Basics of Complexity

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1. **Complexity of Computations**

2. **Asymptotic Complexity**

3. **Working with Asymptotic Complexity**

4. **Complexity Classes**

Complexity of Computations

We want to determine the resource consumption of computations.

- Determine the amount of resources consumed by a computation:
  - Time
  - Space (memory)
- Determine the resource consumption for classes of inputs:
  - The maximum complexity for all inputs of the class.
  - The average complexity for these inputs.

We are going to make these notions precise.

Resource Consumption

Turing machine M with input alphabet Σ that halts for every input.

- Input set $I = \Sigma^*$:
  - The set of input words.
- Input size $|.| : I \to \mathbb{N}$
  - $|i|$: the length of input i.
- Time consumption $t : I \to \mathbb{N}$:
  - $t(i)$: the number of moves that M makes for input i until it halts.
- Space consumption $s : I \to \mathbb{N}$:
  - $s(i)$: the largest distance from the beginning of the tape that the tape head of M reaches for input i until M halts.

For any computational model, I, $|i|$, $t(i)$ and $s(i)$ may be defined.

Worst-case Complexity

Computational model with I, $|i|$, $t(i)$ and $s(i)$ defined.

- Worst-case time complexity $T : \mathbb{N} \to \mathbb{N}$

$$T(n) := \max\{t(i) \mid i \in I \wedge |i| = n\}$$

- Worst-case space complexity $S : \mathbb{N} \to \mathbb{N}$

$$S(n) := \max\{s(i) \mid i \in I \wedge |i| = n\}$$

The maximum amount of resources consumed for any input of size n by computations in a given model.

Average Complexity

- Input distribution *Input*:
  - Family of (discrete) random variables $Input_n$ that describe the distribution of inputs of each size n in I
  - determined by probability function $p_I^n : I \to [0,1]$

    $p_I^n(i)$: probability that, among all inputs of size n, input i occurs.
- Average time/space complexity $\overline{T} : \mathbb{N} \to \mathbb{N}$ and $\overline{S} : \mathbb{N} \to \mathbb{N}$

$$\overline{T}(n) := E[Time_n]$$

$$\overline{S}(n) := E[Space_n]$$

  - Expected values of random variables $Time_n$ and $Space_n$
  - determined by probability functions $p_T^n : \mathbb{N} \to [0,1]$ and $p_S^n : \mathbb{N} \to [0,1]$

    $p_T^n(t)/p_S^n(s)$: probabilities that time t/space s is consumed for input of size n assuming that inputs of size n are distributed according to $Input_n$.

The average amount of resources consumed for inputs of size n (for a given distribution of inputs) by computations in a given model.

Example

Given non-empty integer array a of size $n > 0$, find minimum index j such that $a[j] = \max\{a[i] \mid 0 \leq i < n\}$.

$j := 0;\ m := a[j];\ i := 1$	1
while $i < length(a)$	n
if $a[i] > m$ **then**	$n-1$
$j := i;\ m := a[j]$	$N \leq n-1$
$i := i+1$	$n-1$

- Time: the number of lines executed.

$$T(n) = 1 + n + (n-1) + (n-1) + (n-1) = 4n-2$$

- Space: the number of variables used (including elements of a).

$$S(n) = \overline{S}(n) = n+3$$

We are going to analyze the average time complexity $\overline{T}(n)$.

Average Time Complexity

Assume a holds n distinct values $\{1, \ldots, n\}$.

- Assume all $n!$ permutations of a are equally probable.

$$p_I^n(i) := 1/n!$$

- Quantity N becomes random variable.

  The number of times the corresponding line of the algorithm is executed for each permutation of a.

  - We are interested in the expected value $E[N]$.

- Average time complexity $\overline{T}(n)$:

$$\overline{T}(n) = 1 + n + (n-1) + E[N] + (n-1) = 3n - 1 + E[N]$$

Our goal is to determine the expected value $E[N]$.

