Add : el \times list \rightarrow list, _ \cdot _ : list \times list \rightarrow list\}$.
- LIST-algebra A :
 - $A(el)$... a set of "elements".
 - $A(list)$... the set of finite lists of elements.
 - $A([])$... the empty list.
 - $A(Add)$ adds an element at the front of the list.
 - $A(\cdot)$ concatenates two lists.
- A is generated by $\Omega_c = \{[], Add\}$ in $S_c = \{list\}$:
 - Take arbitrary $l = [e_1, e_2, \dots, e_n] \in A(list)$.
 - Define $X_{el} := \{x_1, x_2, \dots, x_n\}$.
 - Define $\alpha_{el} := [x_1 \mapsto e_1, x_2 \mapsto e_2, \dots, x_n \mapsto e_n]$.
 - Then $l = A(\alpha)(Add(x_1, Add(x_2, \dots, Add(x_n, []))))$.

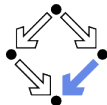
Proofs by Induction



In generated sorts, the principle of structural induction can be applied.

- Take the LIST-algebra A of the previous example.
 - Notation: c_A for $A(c)$.
 - Knowledge: (1) $\forall l \in list_A : []_A \cdot_A l = l$.
(2) $\forall e \in el_A, l, r \in list_A :$
 $Add_A(e, l) \cdot_A r = Add_A(e, l \cdot_A r)$.
- Prove: $\forall l \in list_A : l \cdot_A []_A = l$.
- Induction base $l = []_A$:
 - $l \cdot_A []_A = []_A \cdot_A []_A \stackrel{(1)}{=} []_A = l$.
- Induction step $l = Add_A(e, r)$ (for some $e \in el_A, r \in list_A$).
 - Induction Hypothesis (H): $r \cdot_A []_A = r$.
 - $l \cdot_A []_A = Add_A(e, r) \cdot_A []_A$
 $\stackrel{(2)}{=} Add_A(e, r \cdot_A []_A)$
 $\stackrel{(H)}{=} Add_A(e, r) = l$.

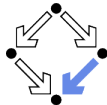
Loose Specifications with Constructors



Take logic L .

- **Loose specification with constructors** $sp = (\Sigma, \Phi, S_c, \Omega_c)$ in L :
 - Signature $\Sigma = (S, \Omega)$, set of formulas $\Phi \subseteq L(\Sigma)$, **generated sorts** $S_c \subseteq S$, **constructors** $\Omega_c \subseteq \Omega$ with target sorts in S_c .
 - **Semantics** $\mathcal{M}(sp) = Mod_{\mathcal{U}, \Sigma}(\Phi)$ where $\mathcal{U} = \{A \in Alg(\Sigma) \mid A \text{ is generated in } S_c \text{ by } \Omega_c\}$.
 - Only generated Σ -algebras are candidates for the specified ADT.

A loose specification with constructors specifies as the ADT the class of all models of its formula set that are generated by the constructors.



loose spec

sorts [**generated**] *sort* ...

opns [**constr**] *operation* ...

vars *variable: sort* ...

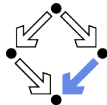
axioms *formula* ...

endspec

- Signature $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$.
- Set of formulas $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$.
- Generated sorts $S_c = \{\mathbf{generated} \textit{sort}, \dots\}$.
- Constructors $\Omega_c = \{\mathbf{constr} \textit{operation}, \dots\}$.

We will only use the concrete syntax to define specifications.

Example



loose spec

sorts *el*

generated *bool*

generated *list*

opns

constr *True* : \rightarrow *bool*

constr *False* : \rightarrow *bool*

constr *[]* : \rightarrow *list*

constr *Add* : *el* \times *list* \rightarrow *list*

.. .. : *list* \times *list* \rightarrow *list*

vars *l, m* : *list*, *e, e₁, e₂* : *el*

axioms

$\neg(\text{True} = \text{False})$

$\neg([\] = \text{Add}(e, l))$

$\text{Add}(e_1, l_1) = \text{Add}(e_2, l_2) \Rightarrow e_1 = e_2$

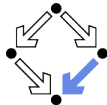
$[\].l = l$

$\text{Add}(e, l).m = \text{Add}(e, l.m)$

endspec

Strictly adequate specification of the “classical” list algebra in *PL*.

