Turing Machines

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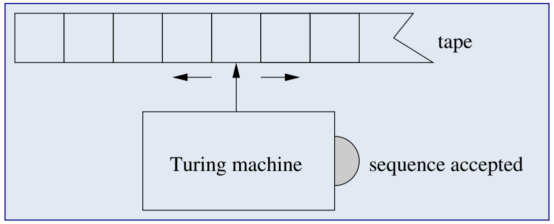
Turing Machine Model



- The machine is always in one of a finite set of states.
  - The machine starts its execution in a fixed start state.
- An infinite tape holds at its beginning the input word.
  - Tape is read and written and arbitrarily moved by the machine.
- The machine proceeds in a sequence of state transitions.
  - Machine reads symbol, overwrites it, and moves tape head left or right.
  - The symbol read and the current state determine the symbol written, the move direction, and the next state.
- If the machine cannot make another transition, it terminates.
  - The machine signals whether it is in an accepting state.

If the machine terminates in an accepting state, the word is *accepted*.

Turing Machines

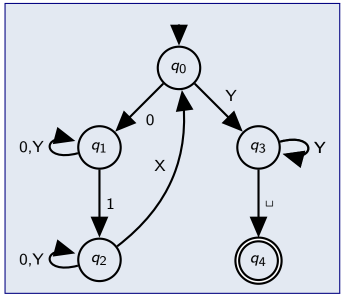
Turing Machine $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$:

- The state set Q, a fine set of states.
- A tape alphabet Γ, a finite set of tape symbols.
- The blank symbol $\sqcup \in \Gamma$.
- An input alphabet $\Sigma \subseteq \Gamma \backslash \{\sqcup\}$.
- The (partial) transition function $\delta : Q \times \Gamma \to_p Q \times \Gamma \times \{\text{'L'}, \text{'R'}\}$,
  - $\delta(q, x) = (q', x', \text{'L'}/\text{'R'})$ ... M reads in state q symbol x, goes to state q', writes symbol x', and moves the tape head left/right.
- The start state $q_0 \in Q$
- A set of accepting states (final states) $F \subseteq Q$.

The crucial difference to an automaton is the infinite tape that can be arbitrarily moved and written.

Example

$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$
$Q = \{q_0, q_1, q_2, q_3, q_4\}$
$\Gamma = \{\sqcup, 0, 1, X, Y\}$
$\Sigma = \{0, 1\}$
$F = \{q_4\}$

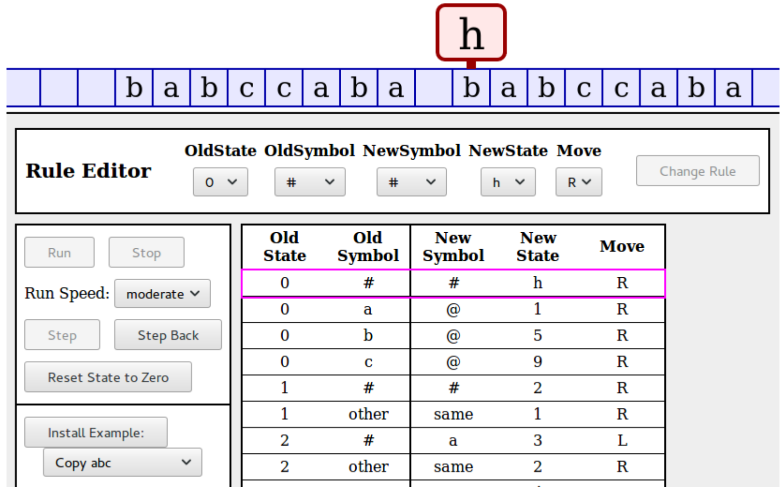


δ	$\sqcup$	0	1	X	Y
q_0	–	(q_1, X, R)	–	–	(q_3, Y, R)
q_1	–	$(q_1, 0, R)$	(q_2, Y, L)	–	(q_1, Y, R)
q_2	–	$(q_2, 0, L)$	–	(q_0, X, R)	(q_2, Y, L)
q_3	$(q_4, \sqcup, R)$	–	–	–	(q_3, Y, R)
q_4	–	–	–	–	–

Machine accepts every word of form $0^n 1^n$ (replacing it by $X^n Y^n$).