Average Time Complexity (Contd)

- p_{nk}: probability that $N = k$ for array of size n.

$$p_{n0} + p_{n1} + p_{n2} + \ldots + p_{n,n-1} = \sum_{k=0}^{n-1} p_{nk} = 1$$

- $p_{nk} = 0$ for $k \geq n$:

$$p_{n0} + p_{n1} + p_{n2} + \ldots = \sum_{k} p_{nk} = 1$$

- $E[N]$ is sum of products of probability of $N = k$ and value k:

$$E[N] = p_{n0} \cdot 0 + p_{n1} \cdot 1 + p_{n2} \cdot 2 + \ldots = \sum_{k} p_{nk} \cdot k$$

Our goal is to determine the value of sum $\sum_k p_{nk} \cdot k$.

Average Time Complexity (Contd)

We apply the technique of "generating functions".

- $G_n(z)$: power series with coefficients $p_{n0}, p_{n1}, \ldots$.

$$G_n(z) := p_{n0} \cdot z^0 + p_{n1} \cdot z^1 + p_{n2} \cdot z^2 + \ldots = \sum_k p_{nk} \cdot z^k$$

- $G'_n(z)$: derivative of $G_n(z)$

$$G'_n(z) = p_{n0} \cdot 0 \cdot z^{-1} + p_{n1} \cdot 1 \cdot z^0 + p_{n2} \cdot 2 \cdot z^1 + \ldots = \sum_k p_{nk} \cdot k \cdot z^{k-1}$$

- $G'_n(1)$:

$$G'_n(1) = p_{n0} \cdot 0 + p_{n1} \cdot 1 + p_{n2} \cdot 2 + \ldots = \sum_k p_{nk} \cdot k$$

Our goal is to determine $G'_n(1)$.

Average Time Complexity (Contd)

We derive a recurrence relation for $G'_n(1)$.

- $n = 1$: $p_{10} = 1$ and $p_{1k} = 0$ for all $k \geq 1$

$$G'_1(1) = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 2 + \ldots = 0$$

- $n > 1$: if the loop has already found the maximum of the first $n-1$ array elements, the last iteration
  - will either increment N (if the last element is the largest one)
    - Probability $1/n$.
    - N becomes k for n, if N was $k-1$ for $n-1$.
  - or will leave N as it is (if the last element is not the largest one).
    - Probability $(n-1)/n$.
    - N becomes k for n, if N was k for $n-1$.

$$p_{nk} = \frac{1}{n} \cdot p_{n-1,k-1} + \frac{n-1}{n} \cdot p_{n-1,k}$$

Our goal is to determine $G'_n(1)$ for $n > 1$.

Average Time Complexity (Contd)

- Determine $G_n(z)$ from p_{nk}:

$$p_{nk} = \frac{1}{n} \cdot p_{n-1,k-1} + \frac{n-1}{n} \cdot p_{n-1,k}$$

$$G_n(z) = \frac{1}{n} \cdot z \cdot G_{n-1}(z) + \frac{n-1}{n} \cdot G_{n-1}(z) = \frac{z+n-1}{n} \cdot G_{n-1}(z)$$

- Compute $G'_n(z)$ by derivation of $G_n(z)$:

$$G'_n(z) = \frac{1}{n} \cdot G_{n-1}(z) + \frac{z+n-1}{n} \cdot G'_{n-1}(z)$$

- Compute $G'_n(1)$:

$$\begin{aligned} G'_n(1) &= \frac{1}{n} \cdot G_{n-1}(1) + \frac{z+n-1}{n} \cdot G'_{n-1}(1) \\ &\overset{(*)}{=} \frac{1}{n} \cdot 1 + \frac{1+n-1}{n} \cdot G'_{n-1}(1) \\ &= \frac{1}{n} + G'_{n-1}(1) \end{aligned}$$

$$(*) \;\; G_n(1) = p_{n0} + p_{n1} + p_{n2} + \ldots = \sum_k p_{nk} = 1$$

Average Time Complexity (Contd)