Example



loose spec

sorts

generated *bool*

generated *nat*

opns

constr *True* :→ *bool*

constr *False* :→ *bool*

constr *0* :→ *nat*

constr *Succ* : *nat* → *nat*

- *+* - : *nat* × *nat* → *nat*

- *** - : *nat* × *nat* → *nat*

- *≤* - : *nat* × *nat* → *bool*

vars *m, n* : *nat*

axioms

$\neg(\text{True} = \text{False})$

$\neg(0 = \text{Succ}(n))$

$\text{Succ}(n) = \text{Succ}(m) \Rightarrow n = m$

$(0 \leq n) = \text{True}$

$(\text{Succ}(n) \leq 0) = \text{False}$

$(\text{Succ}(n) \leq \text{Succ}(m)) = (n \leq m)$

$n + 0 = n$

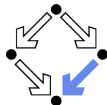
$n + \text{Succ}(m) = \text{Succ}(n + m)$

$n * 0 = 0$

$n * \text{Succ}(m) = n + (n * m)$

endspec

Strictly adequate specification of Peano arithmetic in *PL*.



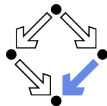
Specified ADT is Strictly Adequate

Proof requires two parts.

- Peano arithmetic satisfies the specified axioms.
 - Can be easily checked.
- Specified ADT is monomorphic: $\forall B, C \in \mathcal{M}(sp) : B \simeq C$.
 - There is an isomorphism $h : B \rightarrow C$.
 - A bijective homomorphism.
 - Definition of unique term representation for every value.
 - Simplifies the remainder of the proof.
 - Definition of bijective mapping h :
 - By pattern matching on term representation.
 - Proof that h is a homomorphism:
 - By using properties expressed with the help of the term representation.

Term representation essential for this kind of proofs.

Values have Unique Term Representations

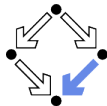


Take arbitrary $A \in \mathcal{M}(sp)$.

- $bool_A = \{True_A, False_A\}$ and $True_A \neq False_A$.
 - A is generated by $\{True, False\}$ in $bool$.
 - **axiom** $\neg(True = False)$.
- $nat_A = \{Succ^k(0)_A : k \in \mathbb{N}\}$ and $\forall k \neq l : Succ^k(0)_A \neq Succ^l(0)_A$.
 - A is generated by $\{0, Succ\}$ in nat .
 - Proof by induction on k : $\forall l \neq k : Succ^k(0)_A \neq Succ^l(0)_A$.
 - $k = 0, l \neq 0$: $0_A \neq Succ^l(0)_A$ (by **axiom** $\neg(0 = Succ(n))$).
 - $k \neq 0, l \neq k$: assume $Succ^k(0)_A = Succ^l(0)_A$, show $k = l$.
Know $l \neq 0$ (by **axiom** $\neg(0 = Succ(n))$).
Thus $k = k' + 1, l = l' + 1$, it suffices to show $k' = l'$.
By assumption, $Succ(Succ^{k'}(0))_A = Succ(Succ^{l'}(0))_A$.
Thus $Succ^{k'}(0)_A = Succ^{l'}(0)_A$ (**axiom** $Succ(n) = Succ(m) \Rightarrow n = m$).
By induction hypothesis, $k' = l'$.

Values are uniquely described by constructor applications.

Definition of Bijective Mapping

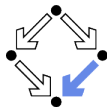


Take arbitrary $B, C \in \mathcal{M}(sp)$.

- h is defined by **pattern matching** on constructor terms:
 - $h_{bool}(True_B) := True_C$.
 - $h_{bool}(False_B) := False_C$.
 - $h_{nat}(Succ^k(0)_B) = Succ^k(0)_C$, for all $k \geq 0$.
- h is consistently defined:
 - $True_B$ and $False_B$ denote different values.
 - $Succ^k(0)_B$ denote different values for different k .
- h is bijective:
 - $True_C$ and $False_C$ denote different values.
 - $Succ^k(0)_C$ denote different values for different k .

One-to-one correspondence between the carriers of B and C .

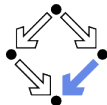
Homomorphism Proof



- Clear for constructors *True*, *False*, *0*, *Succ*:
 - Definition of h already expresses homomorphism condition.
- **Goal:** $\forall m, n \in \text{nat}_B . h(\text{op}_B(m, n)) = \text{op}_C(h(m), h(n))$.
 $\text{op} \dots +, *, \leq$.
 - $\forall k, l \geq 0 . h(\text{op}_B(\text{Succ}^k(0)_B, \text{Succ}^l(0)_B)) = \text{op}_C(h(\text{Succ}^k(0)_B), h(\text{Succ}^l(0)_B))$.
 - B and C are generated by $\{0, \text{Succ}\}$ in nat .
 - $\forall k, l \geq 0 . h(\text{op}_B(\text{Succ}^k(0)_B, \text{Succ}^l(0)_B)) = \text{op}_C(\text{Succ}^k(0)_C, \text{Succ}^l(0)_C)$.
 - By definition of h .
 - $\forall k, l \geq 0 . h(\text{op}(\text{Succ}^k(0), \text{Succ}^l(0))_B) = \text{op}(\text{Succ}^k(0), \text{Succ}^l(0))_C$.
 - By definition of term semantics.