Turing Machine Simulator

http://math.hws.edu/eck/js/turing-machine/TM.html



Generalized Turing Machines

- Infinite tape in both directions.
  - Can be simulated by a machine whose tape is infinite in one direction.
- Multiple tapes.
  - Can be simulated by a machine with a single tape.
- Nondeterministic transitions.
  - We can simulate a nondeterministic M by a deterministic M'.
    - Let r be the maximum number of “choices” that M can make.
  - M' operates with 3 tapes.
    - Tape 1 holds the input (tape is only read).
  - M' writes to tape 2 all finite sequences of numbers $1, \ldots, r$.
    - First all sequences of length 1, then all of length 2, etc.
  - After writing sequence $s_1 s_2 \ldots s_n$ to tape 2, M' simulates M on tape 3.
    - M' copies the input to tape 3 and performs at most n transitions.
    - In transition i, M attempts to perform choice s_i.
    - If choice i is not possible or M terminates after n transitions in a non-accepting state, M' continues with next sequence.
    - If M terminates in accepting state, M' accepts the input.

Every generalized Turing machine can be simulated by the core form.

Turing Machine Configurations

- Configuration $a_1 \ldots a_k \; q \; a_{k+1} \ldots a_m$:
  - q: the current state of M.
  - a_{k+1}: the symbol currently under the tape head.
  - $a_1 \ldots a_k$: the portion of the tape left to the tape head.
  - $a_{k+2} \ldots a_m$: the portion right to the head (followed by $\sqcup \ldots$).
- Move relation: $a_1 \ldots a_k \; q \; a_{k+1} \ldots a_m \vdash b_1 \ldots b_l \; p \; b_{l+1} \ldots b_m$

  If M is a situation described by the left configuration, it can make a transition to the situation described by the right configuration.
  - $a_i = b_i$ for all $i \neq k+1$ and one of the following:
    - $l = k+1$ and $\delta(q, a_{k+1}) = (p, b_l, R)$,
    - $l = k-1$ and $\delta(q, a_{k+1}) = (p, b_{l+2}, L)$.
- Extended move relation: $c_1 \vdash^* c_2$

  M can make in an arbitrary number of moves a transition from the situation described by configuration c_1 to the one described by c_2.

$$c_1 \vdash^0 c_2 :\Leftrightarrow c_1 = c_2$$
$$c_1 \vdash^{i+1} c_2 :\Leftrightarrow \exists c : c_1 \vdash^i c \wedge c \vdash c_2$$
$$c_1 \vdash^* c_2 :\Leftrightarrow \exists i \in \mathbb{N} : c_1 \vdash^i c_2$$

The Language of a Turing Machine

- The language $L(M)$ of Turing machine $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$:

  The set of all inputs that drive M from its initial configuration to a configuration with an accepting state such that from this configuration no further move is possible:

$$L(M) := \left\{ w \in \Sigma^* \;\middle|\; \begin{array}{l} \exists a, b \in \Gamma^*, q \in Q : q_0 \; w \vdash^* a \; q \; b \wedge q \in F \\ \wedge \; \neg \exists a', b' \in \Gamma^*, q' \in Q : a \; q \; b \; \vdash a' \; q' \; b' \end{array} \right\}$$

- L is a recursively enumerable language:
  - There exists a Turing machine M such that $L = L(M)$.
- L is a recursive language:
  - There exists a Turing machine M such that $L = L(M)$ and M terminates for every possible input.

Every recursive language is recursively enumerable; as we will see, the converse does not hold.

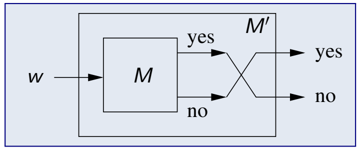
Recursiv. Enumerable/Recursive Languages

Theorem: L is recursive, if and only if both L and its complement $\overline{L}$ are recursively enumerable.

Proof $\Rightarrow$: Let L be a recursive. Since by definition L is recursively enumerable, it remains to be shown that also $\overline{L}$ is recursively enumerable.