- Recurrence relation for $G'_n(1)$:

$$G'_1(1) = 0$$
$$G'_n(1) = \frac{1}{n} + G'_{n-1}(1), \text{ if } n > 1$$

- Solution of $G'_n(1)$:

$$G'_n(1) = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{k=2}^{n} \frac{1}{k} = H_n - 1$$

  - $H(n) = \sum_{k=1}^{n} \frac{1}{k}$: n-th harmonic number
  - $H(n) = \ln n + \gamma + \varepsilon_n$, $\gamma \approx 0.577$, $0 < \varepsilon_n < 1/(2n)$.

- Solution of $E[n]$:

$$E[N] = \ln n + \gamma + \varepsilon_n - 1$$

- Average time complexity $\overline{T}(n)$:

$$\overline{T}(n) = 3n - 1 + E[N] = 3n + \ln n + \varepsilon_n + \gamma - 2$$

Analysis of average complexity is more difficult than that of the worst-case.

Complexity Approximations

Typically, we are only interested to capture the "overall behavior" of a complexity function for large inputs.

- Exact analysis:

  $$\overline{T}(n) = 3n + \ln n + \varepsilon_n + \gamma - 2$$

- Approximation:

  "$\overline{T}(n)$ is of the order $3n + \ln n$"

- Coarser approximation:

  "$\overline{T}(n)$ is of the order $3n$".

- Even coarser approximation:

  "$\overline{T}(n)$ is linear"

- Formalism:

  $$\overline{T}(n) = O(n)$$

We are going to formalize such complexity approximations.

The Landau Symbols

Take $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ from the natural numbers to the non-negative reals.

- $O(g)$: the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\exists c \in \mathbb{R}_{>0}, N \in \mathbb{N} : \forall n \geq N : f(n) \leq c \cdot g(n)$$

  - $f(n) = O(g(n))$: $f \in O(g)$.

    f is bounded from above by g.

- $\Omega(g)$: the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\exists c \in \mathbb{R}_{>0}, N \in \mathbb{N} : \forall n \geq N : f(n) \geq c \cdot g(n)$$

  - $f(n) = \Omega(g(n))$: $f \in \Omega(g)$..

    f is bounded from below by g.

- $\Theta(g)$: the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that

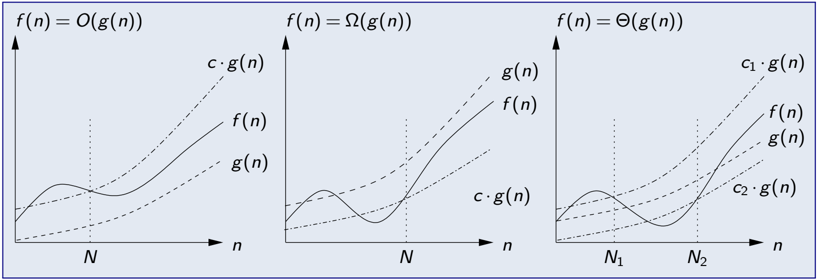
$$f \in O(g) \wedge f \in \Omega(g)$$

  - $f(n) = \Theta(g(n))$: $f \in \Theta(g)$.

    f is bounded from above and below by g.

Understanding the Landau Symbols

$f \in \mathcal{O}(g)$: g represents a bound for f, from above and/or below.



- It suffices, if bound holds from a certain start value N on.
  - Finitely many exceptions are allowed.
- It suffices, if the bound holds up to arbitrarily large constant c.
  - Bounds are independent of "measurement units".

The Landau symbols talk about the *asymptotic* behavior of functions.

Common Practice of the Landau Symbols

We need to understand the historically developed usage of the symbols.

