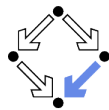


Logic, Checking, and Proving

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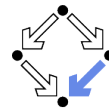
1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

4. The RISC ProofNavigator

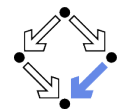
The Language of Logic



Two kinds of syntactic phrases.

- **Term** T denoting an **object**.
 - Variable x
 - Object constant c
 - Function application $f(T_1, \dots, T_n)$ (may be written infix)
 n -ary function constant f
- **Formula** F denoting a **truth value**.
 - Atomic formula $p(T_1, \dots, T_n)$ (may be written infix)
 n -ary predicate constant p .
 - Negation $\neg F$ ("not F ")
 - Conjunction $F_1 \wedge F_2$ (" F_1 and F_2 ")
 - Disjunction $F_1 \vee F_2$ (" F_1 or F_2 ")
 - Implication $F_1 \Rightarrow F_2$ ("if F_1 , then F_2 ")
 - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1 , then F_2 , and vice versa")
 - Universal quantification $\forall x : F$ ("for all x , F ")
 - Existential quantification $\exists x : F$ ("for some x , F ")

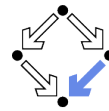
Syntactic Shortcuts



- $\forall x_1, \dots, x_n : F$
 - $\forall x_1 : \dots : \forall x_n : F$
- $\exists x_1, \dots, x_n : F$
 - $\exists x_1 : \dots : \exists x_n : F$
- $\forall x \in S : F$
 - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
 - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

Examples



Terms and formulas may appear in various syntactic forms.

Terms:

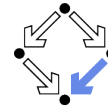
$$\begin{aligned} &\exp(x) \\ &a \cdot b + 1 \\ &a[i] \cdot b \\ &\sqrt{\frac{x^2 + 2x + 1}{(y+1)^2}} \end{aligned}$$

Formulas:

$$\begin{aligned} &a^2 + b^2 = c^2 \\ &n \mid 2n \\ &\forall x \in \mathbb{N} : x \geq 0 \\ &\forall x \in \mathbb{N} : 2 \mid x \vee 2 \mid (x + 1) \\ &\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow \\ &\quad \exists z \in \mathbb{N} : x + z = y \end{aligned}$$

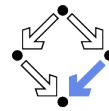
Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas



- **Atomic formula** $p(T_1, \dots, T_n)$
 - True if the predicate denoted by p holds for the values of T_1, \dots, T_n .
- **Negation** $\neg F$
 - True if and only if F is false.
- **Conjunction** $F_1 \wedge F_2$ (“ F_1 and F_2 ”)
 - True if and only if F_1 and F_2 are both true.
- **Disjunction** $F_1 \vee F_2$ (“ F_1 or F_2 ”)
 - True if and only if at least one of F_1 or F_2 is true.
- **Implication** $F_1 \Rightarrow F_2$ (“if F_1 , then F_2 ”)
 - False if and only if F_1 is true and F_2 is false.
- **Equivalence** $F_1 \Leftrightarrow F_2$ (“if F_1 , then F_2 , and vice versa”)
 - True if and only if F_1 and F_2 are both true or both false.
- **Universal quantification** $\forall x : F$ (“for all x , F ”)
 - True if and only if F is true for every possible value assignment of x .
- **Existential quantification** $\exists x : F$ (“for some x , F ”)
 - True if and only if F is true for at least one value assignment of x .

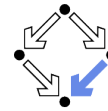
Example



We assume the domain of natural numbers and the “classical” interpretation of constants $1, 2, +, =, <$.

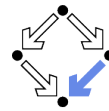
- $1 + 1 = 2$
 - True.
- $1 + 1 = 2 \vee 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \wedge 2 + 2 = 2$
 - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
 - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
 - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
 - True.

Example



- $x + 1 = 1 + x$
 - True, for every assignment of a number a to variable x .
- $\forall x : x + 1 = 1 + x$
 - True (because for every assignment a to x , $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
 - If x is assigned “one”, the formula is true.
 - If x is assigned “two”, the formula is false.
- $\exists x : x + 1 = 2$
 - True (because $x + 1 = 2$ is true for assignment “one” to x).
- $\forall x : x + 1 = 2$
 - False (because $x + 1 = 2$ is false for assignment “two” to x).
- $\forall x : \exists y : x < y$
 - True (because for every assignment a to x , there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
 - False (because for every assignment a to y , there is the assignment $a + 1$ to x which makes $x < y$ false).

Formula Equivalences

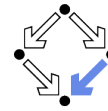


Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \iff F_1$
- $\neg(F_1 \wedge F_2) \iff \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \iff \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \iff F_1 \wedge \neg F_2$
- $\neg\forall x : F \iff \exists x : \neg F$
- $\neg\exists x : F \iff \forall x : \neg F$
- $F_1 \Rightarrow F_2 \iff \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \iff \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \iff \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.

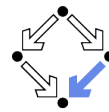
Example



- “All swans are white or black.”
 - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- “There exists a black swan.”
 - $\exists x : swan(x) \wedge black(x)$.
- “A swan is white, unless it is black.”
 - $\forall x : swan(x) \wedge \neg black(x) \Rightarrow white(x)$
 - $\forall x : swan(x) \wedge \neg white(x) \Rightarrow black(x)$
 - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- “Not everything that is white or black is a swan.”
 - $\neg\forall x : white(x) \vee black(x) \Rightarrow swan(x)$.
 - $\exists x : (white(x) \vee black(x)) \wedge \neg swan(x)$.
- “Black swans have at least one black parent”
 - $\forall x : swan(x) \wedge black(x) \Rightarrow \exists y : swan(y) \wedge black(y) \wedge parent(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.

The Usage of Formulas



Precise formulation of statements describing object relationships.

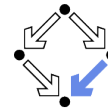
- **Statement:**

If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y .
- **Formula:**
$$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y$$
- **Problem specification:**

Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y .