Since L is recursive, there exists a Turing machine M such that M halts for every input w: if $w \in L$, then M accepts w; if $w \notin L$, then M does not accept w. With the help of M, we can construct the following M' with $L(M') = \overline{L}$:

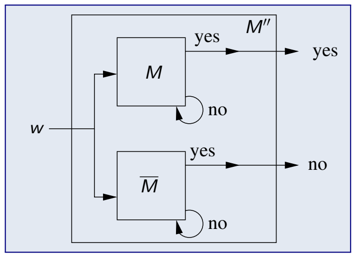
```
function M'(w):
  case M(w) of
    yes: return no
    no: return yes
  end case
end function
```



Recursiv. Enumerable/Recursive Languages

Proof $\Leftarrow$: Let L be such that both L and $\overline{L}$ are recursively enumerable. We show that L is recursive. Since L is r.e., there exists M such that $L = L(M)$ and M halts for $w \in L$ with $M(w) = \text{yes}$. Since $\overline{L}$ is r.e., there exists $\overline{M}$ with $\overline{L} = L(\overline{M})$ and $\overline{M}$ halts for $w \in \overline{L}$ with $\overline{M}(w) = \text{yes}$. We can thus construct M'' with $L(M'') = L$ that always halts:

```
function M''(w):
  parallel
    begin
      if M(w) = yes then
        return yes
      end if
      loop forever
    end
    begin
      if M̄(w) = yes then
        return no
      end if
      loop forever
    end
  end parallel
end function
```



Closure of Recursive Languages

Let L, L_1, L_2 be recursive languages. Then also

- the complement $\overline{L}$,
- the union $L_1 \cup L_2$,
- the intersection $L_1 \cap L_2$

are recursive languages.

Proof by construction of the corresponding Turing machines.

Enumerators

Turing machine $M = (Q, \Gamma, \sqcup, \emptyset, \delta, q_0, F)$ with special symbol $\# \in \Gamma$.

- M is an enumerator, if M has an additional output tape on which
  - M moves its tape head only to the right, and
  - M writes only symbols different from $\sqcup$.
- The generated language $Gen(M)$ of enumerator M is the set of all words that M eventually writes on its output tape.
  - The end of each word is marked by a trailing $\#$.

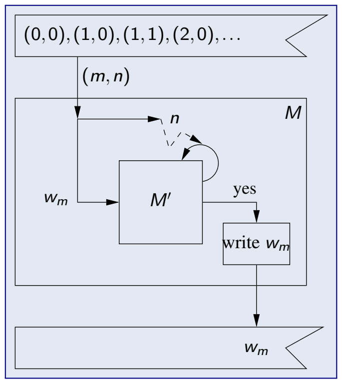
M may run forever and thus $Gen(M)$ may be infinite.

Recognizing versus Generating Languages

Theorem: L is recursively enumerable, if and only if there exists some enumerator M such that $L = Gen(M)$.

Proof $\Rightarrow$: Let L be recursively enumerable, i.e., $L = L(M')$ for some M'. We construct enumerator M such that $L = Gen(M)$.

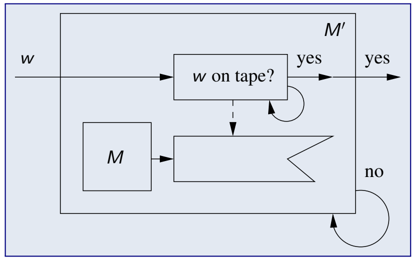
procedure M:
  loop
    produce next (m, n) on working tape
    if $M'(w_m) =$ yes in at most n steps **then**
      write w_m to output tape
    end if
  end loop
end procedure



Recognizing versus Generating Languages

Proof $\Leftarrow$: Let L be such that $L = Gen(M)$ for some enumerator M. We show that there exists some Turing machine M' such that $L = L(M')$.

```
function M'(w):
  while M is not terminated do
    M writes next word w'
    if w = w' then
      return yes
    end if
  end while
  return no
end function
```



Recognizing is possible, if and only if generating is possible.

Functions

Take binary relation $f \subseteq A \times B$.

- $f : A \to B$: f is a total function from A to B.
  - For every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.
- $f : A \to_p B$: f is a partial function from A to B.
  - For every $a \in A$, there is at most one $b \in B$ such that $(a, b) \in f$.
- Auxiliary notions:

$$domain(f) := \{a \mid \exists b : (a, b) \in f\}$$
$$range(f) := \{b \mid \exists a : (a, b) \in f\}$$
$$f(a) := \text{ such } b : (a, b) \in f$$

Every total function $f : A \to B$ is a partial function $f : A \to_p B$; every partial function $f : A \to_p B$ is a total function $f : domain(f) \to B$.