- Most wide spread: $f(n) = O(g(n))$.
  - Often used when actually $f(n) = \Theta(g(n))$ is meant, i.e.,
  - when $g(n)$ is not only a bound from above but also from below.
- Abuse of notation: $f(n) = O(g(n))$
  - $=$ does not denote equality but set inclusion.
  - Notation has nevertheless been universally adopted.
- Ambiguous notation: $f(n) = O(g(n))$
  - Terms $f(n)$ and $g(n)$ with implicit free variable n.
  - To derive $f \in O(g)$, we have to identify the free variable.

"Let $c > 1$. Then $x^c = O(c^x)$."

$$\leadsto$$

"Let $c > 1$, $f(x) := x^c$, and $g(x) := c^x$. Then $f \in O(g)$."

We stick to the common practice.

Duality of O and Ω

- Theorem: for all $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we have

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

- Proof $\Rightarrow$: We assume $f(n) = O(g(n))$ and show $g(n) = \Omega(f(n))$. By the definition of Ω, we have to find constants N_1, c_1 such that

$$\forall n \geq N_1 : g(n) \geq c_1 \cdot f(n)$$

Since $f(n) = O(g(n))$, we have constants N_2, c_2 such that

$$\forall n \geq N_2 : f(n) \leq c_2 \cdot g(n)$$

Take $N_1 := N_2$ and $c_1 := 1/c_2$. Then we have, since $N_1 = N_2$, for all $n \geq N_1$,

$$c_2 \cdot g(n) \geq f(n)$$

and therefore

$$g(n) \geq (1/c_2) \cdot f(n) = c_1 \cdot f(n).$$

- Proof $\Leftarrow$: analogously.

O and Ω are dual.

Example

We prove $3n^2+5n+7 = O(n^2)$.

- We have to find constants c and N such that

$$\forall n \geq N : 3n^2+5n+7 \leq cn^2$$

- For $n \geq 1$, we have

$$3n^2+5n+7 \overset{1\leq n}{\leq} 3n^2+5n+7n = 3n^2+12n$$

- For $n \geq 12$, we also have

$$3n^2+12n \overset{12\leq n}{\leq} 3n^2+n\cdot n = 4n^2$$

- We take $N := 12$ $(= \max\{1,12\})$ and $c := 4$ and have for $n \geq N$

$$3n^2+5n+7 \overset{1\leq n}{\leq} 3n^2+5n+7n = 3n^2+12n \overset{12\leq n}{\leq} 3n^2+n\cdot n = 4n^2 = cn^2$$

Demonstrates general technique for asymptotics of polynomial functions.

Asymptotic Laws

- Theorem: for all $a_0, \ldots, a_m \in \mathbb{R}$, we have

$$a_m n^m + \ldots + a_2 n^2 + a_1 n + a_0 = \Theta(n^m)$$

  - Proof: analogous to previous example.
- Theorem: for all $a, b \in \mathbb{R}_{>0}$, we have

$$log_a \; n = O(log_b \; n)$$

  - Proof: take $c := \log_a b$ and $N := 0$. Then we have for all $n \geq N$

$$log_a \; n = \log_a(b^{\log_b n}) = (\log_a b) \cdot (\log_b n) = c \cdot (\log_b n)$$

Polynomials are dominated by the monomial with the highest exponent; in logarithms, bases don't matter.

Asymptotic Laws

- Theorem: for all $a, b \in \mathbb{R}$ with $b > 1$, we have

$$n^a = O(b^n)$$

  - Proof: we know the Taylor series expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Since $b^n = (e^{\ln b})^n = e^{n \ln b}$, we have for all $n \in \mathbb{N}$

$$b^n = \sum_{i=0}^{\infty} \frac{(n \ln b)^i}{i!} = 1 + (n \ln b) + \frac{(n \ln b)^2}{2!} + \frac{(n \ln b)^3}{3!} + \ldots$$

Since $b > 1$, we have $\ln b > 0$; therefore we know

$$b^n > \frac{(n \ln b)^a}{a!} = \frac{(\ln b)^a}{a!} n^a$$

Consequently

$$n^a < \frac{a!}{(\ln b)^a} b^n$$

Thus we define $N := 0$ and $c := a!/(\ln b)^a$ and are done.

