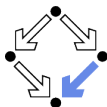


Abstract Datatypes

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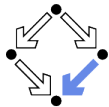
Signatures



Our goal is to model abstract data types.

- A **signature** $\Sigma = (\mathcal{S}, \Omega)$.
 - \mathcal{S} ... set of **sorts**.
 - Ω ... set of **operations** of form $n : s_1 \times \dots \times s_k \rightarrow s$.
 - $s_1, \dots, s_k, s \in \mathcal{S}, k \geq 0$.
 - **operation name** n .
 - **argument sorts** $s_1 \times \dots \times s_k$.
 - **target sort** s .
 - **arity** $s_1 \times \dots \times s_k \rightarrow s$.
 - Case $k = 0$: **constant** $n : \rightarrow s$ of sort s .

A signature models the syntactic interface of an abstract data type.



Example

- $\text{BOOL} = (S_B, \Omega_B)$.
 - $S_B = \{\text{bool}\}$.
 - $\Omega_B = \{\text{True} : \rightarrow \text{bool}, \text{False} : \rightarrow \text{bool}, \neg : \text{bool} \rightarrow \text{bool}, \wedge : \text{bool} \times \text{bool} \rightarrow \text{bool}\}$.
- $\text{NATBOOL} = (S_N, \Omega_N)$.
 - $S_N = \{\text{nat}, \text{bool}\}$.
 - $\Omega_N = \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}, \leq : \text{nat} \times \text{nat} \rightarrow \text{bool}\}$.
- $\text{NATSTACK} = (S, \Omega)$.
 - $S = \{\text{nat}, \text{bool}, \text{stack}\}$.
 - $\Omega = \{\text{Emptystack} : \rightarrow \text{stack}, \text{Push} : \text{stack} \times \text{nat} \rightarrow \text{stack}, \text{Pop} : \text{stack} \rightarrow \text{stack}, \text{Top} : \text{stack} \rightarrow \text{nat}\}$.

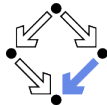
Many-Sorted Algebras



Take signature $\Sigma = (S, \Omega)$.

- A (many-sorted) algebra A for Σ (a Σ -algebra A):
 - A carrier (set) $A(s)$
 - for each sort $s \in S$.
 - A function $A(n : s_1 \times \dots \times s_k \rightarrow s) : A(s_1) \times \dots \times A(s_k) \rightarrow A(s)$
 - for each operation $n : s_1 \times \dots \times s_k \rightarrow s \in \Omega$.
 - (i.e., a value $A(n : \rightarrow s)$ for each constant $n : \rightarrow s \in \Omega$).
- An algebra assigns a meaning to a signature.
 - A set for each sort, a function for each operation.
- $\text{Alg}(\Sigma) := \{A : A \text{ is a } \Sigma\text{-algebra}\}$.
 - The set of all Σ -algebras.

A Σ -algebra models a possible implementation of an abstract datatype.



Example

- Signature $\text{NAT} = (S_N, \Omega_N)$.
 - $S_N = \{\text{nat}\}$.
 - $\Omega_N = \{0 : \rightarrow \text{nat},$
 $\text{Succ} : \text{nat} \rightarrow \text{nat}\}$.
- NAT-algebra A :
 - $A(\text{nat}) = \mathbb{N}$.
 - $A(0) = 0_{\mathbb{N}}$.
 - $A(\text{Succ}) : \mathbb{N} \rightarrow \mathbb{N}$
 $A(\text{Succ})(n) = n + 1$ (i.e., $A(\text{Succ}) = \lambda n.n + 1$).
- NAT-algebra B :
 - $B(\text{nat}) = \{\text{true}, \text{false}\}$.
 - $B(0) = \text{false}$.
 - $B(\text{Succ}) : \{\text{true}, \text{false}\} \rightarrow \{\text{true}, \text{false}\}$
 $B(\text{Succ})(n) = \neg n$.

Not all Σ -algebras behave in the “same” way.

Homomorphisms



Take Σ -algebras A and B for signature $\Sigma = (S, \Omega)$.