Functions

- Let $f : \Sigma^* \to_p \Gamma^*$ where $\sqcup \notin \Sigma \cup \Gamma$.
  - f is a function over words in some alphabets.
- f is Turing computable, if there exists a Turing machine M such that
  - for input w (i.e. initial tape content $w\sqcup\ldots$), M terminates in an accepting state, if and only if $w \in domain(f)$;
  - for input w, M terminates in an accepting state with output w' (i.e. final tape content $w'\sqcup\ldots$), if and only if $w' = f(w)$.
- Not every function $f : \Sigma^* \to_p \Gamma^*$ is Turing computable:
  - The set of all Turing machines is countably infinite: all machines can be ordered in a single list (in the alphabetic order of their definitions).
  - The set of all functions $\Sigma^* \to_p \Gamma^*$ is more than countably infinite (Cantor's diagonalization argument).
  - Consequently, there are more functions than Turing machines.

M computes f, if M terminates for arguments in the domain of f with output $f(a)$ and does not terminate for arguments outside the domain.

Example

We show that natural number subtraction is Turing computable.

- Subtraction $\ominus$ on $\mathbb{N}$:

$$m \ominus n := \begin{cases} m-n & \text{if } m \geq n \\ 0 & \text{else} \end{cases}$$

- Unary representation of $n \in \mathbb{N}$:

$$\underbrace{000\ldots0}_{n \text{ times}} \in L(0^*)$$

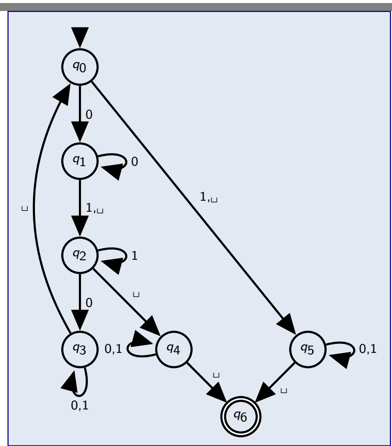
- Input $00\sqcup0$ shall lead to output 0.
  - $2 \ominus 1 = 1$.

Idea: replace every pair of 0 in m and n by $\sqcup$.

Example (Contd)

$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$
$Q = \{q_0, \ldots, q_6\}$
$\Sigma = \{0\}, \Gamma = \{0, 1, \sqcup\}, F = \{q_6\}$

δ	0	1	$\sqcup$
q_0	$(q_1, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	$(q_2, 1, R)$
q_2	$(q_3, 1, L)$	$(q_2, 1, R)$	$(q_4, \sqcup, L)$
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	$(q_0, \sqcup, R)$
q_4	$(q_4, 0, L)$	$(q_4, \sqcup, L)$	$(q_6, 0, R)$
q_5	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_6, \sqcup, R)$
q_6	–	–	–



- In q_0, the leading 0 is replaced by $\sqcup$.
- In q_1, M searches for the next $\sqcup$ and replaces it by a 1.
- In q_2, M searches for the next 0 and replaces it by 1, then moves left.
- In q_3, M searches for previous $\sqcup$, moves right and starts from begin.
- In q_4, M has found a $\sqcup$ instead of 0 and replaces all previous 1 by $\sqcup$.
- In q_5, n is (has become) 0; the rest of the tape is erased.
- In q_6, the computation successfully terminates.

