Abstract Datatypes

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Signatures

Our goal is to model abstract data types.

- A signature $\Sigma = (S, \Omega)$.
  - S ... set of sorts.
  - Ω ... set of operations of form $n : s_1 \times \ldots \times s_k \to s$.
    - $s_1, \ldots, s_k, s \in S, k \geq 0$.
    - operation name n.
    - argument sorts $s_1 \times \ldots \times s_k$.
    - target sort s.
    - arity $s_1 \times \ldots \times s_k \to s$.
    - Case $k = 0$: constant $n :\to s$ of sort s.

A signature models the syntactic interface of an abstract data type.

Example

- $\mathrm{BOOL} = (S_B, \Omega_B)$.
  - $S_B = \{bool\}$.
  - $\Omega_B = \{True :\rightarrow bool, False :\rightarrow bool,$
    $\neg : bool \rightarrow bool,$
    $\wedge : bool \times bool \rightarrow bool\}$.
- $\mathrm{NATBOOL} = (S_N, \Omega_N)$.
  - $S_N = \{nat, bool\}$.
  - $\Omega_N = \{0 :\rightarrow nat,$
    $Succ : nat \rightarrow nat,$
    $\leq : nat \times nat \rightarrow bool\}$.
- $\mathrm{NATSTACK} = (S, \Omega)$.
  - $S = \{nat, bool, stack\}$.
  - $\Omega = \{Emptystack :\rightarrow stack,$
    $Push : stack \times nat \rightarrow stack,$
    $Pop : stack \rightarrow stack,$
    $Top : stack \rightarrow nat\}$.

Many-Sorted Algebras

Take signature $\Sigma = (S, \Omega)$.

- A (many-sorted) algebra A for Σ (a Σ-algebra A):
  - A carrier (set) $A(s)$
    - for each sort $s \in S$.
  - A function $A(n : s_1 \times \ldots \times s_k \to s) : A(s_1) \times \ldots \times A(s_k) \to A(s)$
    - for each operation $n : s_1 \times \ldots \times s_k \to s \in \Omega$.
    - (I.e., a value $A(n :\to s)$ for each constant $n :\to s \in \Omega$).
- An algebra assigns a meaning to a signature.
  - A set for each sort, a function for each operation.
- $Alg(\Sigma) := \{A : A \text{ is a } \Sigma\text{-algebra}\}$.
  - The set of all Σ-algebras.

A Σ-algebra models a possible implementation of an abstract datatype.

Example

- Signature $\mathrm{NAT} = (S_N, \Omega_N)$.
  - $S_N = \{nat\}$.
  - $\Omega_N = \{0 :\to nat,$
    $Succ : nat \to nat\}$.
- NAT-algebra A:
  - $A(\mathrm{nat}) = \mathbb{N}$.
  - $A(0) = 0_{\mathbb{N}}$.
  - $A(Succ) : \mathbb{N} \to \mathbb{N}$
    $A(Succ)(n) = n + 1$ (i.e., $A(Succ) = \lambda n.n + 1$).
- NAT-algebra B:
  - $B(\mathrm{nat}) = \{\mathrm{true}, \mathrm{false}\}$.
  - $B(0) = \mathrm{false}$.
  - $B(Succ) : \{\mathrm{true}, \mathrm{false}\} \to \{\mathrm{true}, \mathrm{false}\}$
    $B(Succ)(n) = \neg n$.

Not all Σ-algebras behave in the "same" way.

Homomorphisms

Take Σ-algebras A and B for signature $\Sigma = (S, \Omega)$.