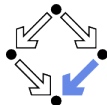
- A Σ -homomorphism $h : A \rightarrow B$ from A to B :
 - $h = (h_s)_{s \in S}$.
 - A function for every sort in the signature.
 - $h_s : A(s) \rightarrow B(s)$.
 - The function maps carrier of A to corresponding carrier of B .
 - $h_s(A(\omega)(a_1, \dots, a_k)) = B(\omega)(h_{s_1}(a_1), \dots, h_{s_k}(a_k))$.
 - for every operation $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega$
 - and every tuple $(a_1, \dots, a_k) \in A(s_1) \times \dots \times A(s_k)$.

$$\begin{array}{ccc} A(s_1) \times \dots \times A(s_k) & \xrightarrow{A(\omega)} & A(s) \\ h_{s_1} \downarrow \cdots h_{s_k} \downarrow & & h_s \downarrow \\ B(s_1) \times \dots \times B(s_k) & \xrightarrow{B(\omega)} & B(s) \end{array}$$

- For constant ω ($k = 0$): $h_s(A(\omega)) = B(\omega)$.

Homomorphism condition: the mappings are “compatible”.

Homomorphisms



How to interpret the existence of a homomorphism $h : A \rightarrow B$?

- Functions $A(\omega)$ and $B(\omega)$ are “compatible”.
 - May first apply $A(\omega)$ to arguments and then map the result to B .
 - Or may first map the arguments to B and then apply $B(\omega)$.
 - Both methods yield the same B -value.
- Carrier of A has (at least) as much structure as carrier of B .
 - If the B -counterparts b_1 and b_2 of the A -values a_1 and a_2 are different, then also a_1 and a_2 are different.

If $b_1 = h(a_1) \neq b_2 = h(a_2)$, we have $h(a_1) \neq h(a_2)$, and thus $a_1 \neq a_2$.
 - Nevertheless, different A -values a_1 and a_2 may have identical B -counterparts b_1 and b_2 .

Also if $a_1 \neq a_2$, it may be the case that $h(a_1) = h(a_2)$.

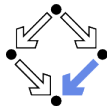
Guidelines for the intuition about homomorphism relation.

Isomorphisms



- An Σ -isomorphism is a bijective Σ -homomorphism.
 - Bijective: one-to-one mapping between A and B .
 - Surjective and injective.
 - Surjective: $\forall b \in B(s) : \exists a \in A(s) : h_s(a) = b$.
 - Every value of $B(s)$ is the counterpart of some value of $A(s)$.
 - Injective: $\forall a, a' \in A(s) : h_s(a) = h_s(a') \Rightarrow a = a'$.
 - Different values of $A(s)$ are mapped to different values of $B(s)$.
- Two Σ -algebras A and B are **isomorphic** ($A \simeq B$):
 - There exists a Σ -isomorphism between A and B .
- The isomorphism-relation \simeq is an equivalence relation.
 - Has reflexivity, symmetry, transitivity.

Isomorphic Σ -algebras are “identical up to renaming”.



Example

- Signature $\text{BOOL} = (\{\text{bool}\}, \{ \text{True} : \rightarrow \text{bool}, \text{False} : \rightarrow \text{bool}, \neg : \text{bool} \rightarrow \text{bool}, \wedge : \text{bool} \times \text{bool} \rightarrow \text{bool} \})$.

- BOOL -algebra A :

$$A(\text{bool}) = \{\text{true}, \text{false}\}$$

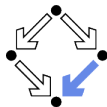
$$A(\text{True}) = \text{true}$$

$$A(\text{False}) = \text{false}$$

$$A(\neg)(n) := \text{not}(n) = \begin{cases} \text{false}, & \text{if } n = \text{true} \\ \text{true}, & \text{if } n = \text{false} \end{cases}$$

$$A(\wedge)(n, m) := \text{and}(n, m) = \begin{cases} \text{true}, & \text{if } n = m = \text{true} \\ \text{false}, & \text{otherwise} \end{cases}$$

The “classical” BOOL -algebra.



Example (Contd)

- BOOL-algebra B :

$$B(\text{bool}) = \{\#\}$$

$$B(\text{True}) = B(\text{False}) = B(\neg)(\#) = B(\wedge)(\#, \#) = \#.$$

- BOOL-algebra C :

$$C(\text{bool}) = \{0, 1\}$$

$$C(\text{True}) = 1$$

$$C(\text{False}) = 0$$

$$C(\neg)(n) = 1 - n$$

$$C(\wedge)(n, m) = n * m$$

- BOOL-algebra D :

$$D(\text{bool}) = \mathbb{N}$$

$$D(\text{True}) = 1$$

$$D(\text{False}) = 0$$

$$D(\neg)(n) = \begin{cases} n + 1, & \text{if } n \text{ is even} \\ n - 1, & \text{otherwise} \end{cases}$$

$$D(\wedge)(n, m) = n * m$$

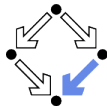


Example (Contd'2)

How can all these BOOL-algebras be related?