 - Inputs: x, y
 - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
 - Output: q
 - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y$

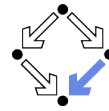
Problem Specifications



- The **specification** of a computation problem:
 - Input: variables $x_1 \in S_1, \dots, x_n \in S_n$
 - Input condition (“precondition”): formula $I(x_1, \dots, x_n)$.
 - Output: variables $y_1 \in T_1, \dots, y_m \in T_m$
 - Output condition (“postcondition”): $O(x_1, \dots, x_n, y_1, \dots, y_m)$.
 - $F(x_1, \dots, x_n)$: only x_1, \dots, x_n are free in formula F .
 - x is *free* in F , if not every occurrence of x is inside the scope of a quantifier (such as \forall or \exists) that binds x .
- An **implementation** of the specification:
 - A function (program) $f : S_1 \times \dots \times S_n \rightarrow T_1 \times \dots \times T_m$ such that
$$\forall x_1 \in S_1, \dots, x_n \in S_n : I(x_1, \dots, x_n) \Rightarrow \mathbf{let} (y_1, \dots, y_m) = f(x_1, \dots, x_n) \mathbf{in} O(x_1, \dots, x_n, y_1, \dots, y_m)$$
 - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.

Example: A Problem Specification



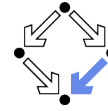
Given an integer array a , a position p in a , and a length l , return the array b derived from a by removing $a[p], \dots, a[p + l - 1]$.

- **Input:** $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- **Input condition:**
 $p + l \leq \text{length}(a)$
- **Output:** $b \in \mathbb{Z}^*$
- **Output condition:**
let $n = \text{length}(a)$ in
 $\text{length}(b) = n - l \wedge$
 $(\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge$
 $(\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i + l])$

Mathematical theory:

$$T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \rightarrow T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\}$$
$$\text{length} : T^* \rightarrow \mathbb{N}, \text{length}(a) = \text{such } i \in \mathbb{N} : a \in T^i$$

Validating Problem Specifications



Do formal input condition $I(x)$ and output condition $O(x, y)$ really capture our informal intentions?

- **Do concrete inputs/output satisfy/violate these conditions?**
 - $I(a_1), \neg I(a_2), O(a_1, b_1), \neg O(a_1, b_2)$.
- **Is input condition satisfiable?**
 - $\exists x : I(x)$.
- **Is input condition not trivial?**
 - $\exists x : \neg I(x)$.
- **Is output condition satisfiable for every input?**
 - $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- **Is output condition for all (at least some) inputs not trivial?**
 - $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.
 - $\exists x : I(x) \wedge \exists y : \neg O(x, y)$.
- **Is for every legal input at most one output legal?**
 - $\forall x : I(x) \Rightarrow \forall y_1, y_2 : O(x, y_1) \wedge O(x, y_2) \Rightarrow y_1 = y_2$.

Validate specification to increase our confidence in its meaning!

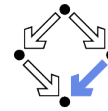
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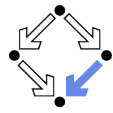
The RISC Algorithm Language (RISCAL)



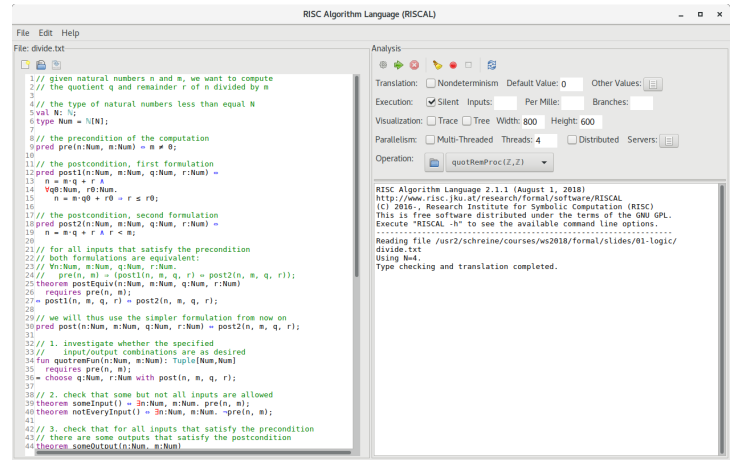
- **A system for formally specifying and checking algorithms.**
 - Research Institute for Symbolic Computation (RISC), 2016–.
<http://www.risc.jku.at/research/formal/software/RISCAL>.
 - Implemented in Java with SWT library for the GUI.
 - Tested under Linux only; freely available as open source (GPL3).
- **A language for the defining mathematical theories and algorithms.**
 - A static type system with only finite types (of parameterized sizes).
 - Predicates, explicitly (also recursively) and implicitly def.d functions.
 - Theorems (universally quantified predicates expected to be true).
 - Procedures (also recursively defined).
 - Pre- and post-conditions, invariants, termination measures.
- **A framework for evaluating/executing all definitions.**
 - Model checking: predicates, functions, theorems, procedures, annotations may be evaluated/executed for all possible inputs.
 - All paths of a non-deterministic execution may be elaborated.
 - The execution/evaluation may be visualized.

Validating algorithms by automatically verifying finite approximations.

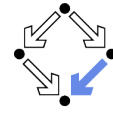
The RISC Algorithm Language (RISCAL)







RISCAL divide.txt &



Using RISCAL

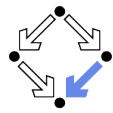


See also the (printed/online) "Tutorial and Reference Manual".

- Press button  (or <Ctrl>-s) to save specification.
 - Automatically processes (parses and type-checks) specification.
 - Press button  to re-process specification.
- Choose values for undefined constants in specification.
 - Natural number for val *const*: \mathbb{N} .
 - *Default Value*: used if no other value is specified.
 - *Other Values*: specific values for individual constants.
- Select *Operation* from menu and then press button .
 - Executes operation for chosen constant values and all possible inputs.
 - Option *Silent*: result of operation is not printed.
 - Option *Nondeterminism*: all execution paths are taken.
 - Option *Multi-threaded*: multiple threads execute different inputs.
 - Press button  to abort execution.