Polynomials are dominated by all exponentials with base greater one.

Asymptotic Laws

$$c \cdot f(n) = O(f(n))$$
$$n^m = O(n^{m'}), \text{ for all } m \leq m'$$
$$a_m n^m + \ldots + a_2 n^2 + a_1 n + a_0 = \Theta(n^m)$$
$$\log_a n = O(\log_b n), \text{ for all } a, b > 0$$
$$n^a = O(b^n), \text{ for all } a, b \text{ with } b > 1$$

- **Reflexivity:**
  $f = O(f)$, $f = \Omega(f)$, $f = \Theta(f)$.
- **Symmetry:**
  - If $f = O(g)$, then $g = \Omega(f)$.
  - If $f = \Omega(g)$, then $g = O(f)$.
  - If $f = \Theta(g)$, then $g = \Theta(f)$.
- **Transitivity:**
  - If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.
  - If $f = \Omega(g)$ and $g = \Omega(h)$, then $f = \Omega(h)$.
  - If $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

The proof of reflexivity/symmetry/transitivity is an easy exercise.

Asymptotic Notation in Equations

A more general form of "syntactic abuse".

- Equation:

$$A[\mathcal{O}_1(f(n))] = B[\mathcal{O}_2(g(n))]$$

  - (Possibly multiple) occurrences of $\mathcal{O}_1, \mathcal{O}_2 \in \{O, \Omega, \Theta\}$.

- Interpretation:

$$\forall f' \in \mathcal{O}_1(f) : \exists g' \in \mathcal{O}_2(g) :$$
$$\forall n \in \mathbb{N} : A[f'(n)] = B[g'(n)]$$

  - Every occurrence of $\mathcal{O}$ is replaced by a function in the corresponding asymptotic complexity class.
  - Functions on the left side are universally quantified, functions on the right side are existentially quantified.

A convenient shortcut to express asymptotic relationships.

Example

- Example:

$$H_n = \ln n + \gamma + O(\frac{1}{n})$$

  - There is a function $f \in O(1/n)$ such that, for all $n \in \mathbb{N}$, $H_n = \ln n + \gamma + f(n)$.
- Example:

$$2n^2 + 3n + 1 = O(2n^2) + O(n) = O(n^2)$$

  - Equation $2n^2 + 3n + 1 = O(2n^2) + O(n)$

$$\begin{array}{l} \exists f \in O(2n^2), g \in O(n) : \\ \forall n \in \mathbb{N} : 2n^2 + 3n + 1 = f(n) + g(n) \end{array}$$

  - Equation $O(2n^2) + O(n) = O(n^2)$

$$\begin{array}{l} \forall f \in O(2n^2), g \in O(n) : \exists h \in O(n^2) : \\ \forall n \in \mathbb{N} : f(n) + g(n) = h(n) \end{array}$$

Further Asymptotic Equations

We thus can express further asymptotic relationships.

$$
\begin{aligned}
O(O(f(n))) &= O(f(n)) \\
O(f(n)) + O(g(n)) &= O(f(n) + g(n)) \\
O(f(n)) \cdot O(g(n)) &= O(f(n) \cdot g(n)) \\
O(f(n) \cdot g(n)) &= f(n) \cdot O(g(n)) \\
O(f(n)^m) &= O(f(n))^m, \text{ for all } m \geq 0
\end{aligned}
$$

The proofs are simple exercises.

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Further Landau Symbols

Take $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ from the natural numbers to the non-negative reals.

- $o(g)$: the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall c \in \mathbb{R}_{>0} : \exists N \in \mathbb{N} : \forall n \geq N : f(n) \leq c \cdot g(n)$$

  - $f(n) = o(g(n))$: $f \in o(g)$.

    f is asymptotically smaller than g.