Example (Contd)

- $2 \ominus 1 = 1$:

$$q_0 00\sqcup0 \vdash \sqcup q_1 0\sqcup0 \vdash \sqcup0 q_1 \sqcup0 \vdash \sqcup01 q_2 0$$
$$\vdash \sqcup0 q_3 11 \vdash \sqcup q_3 011 \vdash q_3 \sqcup011 \vdash \sqcup q_0 011$$
$$\vdash \sqcup\sqcup q_1 11 \vdash \sqcup\sqcup1 q_2 1 \vdash \sqcup\sqcup11 q_2 \vdash \sqcup\sqcup1 q_4 1$$
$$\vdash \sqcup\sqcup q_4 1 \vdash \sqcup q_4 \vdash \sqcup0 q_6$$

- $1 \ominus 2 = 0$:

$$q_0 0\sqcup00 \vdash \sqcup q_1 \sqcup00 \vdash \sqcup1 q_2 00 \vdash \sqcup q_3 110$$
$$\vdash q_3 \sqcup110 \vdash \sqcup q_0 110 \vdash \sqcup\sqcup q_5 10 \vdash \sqcup\sqcup\sqcup q_5 0$$
$$\vdash \sqcup\sqcup\sqcup\sqcup q_5 \vdash \sqcup\sqcup\sqcup\sqcup\sqcup q_6.$$

For $m > n$, leading blanks still have to be removed.

Turing Computability

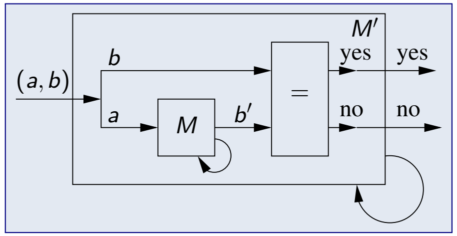
Theorem: $f : \Sigma^* \to_p \Gamma^*$ is Turing computable, if and only if

$$L_f := \{(a, b) \in \Sigma^* \times \Gamma^* \mid a \in domain(f) \wedge b = f(a)\}$$

is recursively enumerable.

Proof $\Rightarrow$: Since $f : \Sigma^* \to_p \Gamma^*$ is Turing computable, there exists a Turing machine M which computes f. To show that L_f is r.e., we construct M' with $L(M') = L_f$:

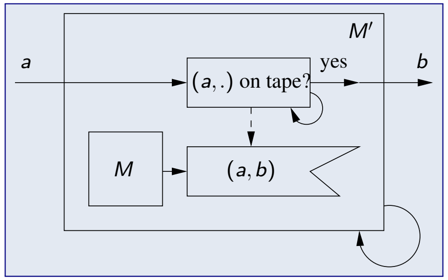
function $M'(a, b)$:
  $b' \leftarrow M(a)$
  if $b' = b$ **then**
    return yes
  else
    return no
  end if
end function



Turing Computability

Proof $\Leftarrow$: Since L_f is recursively enumerable, there exists an enumerator M with $Gen(M) = L_f$. We construct the following Turing machine M' which computes f:

```
function M'(a):
    while M is not terminated do
        M writes (a', b') to tape
        if a = a' then
            return b'
        end if
    end while
    loop forever
end function
```



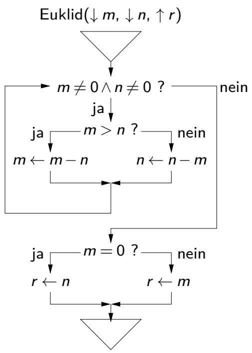
Computing is possible, if and only if recognizing is possible.

Algorithms

Computer science is based on algorithms.

Compute as follows the greatest common divisor of two natural numbers m, n that are not both 0:

1. If $m = 0$, the result is n.
2. If $n = 0$, the result is m.
3. If $m > n$, subtract n from m and continue with step 1.
4. Otherwise subtract m from n and continue with step 1.




```
Euklid(↓ m, ↓ n, ↑ r):
  while m ≠ 0 ∧ n ≠ 0 do
    if m > n
      then m ← m − n
      else n ← n − m
  if m = 0
    then r ← n
    else r ← m
end Euklid.
```

What is an "algorithm" and what is computable by an algorithm?

The Church-Turing Thesis

Church-Turing Thesis: Every problem that is solvable by an algorithm (in an intuitive sense) is solvable by a Turing machine. Thus the set of intuitively computable functions is identical with the set of Turing computable functions.

- Replaces fuzzy notion "algorithm" by precise notion "Turing machine".
- Unprovable thesis, exactly because the notion "algorithm" is fuzzy.
- Substantially validated, because many different computational models have no more computational power than Turing machines.
  - Random access machines, loop programs, recursive functions, goto programs, λ-calculus, rewriting systems, grammars, ...

Turing machines represent the most powerful computational model known, but there are many other equally powerful ("Turing complete") models.