During evaluation all annotations (pre/postconditions, etc.) are checked.

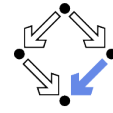
Typing Mathematical Symbols



ASCII String	Unicode Character	ASCII String	Unicode Character
Int	\mathbb{Z}	\neq	\neq
Nat	\mathbb{N}	\leq	\leq
:=	$:=$	\geq	\geq
true	\top	*	\cdot
false	\perp	times	\times
~	\neg	{}	\emptyset
\wedge	\wedge	intersect	\cap
\vee	\vee	union	\cup
\Rightarrow	\Rightarrow	Intersect	\cap
\Leftrightarrow	\Leftrightarrow	Union	\cup
forall	\forall	isin	\in
exists	\exists	subseteq	\subseteq
sum	\sum	\ll	\ll
product	\prod	\gg	\gg

Type the ASCII string and press <Ctrl>-# to get the Unicode character.

Example: Quotient and Remainder



Given natural numbers n and m , we want to compute the quotient q and remainder r of n divided by m .

```
// the type of natural numbers less than equal N
val N:  $\mathbb{N}$ ;
type Num =  $\mathbb{N}[N]$ ;

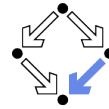
// the precondition of the computation
pred pre(n:Num, m:Num)  $\Leftrightarrow$  m  $\neq$  0;

// the postcondition, first formulation
pred post1(n:Num, m:Num, q:Num, r:Num)  $\Leftrightarrow$ 
  n = m*q + r  $\wedge$ 
   $\forall$ q0:Num, r0:Num.
    n = m*q0 + r0  $\Rightarrow$  r  $\leq$  r0;

// the postcondition, second formulation
pred post2(n:Num, m:Num, q:Num, r:Num)  $\Leftrightarrow$ 
  n = m*q + r  $\wedge$  r < m;
```

We will investigate this specification.

Example: Quotient and Remainder

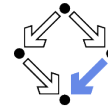


```
// for all inputs that satisfy the precondition
// both formulations are equivalent:
//  $\forall n:\text{Num}, m:\text{Num}, q:\text{Num}, r:\text{Num}.$ 
//  $\text{pre}(n, m) \Rightarrow (\text{post1}(n, m, q, r) \Leftrightarrow \text{post2}(n, m, q, r));$ 
theorem postEquiv(n:Num, m:Num, q:Num, r:Num)
  requires pre(n, m);
 $\Leftrightarrow \text{post1}(n, m, q, r) \Leftrightarrow \text{post2}(n, m, q, r);$ 

// we will thus use the simpler formulation from now on
pred post(n:Num, m:Num, q:Num, r:Num)  $\Leftrightarrow \text{post2}(n, m, q, r);$ 
```

Check equivalence for all values that satisfy the precondition.

Example: Quotient and Remainder



Choose e.g. value 5 for N .

Switch option *Silent* off:

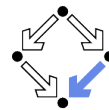
```
Executing postEquiv( $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}$ ) with all 1296 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function postEquiv(0,1,0,0):
Result (0 ms): true
Run 7 of deterministic function postEquiv(1,1,0,0):
Result (0 ms): true
...
Run 1295 of deterministic function postEquiv(5,5,5,5):
Result (0 ms): true
Execution completed for ALL inputs (6314 ms, 1080 checked, 216 inadmissible).
```

Switch option *Silent* on:

```
Executing postEquiv( $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}$ ) with all 1296 inputs.
Execution completed for ALL inputs (244 ms, 1080 checked, 216 inadmissible).
```

If theorem is false for some input, an error message is displayed.

Example: Quotient and Remainder



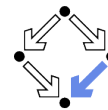
Drop precondition from theorem.

```
theorem postEquiv(n:Num, m:Num, q:Num, r:Num)  $\Leftrightarrow$ 
  // requires pre(n, m);
  post1(n, m, q, r)  $\Leftrightarrow$  post2(n, m, q, r);
```

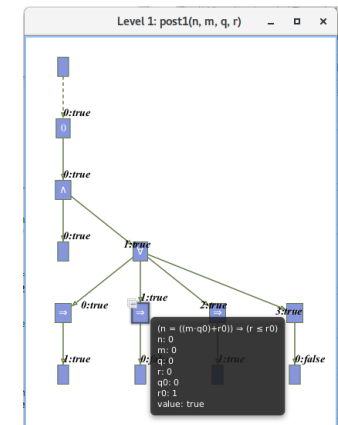
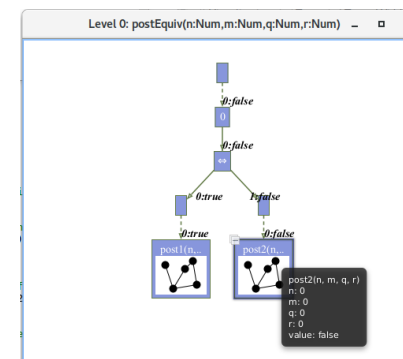
```
Executing postEquiv( $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}$ ) with all 1296 inputs.
Run 0 of deterministic function postEquiv(0,0,0,0):
ERROR in execution of postEquiv(0,0,0,0): evaluation of
postEquiv
at line 25 in file divide.txt:
theorem is not true
ERROR encountered in execution.
```

For $n = 0, m = 0, q = 0, r = 0$, the modified theorem is not true.

Visualizing the Formula Evaluation

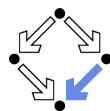


Select $N = 1$ and visualization option "Tree".



Investigate the (pruned) evaluation tree to determine how the truth value of a formula was derived (double click to zoom into/out of predicates).

Example: Quotient and Remainder



Switch option “Nondeterminism” on.

```
// 1. investigate whether the specified input/output combinations are as desired
fun quotremFun(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
= choose q:Num, r:Num with post(n, m, q, r);
```

Executing quotremFun(\mathbb{Z} , \mathbb{Z}) with all 36 inputs.