- $\omega(g)$: the set of all functions $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall c \in \mathbb{R}_{>0} : \exists N \in \mathbb{N} : \forall n \geq N : g(n) \leq c \cdot f(n)$$

  - $f(n) = \omega(g(n))$: $f \in \omega(g)$.

    f is asymptotically larger than g.

- Theorem: for all $f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$, we have

$$f \in o(g) \Leftrightarrow g \in \omega(f)$$
$$f \in o(g) \Rightarrow f \in O(g) \land f \notin \Theta(g)$$
$$f \in \omega(g) \Rightarrow f \in \Omega(g) \land f \notin \Theta(g)$$

Useful to create a hierarchy of asymptotic growth functions.

Hierarchy of Complexity Classes

Define $f \prec g \;:\Leftrightarrow\; f = o(g)$.

$$
\begin{aligned}
1 &\prec \log\log\log n \prec \log\log n \prec \sqrt{\log n} \prec \log n \prec (\log n)^2 \prec (\log n)^3 \\
&\prec \sqrt[3]{n} \prec \sqrt{n} \prec n \prec n\log n \prec n\sqrt{n} \prec n^2 \prec n^3 \\
&\prec n^{\log n} \prec 2^{\sqrt{n}} \prec 2^n \prec 3^n \prec n! \prec n^n \prec 2^{n^2} \prec 2^{2^n} \\
&\prec 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \ (n \text{ times})
\end{aligned}
$$

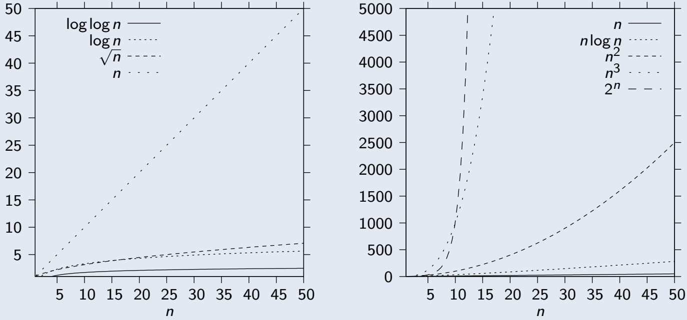
Fundamental knowledge about complexity classes.

Hierarchy of Complexity Classes

- $O(1)$ **(Constant):** upper limit on function values.
  - Space complexity of algorithms that work with fixed memory size.
- $O(\log n)$ **(Logarithmic):** values grow very slowly.
  - Time complexity of binary search.
- $O(n)$ **(Linear):** values grow proportionally with argument.
  - Time complexity of linear search.
- $O(n \log n)$ **(Linear-Logarithmic):** value growth is reasonably well behaved.
  - Time complexity of fast sorting algorithms, e.g., Mergesort.
- $O(n^c)$ **(Polynomial):** values grow rapidly but with polynomial bound.
  - Executions still "feasible" for large inputs, e.g., matrix multiplication.
- $O(c^n)$ **(Exponential):** values grow extremely rapidly.
  - Executions only reasonable for small inputs; e.g, finding exact solutions to many optimization problems ("traveling salesman problem").
- $O(c^{d^n})$ **(Double Exponential):** Function values grow overwhelmingly rapidly.
  - Decision of statements about real numbers ("quantifier elimination"), solving multivariate polynomial equations ("Buchberger's algorithm").

Only computations up to polynomial complexity are considered "feasible".

Complexity Classes



Time complexity	Maximum n for time bound 1s	1m	1h	Improvement for input size n
n	1000	60000	3600000	$*60$
$n \log n$	140	4895	204094	$*40$ ($\approx$)
n^2	31	244	1897	$*7.75$ ($\sqrt{60}$)
n^3	10	39	153	$*4.91$ ($\sqrt[3]{60}$)
2^n	9	15	21	$+5.9$ ($\log_2 60$)

Improvement in asymptotic complexity outperforms technological speedup.