Proof goal is expressed with the help of constructor terms.

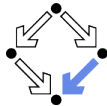
Homomorphism Proof



The core of the homomorphism proof.

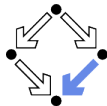
- **Goal:** $h((\text{Succ}^k(0) + \text{Succ}^l(0))_B) = (\text{Succ}^k(0) + \text{Succ}^l(0))_C$.
 - First simplify left and right hand side of the equation.
- **Lemma:** $\forall A \in \mathcal{M}(sp) : (\text{Succ}^k(0) + \text{Succ}^l(0))_A = \text{Succ}^{k+l}(0)_A$.
 - Induction base $l = 0$: by **axiom** $n + 0 = n$.
 - Induction step $l = l' + 1$:
$$\begin{aligned} & (\text{Succ}^k(0) + \text{Succ}^{l'+1}(0))_A \\ &= \text{Succ}(\text{Succ}^k(0) + \text{Succ}^{l'}(0))_A \\ &= \text{Succ}(\text{Succ}^{k+l'}(0))_A \\ &= \text{Succ}^{k+l'+1}(0)_A. \end{aligned}$$
- **Simplified goal:** $h(\text{Succ}^{k+l}(0)_B) = \text{Succ}^{k+l}(0)_C$.
 - By definition of h .

Similar for the homomorphism proofs of the other operations.



-
1. General Remarks
 2. Loose Specifications
 3. Loose Specifications with Constructors
 - 4. Loose Specifications with Free Constructors**
 5. Summary

Freely Generated Algebras

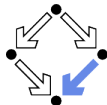


Take signature $\Sigma = (S, \Omega)$, Σ -algebra A .

- Define set of operations $\Omega_c \subseteq \Omega$ (the **constructors**).
 - Restricted signature $\Sigma_c = (S, \Omega_c)$.
- A is **freely generated by** Ω_c :
 - For each sort $s \in S$ and $a \in A(s)$, there exists **exactly one** ground term $t \in T_{\Sigma_c, s}$ with $a = A(t)$.
 - Value a can be described by a **unique** term t that involves only constructors.
 - A is **freely generated** if it is generated by Ω .
- A is **freely generated by** Ω_c **in** S_c :
 - Analogous definition as for **generated by ... in ...**

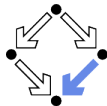
Freely generated algebras have unique constructor term representations for the values of the freely generated sorts (no “junk” in carriers and no “confusion” among constructor terms).

Example



- The “classical” BOOL-algebra $(\{true, false\}, \dots)$:
 - Freely generated by $\{True, False\}$.
 - Not freely generated by $\{True, False, \neg\}$.
- The “one-element” BOOL-algebra $(\{\#\}, \dots)$.
 - Freely generated by $\{True\}$ and by $\{False\}$.
 - Not freely generated by $\{True, False\}$.
- The “classical” NAT-algebra (\mathbb{N}, \dots) :
 - Freely generated by $\{0, Succ\}$.
 - Not freely generated by $\{0, Succ, +\}$.
- The “classical” INT-algebra (\mathbb{Z}, \dots) :
 - $INT = (int, \{0 : \rightarrow int, Succ : int \rightarrow int, Pred : int \rightarrow int\})$.
 - Not freely generated by (any subset of) operations.

A set of free constructors cannot be extended.



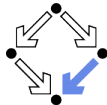
Inductive Function Definitions

Freely generated algebras allow inductive function definitions.

- Signature $\text{LIST} = (S, \Omega)$:
 - $S = \{el, list\}$.
 - $\Omega = \{[] : \rightarrow list, Add : el \times list \rightarrow list, _ \cdot _ : list \times list \rightarrow list\}$.
- Classical LIST-algebra A as in the previous example.
 - A is freely generated by $\Omega_c = \{[], Add\}$ in $S_c = \{list\}$:
- Inductive definition of function $g : A(list) \rightarrow \mathbb{N}$.
 - $g([]_A) = 0$.
 - $g(Add(x, t)_A) = g(t_A) + 1$ for all $x \in X, t \in T_{\Sigma_c(X), list}$.

Inductive definition by “pattern matching” on constructor terms (independent of the nature of the carrier).

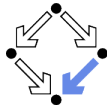
Loose Specifications with Free Constructors



Take logic L .

- **Loose specification with free constructors** $sp = (\Sigma, \Phi, S_c, \Omega_c)$ in L :
 - Signature $\Sigma = (S, \Omega)$, set of formulas $\Phi \subseteq L(\Sigma)$, **freely generated sorts** $S_c \subseteq S$, **constructors** $\Omega_c \subseteq \Omega$ with target sorts in S_c .
 - **Semantics** $\mathcal{M}(sp) = \text{Mod}_{\mathcal{U}, \Sigma}(\Phi)$ where
$$\mathcal{U} = \{A \in \text{Alg}(\Sigma) \mid A \text{ is freely generated in } S_c \text{ by } \Omega_c\}.$$
 - Only freely generated Σ -algebras are candidates for the specified ADT.