Ignoring inadmissible inputs...

Branch 0:6 of nondeterministic function quotremFun(0,1):

Result (0 ms): [0,0]

Branch 1:6 of nondeterministic function quotremFun(0,1):

No more results (8 ms).

...

Branch 0:35 of nondeterministic function quotremFun(5,5):

Result (0 ms): [1,0]

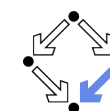
Branch 1:35 of nondeterministic function quotremFun(5,5):

No more results (14 ms).

Execution completed for ALL inputs (413 ms, 30 checked, 6 inadmissible).

First validation by inspecting the values determined by output condition
(nondeterminism may produce for some inputs multiple outputs).

Example: Quotient and Remainder

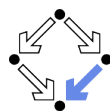


```
// 2. check that some but not all inputs are allowed
theorem someInput()  $\Leftrightarrow$   $\exists$ n:Num, m:Num. pre(n, m);
theorem notEveryInput()  $\Leftrightarrow$   $\exists$ n:Num, m:Num.  $\neg$ pre(n, m);
```

Executing someInput().
Execution completed (0 ms).
Executing notEveryInput().
Execution completed (0 ms).

A very rough validation of the input condition.

Example: Quotient and Remainder



```
// 3. check whether for all inputs that satisfy the precondition
// there are some outputs that satisfy the postcondition
theorem someOutput(n:Num, m:Num)
  requires pre(n, m);
 $\Leftrightarrow$   $\exists$ q:Num, r:Num. post(n, m, q, r);
```

```
// 4. check that not every output satisfies the postcondition
theorem notEveryOutput(n:Num, m:Num)
  requires pre(n, m);
 $\Leftrightarrow$   $\exists$ q:Num, r:Num.  $\neg$ post(n, m, q, r);
```

Executing someOutput(\mathbb{Z} , \mathbb{Z}) with all 36 inputs.

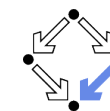
Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).

Executing notEveryOutput(\mathbb{Z} , \mathbb{Z}) with all 36 inputs.

Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).

A very rough validation of the output condition.

Example: Quotient and Remainder



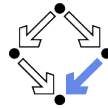
```
// 5. check that the output is uniquely defined
// (optional, need not generally be the case)
theorem uniqueOutput(n:Num, m:Num)
  requires pre(n, m);
 $\Leftrightarrow$ 
 $\forall$ q:Num, r:Num. post(n, m, q, r)  $\Rightarrow$ 
 $\forall$ q0:Num, r0:Num. post(n, m, q0, r0)  $\Rightarrow$ 
  q = q0  $\wedge$  r = r0;
```

Executing uniqueOutput(\mathbb{Z} , \mathbb{Z}) with all 36 inputs.

Execution completed for ALL inputs (18 ms, 30 checked, 6 inadmissible).

The output condition indeed determines the outputs uniquely.

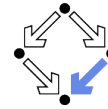
Example: Quotient and Remainder



```
// 6. check whether the algorithm satisfies the specification
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures let q=result.1, r=result.2 in post(n, m, q, r);
{
  var q: Num = 0;
  var r: Num = n;
  while r ≥ m do
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

Check whether the algorithm satisfies the specification.

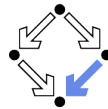
Example: Quotient and Remainder



```
Executing quotRemProc( $\mathbb{Z}, \mathbb{Z}$ ) with all 36 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function quotRemProc(0,1):
Result (0 ms): [0,0]
Run 7 of deterministic function quotRemProc(1,1):
Result (0 ms): [1,0]
...
Run 31 of deterministic function quotRemProc(1,5):
Result (1 ms): [0,1]
Run 32 of deterministic function quotRemProc(2,5):
Result (0 ms): [0,2]
Run 33 of deterministic function quotRemProc(3,5):
Result (0 ms): [0,3]
Run 34 of deterministic function quotRemProc(4,5):
Result (0 ms): [0,4]
Run 35 of deterministic function quotRemProc(5,5):
Result (1 ms): [1,0]
Execution completed for ALL inputs (161 ms, 30 checked, 6 inadmissible).
```

A verification of the algorithm by checking all possible executions.

Example: Quotient and Remainder

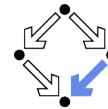


```
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures post(n, m, result.1, result.2);
{
  var q: Num = 0;
  var r: Num = n;
  while r > m do // error!
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

```
Executing quotRemProc( $\mathbb{Z}, \mathbb{Z}$ ) with all 36 inputs.
ERROR in execution of quotRemProc(1,1): evaluation of
  ensures let q = result.1, r = result.2 in post(n, m, q, r);
at line 65 in file divide.txt:
  postcondition is violated by result [0,1]
ERROR encountered in execution.
```

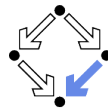
A falsification of an incorrect algorithm.

Example: Sorting an Array



```
val N:Nat; val M:Nat;
type nat = Nat[M]; type array = Array[N,nat]; type index = Nat[N-1];

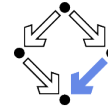
proc sort(a:array): array
  ensures  $\forall i:\text{nat}. i < N-1 \Rightarrow \text{result}[i] \leq \text{result}[i+1]$ ;
  ensures  $\exists p:\text{Array}[N,\text{index}].$ 
    ( $\forall i:\text{index}, j:\text{index}. i \neq j \Rightarrow p[i] \neq p[j]$ )  $\wedge$ 
    ( $\forall i:\text{index}. a[i] = \text{result}[p[i]]$ );
{
  var b:array = a;
  for var i:Nat[N]:=1; i<N; i:=i+1 do {
    var x:nat := b[i];
    var j:Int[-1,N] := i-1;
    while j ≥ 0  $\wedge$  b[j] > x do {
      b[j+1] := b[j];
      j := j-1;
    }
    b[j+1] := x;
  }
  return b;
}
```