- A Σ-homomorphism $h : A \to B$ from A to B:
  - $h = (h_s)_{s \in S}$.
    - A function for every sort in the signature.
  - $h_s : A(s) \to B(s)$.
    - The function maps carrier of A to corresponding carrier of B.
  - $h_s(A(\omega)(a_1, \ldots, a_k)) = B(\omega)(h_{s_1}(a_1), \ldots, h_{s_k}(a_k))$.
    - for every operation $\omega = (n : s_1 \times \ldots \times s_k \to s) \in \Omega$
    - and every tuple $(a_1, \ldots, a_k) \in A(s_1) \times \ldots \times A(s_k)$.

$$\begin{array}{ccc}
A(s_1) \times \ldots \times A(s_k) & \xrightarrow{A(\omega)} & A(s) \\
h_{s_1} \downarrow \cdots h_{s_k} \downarrow & & h_s \downarrow \\
B(s_1) \times \ldots \times B(s_k) & \xrightarrow{B(\omega)} & B(s)
\end{array}$$

    - For constant ω $(k = 0)$: $h_s(A(\omega)) = B(\omega)$.

Homomorphism condition: the mappings are "compatible".

Homomorphisms

How to interpret the existence of a homomorphism $h : A \to B$?

- Functions $A(\omega)$ and $B(\omega)$ are "compatible".
  - May first apply $A(\omega)$ to arguments and then map the result to B.
  - Or may first map the arguments to B and then apply $B(\omega)$.
  - Both methods yield the same B-value.
- Carrier of A has (at least) as much structure as carrier of B.
  - If the B-counterparts b_1 and b_2 of the A-values a_1 and a_2 are different, then also a_1 and a_2 are different.

    If $b_1 = h(a_1) \neq b_2 = h(a_2)$, we have $h(a_1) \neq h(a_2)$, and thus $a_1 \neq a_2$.
  - Nevertheless, different A-values a_1 and a_2 may have identical B-counterparts b_1 and b_2.

    Also if $a_1 \neq a_2$, it may be the case that $h(a_1) = h(a_2)$.

Guidelines for the intuition about homomorphism relation.

Isomorphisms

- An Σ-isomorphism is a bijective Σ-homomorphism.
  - Bijective: one-to-one mapping between A and B.
    - Surjective and injective.
  - Surjective: $\forall b \in B(s) : \exists a \in A(s) : h_s(a) = b$.
    - Every value of $B(s)$ is the counterpart of some value of $A(s)$.
  - Injective: $\forall a, a' \in A(s) : h_s(a) = h_s(a') \Rightarrow a = a'$.
    - Different values of $A(s)$ are mapped to different values of $B(s)$.
- Two Σ-algebras A and B are isomorphic ($A \simeq B$):
  - There exists a Σ-isomorphism between A and B.
- The isomorphism-relation $\simeq$ is an equivalence relation.
  - Has reflexivity, symmetry, transitivity.

Isomorphic Σ-algebras are "identical up to renaming".

Example

- Signature $\mathrm{BOOL} = (\{bool\},$
  $\{True :\rightarrow bool, False :\rightarrow bool,$
  $\neg : bool \rightarrow bool, \wedge : bool \times bool \rightarrow bool\}).$
- BOOL-algebra A:

$$
\begin{aligned}
&A(bool) = \{true, false\}\\
&A(True) = true\\
&A(False) = false\\
&A(\neg)(n) := not(n) = \begin{cases} false, & \text{if } n = true \\ true, & \text{if } n = false \end{cases}\\
&A(\wedge)(n, m) := and(n, m) = \begin{cases} true, & \text{if } n = m = true \\ false, & \text{otherwise} \end{cases}
\end{aligned}
$$

The "classical" BOOL-algebra.

Example (Contd)

- BOOL-algebra B:
  $$B(\mathit{bool}) = \{\#\}$$
  $$B(\mathit{True}) = B(\mathit{False}) = B(\neg)(\#) = B(\wedge)(\#,\#) = \#.$$
- BOOL-algebra C:
  $$C(\mathit{bool}) = \{0,1\}$$
  $$C(\mathit{True}) = 1$$
  $$C(\mathit{False}) = 0$$
  $$C(\neg)(n) = 1 - n$$
  $$C(\wedge)(n,m) = n * m$$
- BOOL-algebra D:
  $$D(\mathit{bool}) = \mathbb{N}$$
  $$D(\mathit{True}) = 1$$
  $$D(\mathit{False}) = 0$$
  $$D(\neg)(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{otherwise} \end{cases}$$
  $$D(\wedge)(n,m) = n * m$$

Example (Contd'2)

How can all these BOOL-algebras be related?