A loose specification with free constructors specifies the class of all models of its formula set that are freely generated by the constructors.



loose spec

sorts [**freely generated**] *sort* ...

opns [**constr**] *operation* ...

vars *variable: sort* ...

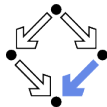
axioms *formula* ...

endspec

- Signature $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$.
- Set of formulas $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$.
- Generated sorts $S_c = \{\mathbf{freely\ generated\ sort}, \dots\}$.
- Constructors $\Omega_c = \{\mathbf{constr\ operation}, \dots\}$.

Also mixing of generated sorts with freely generated sorts possible.

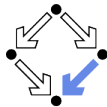
Example



```
loose spec
  sorts el
    freely generated bool
    freely generated list
  opns
    constr True :→ bool
    constr False :→ bool
    constr [ ] :→ list
    constr Add : el × list → list
    . . . : list × list → list
  vars l, m : list, e, e1, e2 : el
  axioms
    [ ].l = l
    Add(e, l).m = Add(e, l.m)
endspec
```

Strictly adequate specification of the “classical” list algebra in EL; the non-constructor operation is inductively defined.

Example



loose spec

sorts

freely generated *bool*

freely generated *nat*

opns

constr *True* : \rightarrow *bool*

constr *False* : \rightarrow *bool*

constr *0* : \rightarrow *nat*

constr *Succ* : *nat* \rightarrow *nat*

- $+$ - : *nat* \times *nat* \rightarrow *nat*

- $*$ - : *nat* \times *nat* \rightarrow *nat*

- \leq - : *nat* \times *nat* \rightarrow *bool*

vars *m, n* : *nat*

axioms

$(0 \leq n) = True$

$(Succ(n) \leq 0) = False$

$(Succ(n) \leq Succ(m)) = (n \leq m)$

$n + 0 = n$

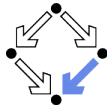
$n + Succ(m) = Succ(n + m)$

$n * 0 = 0$

$n * Succ(m) = n + (n * m)$

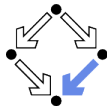
endspec

Strictly adequate specification of the “classical” list algebra in EL; the non-constructor operations are inductively defined.



-
1. General Remarks
 2. Loose Specifications
 3. Loose Specifications with Constructors
 4. Loose Specifications with Free Constructors
 5. Summary

Summary

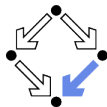


A couple of core messages. . .

- A loose specification describes a class of models as an ADT.
 - To check whether a given algebra implements the specification (i.e., whether it is an element of the specified ADT):
 - Check whether the algebra satisfies the specification axioms.
 - There may exist “confusion” among terms.
 - Carriers may collapse to singletons (or be too “small”).
 - In *PL*, additional axioms can prevent this.
 - Non-equalities of operation results (injectiveness of operations).
 - Carriers may contain “junk”.
 - In *PL*, an additional axiom can prevent this for a finite carrier.
 - Axiom enumerates constants that denote all *s* of the sort.

Without constructors, loose specifications are generally clumsy because many “boring” axioms are needed.

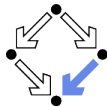
Summary (Contd)



- Loose specifications with **constructors**.
 - Every value is denoted by **some** constructor term.
 - Thus junk is removed from (also infinite) carriers.
 - Induction proofs on term representation of s become possible.
 - Problem: not all carriers allow term representations.
 - ADT “real” (carrier is not countable).
- Loose specifications with **free constructors**.
 - Every value is denoted by **exactly one** constructor term.
 - Thus there is no “confusion” among constructor terms and the collapse of carriers is prevented.
 - Inductive function definitions by pattern matching on term representations of s become possible.
 - Problem: not all carriers have unique term representations.
 - ADT “set” (no unique representation at all).
 - ADT “integer” (unique representation is unconvient).

With constructors, loose specifications become easy to use.

Summary (Contd)



So what is the role of loose specifications. . .

- Loose specifications are **good** for specifying **requirements**.
 - May specify zero, one, many datatypes (polymorphic ADTs).
 - Thus allow arbitrarily many implementations.
 - A loose specification may not have any model (implementation) at all!
 - Specification axioms can (should) be abstract.
 - Later verification that concrete implementation satisfies the axioms.
- Loose specifications are **not good** for specifying **designs**.
 - Not descriptions of concrete algorithms/implementations.
- Loose specifications are generally **not executable**.
 - No engines to execute loose specifications for rapid prototyping.

Loose specifications are for *reasoning*, not for *executing*; they are the basis of program specification languages such as Larch/C++.