Example: Sorting an Array

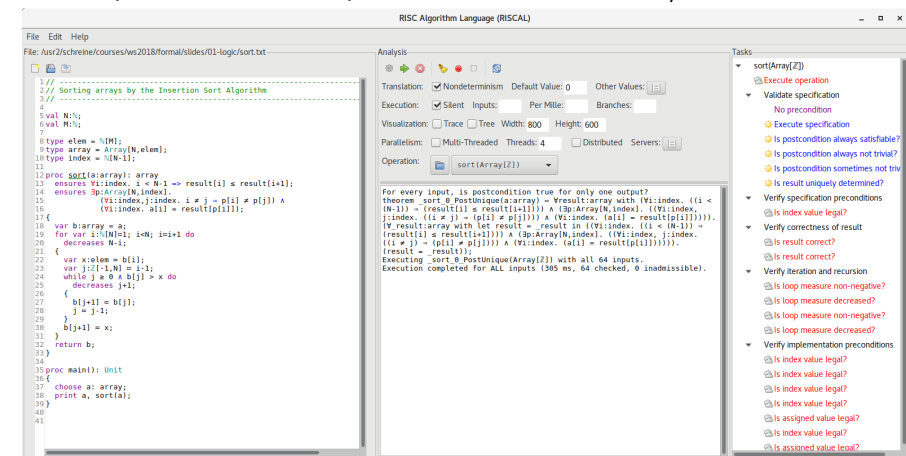
Using N=5.
Using M=5.
Type checking and translation completed.
Executing sort(Array[Z]) with all 7776 inputs.
1223 inputs (1223 checked, 0 inadmissible, 0 ignored)...
2026 inputs (2026 checked, 0 inadmissible, 0 ignored)...
...
5114 inputs (5114 checked, 0 inadmissible, 0 ignored)...
5467 inputs (5467 checked, 0 inadmissible, 0 ignored)...
5792 inputs (5792 checked, 0 inadmissible, 0 ignored)...
6118 inputs (6118 checked, 0 inadmissible, 0 ignored)...
6500 inputs (6500 checked, 0 inadmissible, 0 ignored)...
6788 inputs (6788 checked, 0 inadmissible, 0 ignored)...
7070 inputs (7070 checked, 0 inadmissible, 0 ignored)...
7354 inputs (7354 checked, 0 inadmissible, 0 ignored)...
7634 inputs (7634 checked, 0 inadmissible, 0 ignored)...
Execution completed for ALL inputs (32606 ms, 7776 checked, 0 inadmissible).
Not all nondeterministic branches may have been considered.

Also this algorithm can be automatically checked.

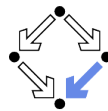


Example: Sorting an Array

Select operation sort and press the button "Show/Hide Tasks".



Automatically generated formulas to validate procedure specifications.



Example: Sorting an Array

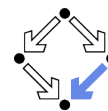
Right-click to print definition of a formula, double-click to check it.

For every input, is postcondition true for only one output?

```
theorem _sort_0_PostUnique(a:array) ⇔
  ∀result:array with
    (∀i:index. ((i < (N-1)) ⇒ (result[i] ≤ result[i+1]))) ∧
    (∃p:Array[N,index]. ((∀i:index, j:index. ((i ≠ j) ⇒ (p[i] ≠ p[j]))) ∧
      (∀i:index. (a[i] = result[p[i]]))))).
  (∀_result:array with let result = _result in #
    ((∀i:index. ((i < (N-1)) ⇒ (result[i] ≤ result[i+1]))) ∧
      (∃p:Array[N,index]. ((∀i:index, j:index. ((i ≠ j) ⇒ (p[i] ≠ p[j]))) ∧
        (∀i:index. (a[i] = result[p[i]])))))).
    (result = _result));
```

Using N=3.
Using M=3.
Type checking and translation completed.
Executing _sort_0_PostUnique(Array[Z]) with all 64 inputs.
Execution completed for ALL inputs (529 ms, 64 checked, 0 inadmissible).

The output is indeed uniquely defined by the output condition.

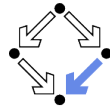


Model Checking versus Proving

Two fundamental techniques for the verification of computer programs.

- **Checking Program Executions**
 - Enumeration of all possible executions and evaluation of formulas (e.g. postconditions) on the resulting states.
 - Fully automatic, no human interaction is required.
 - Only possible if there are only finitely many executions (and finitely many values for the quantified variables in the formulas).
 - State space explosion: "finitely many" means "not too many".
- **Proving Verification Conditions**
 - Logic formulas that are valid if and only if program is correct with respect to its specification.
 - Also possible if there are infinitely many executions and infinitely many values for the quantified variables.
 - Many conditions can be automatically proved (automated reasoners); in general interaction with human is required (proof assistants).

General verification requires the proving of logic formulas.



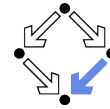
1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

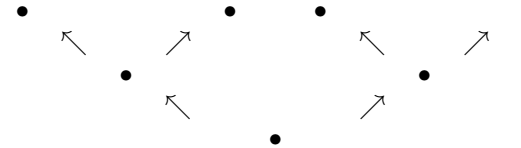
4. The RISC ProofNavigator

Proofs



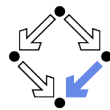
A **proof** is a structured argument that a formula is true.

- A tree whose nodes represent **proof situations (states)**.



- Each proof situation consists of **knowledge** and a **goal**.
 - $K_1, \dots, K_n \vdash G$
 - Knowledge K_1, \dots, K_n : formulas assumed to be true.
 - Goal G : formula to be proved relative to knowledge.
- The **root** of the tree is the initial proof situation.
 - K_1, \dots, K_n : axioms of mathematical background theories.
 - G : formula to be proved.