- Homomorphism $h : A \to B$:
  - $h(\mathit{true}) = h(\mathit{false}) = \#$.
- No homomorphism from B to A.

  Assume homomorphism $h : B \to A$.
  Then $h(\#) = h(B(\neg)(\#)) = A(\neg)(h(\#)) = \mathit{not}(h(\#)) \neq h(\#)$.
- Isomorphism $g : A \to C$:
  - $g(\mathit{true}) = 1, g(\mathit{false}) = 0$.
  - $g^{-1}(1) = \mathit{true}, g^{-1}(0) = \mathit{false}$.
- A and D are not isomorphic.
  - No bijection between $\{\mathit{true}, \mathit{false}\}$ and $\mathbb{N}$.
- Homomorphisms $k : A \to D$ and $l : D \to A$.
  - $k(\mathit{true}) = 1, k(\mathit{false}) = 0$.
  - $l(n) = (n \text{ is even})$.

Example (Contd'3)

- BOOL-algebra E:

  $$E(bool) = \mathbb{N}$$
  $$E(True) = 1$$
  $$E(False) = 0$$
  $$E(\neg)(n) = n + 1$$
  $$E(\wedge)(n, m) = n + m$$

- Neither a homorphism from A to E nor one from E to A.
  - Assume homomorphism $h : A \to E$.

    Then $h(false) = h(A(\neg)(true)) = E(\neg)(h(true)) = h(true) + 1$.
    Also $h(true) = h(A(\neg)(false)) = E(\neg)(h(false)) = h(false) + 1$.

    But then $h(false) = h(true) + 1 = (h(false) + 1) + 1 = h(false) + 2$.
  - Assume homomorphism $g : E \to A$.

    Then $g(1) = g(E(\neg)(0)) = A(\neg)(g(0)) = not(g(0))$.
    Also $g(1) = g(E(\wedge)(1, 0)) = A(\wedge)(g(1), g(0)) = and(not(g(0)), g(0)) = false$.
    Also $g(2) = g(E(\neg)(1)) = A(\neg)(g(1)) = not(g(1)) = true$.
    But also $g(2) = g(E(\wedge)(1, 1)) = A(\wedge)(g(1), g(1)) = and(false, false) = false$.

Abstract Data Types

- A datatype:
  - An equivalence class of isomorphic Σ-algebras.
    - A class $[A] = \{B \in Alg(\Sigma) : B \simeq A\}$ (for some Σ-algebra A).
  - The elements of such a class are identical up to renaming.
    - Thus we do not consider individual Σ-algebras as datatypes.
- An abstract data type (ADT):
  - A class of Σ-algebras closed under isomorphism.
    - A class $\mathcal{C} \subseteq Alg(\Sigma)$.
    - If $A \in \mathcal{C}$ and $A \simeq B$, then $B \in \mathcal{C}$ (for any Σ-algebras A and B).
  - Every ADT $\mathcal{C}$ can be decomposed into datatypes:
    - $\mathcal{C} = \bigcup\{[A] : A \in \mathcal{C}\}$.
    - All the datatypes that can implement the ADT.
  - An ADT is monomorphic if all its elements are isomorphic.
    - The ADT can be implemented by a single datatype.
  - A non-monomorphic ADT is polymorphic.
    - The ADT can be implemented by multiple datatypes.

Example

Take the BOOL-algebras of the previous example.

- ADT $\mathcal{A} := \{J \in Alg(\mathrm{BOOL}) : J \simeq A\}$
  - All algebras isomorphic to the classical the BOOL-algebra A.
  - A monomorphic ADT with a single datatype $[A]$ containing C.
- ADT $\mathcal{B} := \{J \in Alg(\mathrm{BOOL}) : J \simeq A \vee J \simeq B\}$
  - A polymorphic ADT with two datatypes $[A]$ and $[B]$.

We need a language to specify abstract datatypes in a convenient way.