- Homomorphism $h : A \rightarrow B$:
 - $h(true) = h(false) = \#$.
- No homomorphism from B to A .

Assume homomorphism $h : B \rightarrow A$.
Then $h(\#) = h(B(\neg)(\#)) = A(\neg)(h(\#)) = not(h(\#)) \neq h(\#)$.
- Isomorphism $g : A \rightarrow C$:
 - $g(true) = 1, g(false) = 0$.
 - $g^{-1}(1) = true, g^{-1}(0) = false$.
- A and D are not isomorphic.
 - No bijection between $\{true, false\}$ and \mathbb{N} .
- Homomorphisms $k : A \rightarrow D$ and $l : D \rightarrow A$.
 - $k(true) = 1, k(false) = 0$.
 - $l(n) = (n \text{ is even})$.



Example (Contd'3)

- BOOL-algebra E :

$$E(\text{bool}) = \mathbb{N}$$

$$E(\text{True}) = 1$$

$$E(\text{False}) = 0$$

$$E(\neg)(n) = n + 1$$

$$E(\wedge)(n, m) = n + m$$

- Neither a homomorphism from A to E nor one from E to A .

- Assume homomorphism $h : A \rightarrow E$.

$$\text{Then } h(\text{false}) = h(A(\neg)(\text{true})) = E(\neg)(h(\text{true})) = h(\text{true}) + 1.$$

$$\text{Also } h(\text{true}) = h(A(\neg)(\text{false})) = E(\neg)(h(\text{false})) = h(\text{false}) + 1.$$

$$\text{But then } h(\text{false}) = h(\text{true}) + 1 = (h(\text{false}) + 1) + 1 = h(\text{false}) + 2.$$

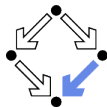
- Assume homomorphism $g : E \rightarrow A$.

$$\text{Then } g(1) = g(E(\neg)(0)) = A(\neg)(g(0)) = \text{not}(g(0)).$$

$$\text{Also } g(1) = g(E(\wedge)(1, 0)) = A(\wedge)(g(1), g(0)) = \text{and}(\text{not}(g(0)), g(0)) = \text{false}.$$

$$\text{Also } g(2) = g(E(\neg)(1)) = A(\neg)(g(1)) = \text{not}(g(1)) = \text{true}.$$

$$\text{But also } g(2) = g(E(\wedge)(1, 1)) = A(\wedge)(g(1), g(1)) = \text{and}(\text{false}, \text{false}) = \text{false}.$$



Abstract Data Types

- A **datatype**:
 - An equivalence class of isomorphic Σ -algebras.
 - A class $[A] = \{B \in Alg(\Sigma) : B \simeq A\}$ (for some Σ -algebra A).
 - The elements of such a class are identical up to renaming.
 - Thus we do not consider individual Σ -algebras as datatypes.
- An **abstract data type (ADT)**:
 - A class of Σ -algebras closed under isomorphism.
 - A class $\mathcal{C} \subseteq Alg(\Sigma)$.
 - If $A \in \mathcal{C}$ and $A \simeq B$, then $B \in \mathcal{C}$ (for any Σ -algebras A and B).
 - Every ADT \mathcal{C} can be decomposed into datatypes:
 - $\mathcal{C} = \bigcup\{[A] : A \in \mathcal{C}\}$.
 - All the datatypes that can implement the ADT.
 - An ADT is **monomorphic** if all its elements are isomorphic.
 - The ADT can be implemented by a single datatype.
 - A non-monomorphic ADT is **polymorphic**.
 - The ADT can be implemented by multiple datatypes.

Example



Take the BOOL-algebras of the previous example.

- ADT $\mathcal{A} := \{J \in Alg(\text{BOOL}) : J \simeq A\}$
 - All algebras isomorphic to the classical the BOOL-algebra A .
 - A monomorphic ADT with a single datatype $[A]$ containing C .
- ADT $\mathcal{B} := \{J \in Alg(\text{BOOL}) : J \simeq A \vee J \simeq B\}$
 - A polymorphic ADT with two datatypes $[A]$ and $[B]$.

We need a language to specify abstract datatypes in a convenient way.