Proof Rules



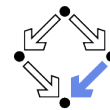
A **proof rules** describes how a proof situation can be reduced to zero, one, or more “subsituations”.

$$\frac{\dots \vdash \dots \quad \dots \vdash \dots}{K_1, \dots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
 - Zero subsituations: G has been proved, (sub)proof is closed.
 - One or more subsituations: G is proved, if all subgoals are proved.
- **Top-down rules**: focus on G .
 - G is decomposed into simpler goals G_1, G_2, \dots
- **Bottom-up rules**: focus on K_1, \dots, K_n .
 - Knowledge is extended to K_1, \dots, K_n, K_{n+1} .

In each proof situation, we aim at showing that the goal is “apparently” true with respect to the given knowledge.

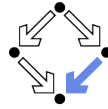
Conjunction $F_1 \wedge F_2$



$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2} \quad \frac{\dots, K_1 \wedge K_2, K_1, K_2 \vdash G}{\dots, K_1 \wedge K_2 \vdash G}$$

- **Goal $G_1 \wedge G_2$** .
 - Create two subsituations with goals G_1 and G_2 .
We have to show $G_1 \wedge G_2$.
 - *We show G_1 : ... (proof continues with goal G_1)*
 - *We show G_2 : ... (proof continues with goal G_2)*
- **Knowledge $K_1 \wedge K_2$** .
 - Create one subsituation with K_1 and K_2 in knowledge.
We know $K_1 \wedge K_2$. We thus also know K_1 and K_2 . (proof continues with current goal and additional knowledge K_1 and K_2)

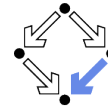
Disjunction $F_1 \vee F_2$



$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \quad \frac{\dots, K_1 \vdash G \quad \dots, K_2 \vdash G}{\dots, K_1 \vee K_2 \vdash G}$$

- **Goal $G_1 \vee G_2$.**
 - Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):
We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- **Knowledge $K_1 \vee K_2$.**
 - Create two subsituations, one with K_1 and one with K_2 in knowledge.
We know $K_1 \vee K_2$. We thus proceed by case distinction:
 - Case K_1 : ... (proof continues with current goal and additional knowledge K_1).
 - Case K_2 : ... (proof continues with current goal and additional knowledge K_2).

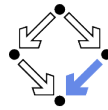
Implication $F_1 \Rightarrow F_2$



$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \quad \frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

- **Goal $G_1 \Rightarrow G_2$**
 - Create one subsituation where G_2 is proved under the assumption that G_1 holds:
We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)
- **Knowledge $K_1 \Rightarrow K_2$**
 - Create two subsituations, one with goal K_1 and one with knowledge K_2 .
We know $K_1 \Rightarrow K_2$.
 - We show K_1 : ... (proof continues with goal K_1)
 - We know K_2 : ... (proof continues with current goal and additional knowledge K_2).

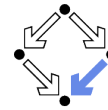
Equivalence $F_1 \Leftrightarrow F_2$



$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \quad \frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

- **Goal $G_1 \Leftrightarrow G_2$**
 - Create two subsituations with implications in both directions as goals:
We have to show $G_1 \Leftrightarrow G_2$.
 - We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$)
 - We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- **Knowledge $K_1 \Leftrightarrow K_2$**
 - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.
We know $K_1 \Leftrightarrow K_2$.
 - We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$)
 - We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)

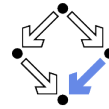
Universal Quantification $\forall x : F$



$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G} \quad (x_0 \text{ new for } K, G) \quad \frac{\dots, \forall x : K, K[T/x] \vdash G}{\dots, \forall x : K \vdash G}$$

- **Goal $\forall x : G$**
 - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x_0/x]$.
We have to show $\forall x : G$. Take arbitrary x_0 .
We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)
- **Knowledge $\forall x : K$**
 - Choose term T to create one subsituation with formula $K[T/x]$ added to the knowledge.
We know $\forall x : K$ and thus also $K[T/x]$.
(proof continues with current goal and additional knowledge $K[T/x]$)

Existential Quantification $\exists x : F$



$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \quad \frac{\dots, K[x_0/x] \vdash G}{\dots, \exists x : K \vdash G} \quad (x_0 \text{ new for } K, G)$$

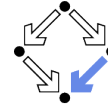
■ Goal $\exists x : G$

- Choose term T to create one subsituation with goal $G[T/x]$.
*We have to show $\exists x : G$. It suffices to show $G[T/x]$.
(proof continues with goal $G[T/x]$)*

■ Knowledge $\exists x : K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K[x_0/x]$.
*We know $\exists x : K$. Let x_0 be such that $K[x_0/x]$.
(proof continues with current goal and additional knowledge $K[x_0/x]$)*

Example



We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

$$(b) \forall y : \exists x : P(x, y)$$

Take arbitrary y_0 . We show

$$(c) \exists x : P(x, y_0)$$

From (1) we know for some x_0

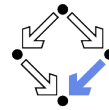
$$(2) \forall y : P(x_0, y)$$

From (2) we know

$$(3) P(x_0, y_0)$$

From (3), we know (c). QED.

Example



We show

$$(a) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y)) \Rightarrow (\exists x, y : q(x, y))$$

We assume

$$(1) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y))$$

and show

$$(b) \exists x, y : q(x, y)$$

From (1), we know

$$(2) \exists x : p(x)$$

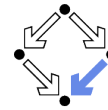
$$(3) \forall x : p(x) \Rightarrow \exists y : q(x, y)$$

From (2) we know for some x_0

$$(4) p(x_0)$$

...

Example (Contd)



...

From (3), we know

$$(5) p(x_0) \Rightarrow \exists y : q(x_0, y)$$

From (4) and (5), we know

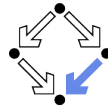
$$(6) \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7) q(x_0, y_0)$$

From (7), we know (b). QED.

Indirect Proofs

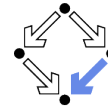


$$\frac{K, \neg G \vdash \text{false}}{K \vdash G} \quad \frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G} \quad \frac{\dots, \neg G \vdash \neg K}{\dots, K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.
 - Prove that “false” is true.
 - Prove that a formula F is true and also prove that it is false.
 - Prove that some knowledge K is false, i.e. that $\neg K$ is true.
 - Switches goal G and knowledge K (negating both).

Sometimes simpler than a direct proof.

Example



We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

$$(b) \forall y : \exists x : P(x, y)$$

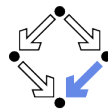
We assume

$$(2) \neg \forall y : \exists x : P(x, y)$$

and show a contradiction.

...

Example



...

From (2), we know

$$(3) \exists y : \forall x : \neg P(x, y)$$

Let y_0 be such that

$$(4) \forall x : \neg P(x, y_0)$$

From (1) we know for some x_0

$$(5) \forall y : P(x_0, y)$$

From (5) we know

$$(6) P(x_0, y_0)$$

From (4), we know

$$(7) \neg P(x_0, y_0)$$

From (6) and (7), we have a contradiction. QED.

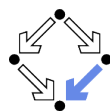
1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

4. The RISC ProofNavigator

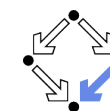
The RISC ProofNavigator



- **An interactive proving assistant for program verification.**
 - Research Institute for Symbolic Computation (RISC), 2005–.
<http://www.risc.jku.at/research/formal/software/ProofNavigator>.
 - Development based on prior experience with PVS (SRI, 1993–).
 - Kernel and GUI implemented in Java.
 - Uses external SMT (satisfiability modulo theories) solver.
 - CVCL (Cooperating Validity Checker Lite) 2.0, CVC3, CVC4 1.4.
 - Runs under Linux (only); freely available as open source (GPL).
- **A language for the definition of logical theories.**
 - Based on a strongly typed higher-order logic (with subtypes).
 - Introduction of types, constants, functions, predicates.
- **Computer support for the construction of proofs.**
 - Commands for basic inference rules and combinations of such rules.
 - Applied interactively within a sequent calculus framework.
 - Top-down elaboration of proof trees.

Designed for simplicity of use; applied to non-trivial verifications.

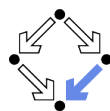
Using the Software



For survey, see “Program Verification with the RISC ProofNavigator”. For details, see “The RISC ProofNavigator: Tutorial and Manual”.

- **Develop a theory.**
 - Text file with declarations of types, constants, functions, predicates.
 - Axioms (propositions assumed true) and formulas (to be proved).
- **Load the theory.**
 - File is read; declarations are parsed and type-checked.
 - Type-checking conditions are generated and proved.
- **Prove the formulas in the theory.**
 - Human-guided top-down elaboration of proof tree.
 - Steps are recorded for later replay of proof.
 - Proof status is recorded as “open” or “completed”.
- **Modify theory and repeat above steps.**
 - Software maintains dependencies of declarations and proofs.
 - Proofs whose dependencies have changed are tagged as “untrusted”.

Starting the Software



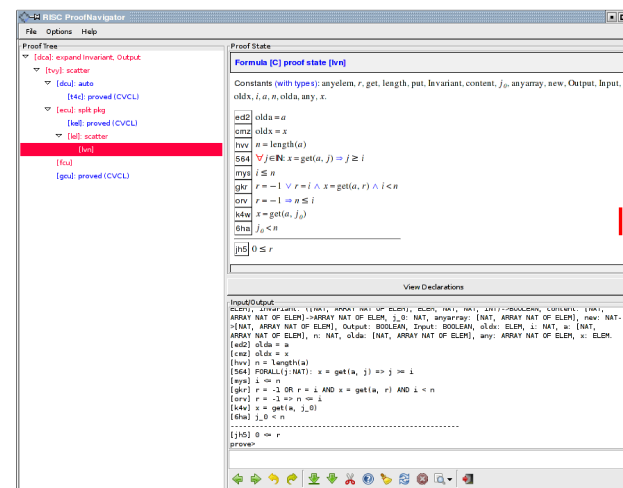
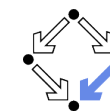
- **Starting the software:**

```
module load ProofNavigator (users at RISC)
ProofNavigator &
```
- **Command line options:**

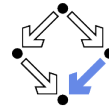
```
Usage: ProofNavigator [OPTION]... [FILE]
FILE: name of file to be read on startup.
OPTION: one of the following options:
  -n, --nogui: use command line interface.
  -c, --context NAME: use subdir NAME to store context.
  --cvcl PATH: PATH refers to executable "cvcl".
  -s, --silent: omit startup message.
  -h, --help: print this message.
```
- **Repository stored in subdirectory of current working directory:**

```
ProofNavigator/
  ■ Option -c dir or command newcontext "dir" :
    ■ Switches to repository in directory dir.
```

The Graphical User Interface



A Theory



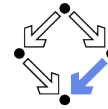
```
% switch repository to "sum"
newcontext "sum";

% the recursive definition of the sum from 0 to n
sum: NAT->NAT;
S1: AXIOM sum(0)=0;
S2: AXIOM FORALL(n:NAT): n>0 => sum(n)=n+sum(n-1);

% proof that explicit form is equivalent to recursive definition
S: FORMULA FORALL(n:NAT): sum(n) = (n+1)*n/2;
```

Declarations written with an external editor in a text file.

Proving a Formula



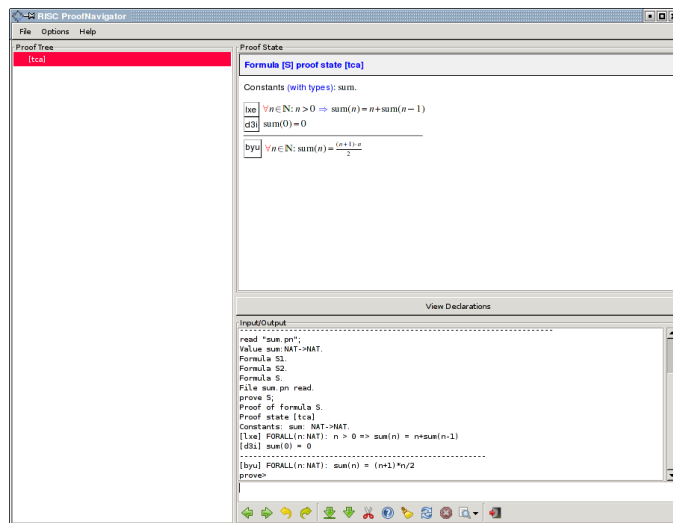
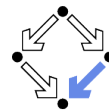
When the file is loaded, the declarations are pretty-printed:

```
sum ∈ ℕ → ℕ
axiom S1 ≡ sum(0) = 0
axiom S2 ≡ ∀ n ∈ ℕ: n > 0 ⇒ sum(n) = n + sum(n - 1)
S ≡ ∀ n ∈ ℕ: sum(n) =  $\frac{(n+1) \cdot n}{2}$ 
```

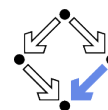
The proof of a formula is started by the prove command.

```
Formula S
prove S: Construct Proof
proof S: Show Proof
formula S: Print Formula
```

Proving a Formula



Proving a Formula

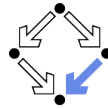


- Proof of formula F is represented as a **tree**.
 - Each tree node denotes a **proof state (goal)**.
 - Logical sequent: $A_1, A_2, \dots \vdash B_1, B_2, \dots$
 - Interpretation: $(A_1 \wedge A_2 \wedge \dots) \Rightarrow (B_1 \vee B_2 \vee \dots)$
 - Initially single node $Axioms \vdash F$.
- The **tree must be expanded to completion**.
 - Every leaf must denote an obviously valid formula.
 - Some A_i is false or some B_j is true.
- A proof step consists of the **application of a proving rule to a goal**.
 - Either the goal is recognized as true.
 - Or the goal becomes the parent of a number of children (subgoals).

The conjunction of the subgoals implies the parent goal.

$$\begin{array}{l} \text{Constants: } x_0 \in S_0, \dots \\ [L_1] \quad A_1 \\ \dots \\ [L_n] \quad A_n \\ \hline [L_{n+1}] \quad B_1 \\ \dots \\ [L_{n+m}] \quad B_m \end{array}$$

An Open Proof Tree



Proof Tree

- ▾ [tca]: induction n in byu
 - [dbj]: proved (CVCL)
 - [ebj]

Formula [S] proof state [dbj]

Constants (with types): sum.

lxe $\forall n \in \mathbb{N}: n > 0 \Rightarrow \text{sum}(n) = n + \text{sum}(n-1)$

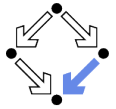
d3i $\text{sum}(0) = 0$

nfq $\text{sum}(0) = \frac{(0+1) \cdot 0}{2}$

Parent: [tca]

Closed goals are indicated in blue; goals that are open (or have open subgoals) are indicated in red. The red bar denotes the “current” goal.

A Completed Proof Tree

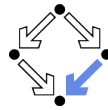


Proof Tree

- ▾ [tca]: induction n in byu
 - [dbj]: proved (CVCL)
 - ▾ [ebj]: instantiate n_0+1 in lxe
 - [k5f]: proved (CVCL)

The visual representation of the complete proof structure; by clicking on a node, the corresponding proof state is displayed.

Navigation Commands

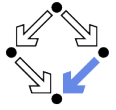


Various buttons support navigation in a proof tree.

- : prev
 - Go to previous open state in proof tree.
- : next
 - Go to next open state in proof tree.
- : undo
 - Undo the proof command that was issued in the parent of the current state; this discards the whole proof tree rooted in the parent.
- : redo
 - Redo the proof command that was previously issued in the current state but later undone; this restores the discarded proof tree.

Single click on a node in the proof tree displays the corresponding state; double click makes this state the current one.

Proving Commands

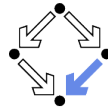


The most important proving commands can be also triggered by buttons.

- (scatter)
 - Recursively applies decomposition rules to the current proof state and to all generated child states; attempts to close the generated states by the application of a validity checker.
- (decompose)
 - Like scatter but generates a single child state only (no branching).
- (split)
 - Splits current state into multiple children states by applying rule to current goal formula (or a selected formula).
- (auto)
 - Attempts to close current state by instantiation of quantified formulas.
- (autostar)
 - Attempts to close current state and its siblings by instantiation.

Automatic decomposition of proofs and closing of proof states.

Proving Commands

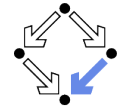


More commands can be selected from the menus.





- `assume`
 - Introduce a new assumption in the current state; generates a sibling state where this assumption has to be proved.
- `case`:
 - Split current state by a formula which is assumed as true in one child state and as false in the other.
- `expand`:
 - Expand the definitions of denoted constants, functions, or predicates.
- `lemma`:
 - Introduce another (previously proved) formula as new knowledge.
- `instantiate`:
 - Instantiate a universal assumption or an existential goal.
- `induction`:
 - Start an induction proof on a goal formula that is universally quantified over the natural numbers.

Here the creativity of the user is required!

Auxiliary Commands

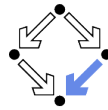


Some buttons have no command counterparts.

- : `counterexample`
 - Generate a “counterexample” for the current proof state, i.e. an interpretation of the constants that refutes the current goal.
- :
 - Abort current prover activity (proof state simplification or counterexample generation).
- :
 - Show menu that lists all commands and their (optional) arguments.
- :
 - Simplify current state (if automatic simplification is switched off).

More facilities for proof control.

Proving Strategies



- Initially: semi-automatic proof decomposition.
 - `expand` expands constant, function, and predicate definitions.
 - `scatter` aggressively decomposes a proof into subproofs.
 - `decompose` simplifies a proof state without branching.
 - `induction` for proofs over the natural numbers.
- Later: critical hints given by user.
 - `assume` and `case` cut proof states by conditions.
 - `instantiate` provide specific formula instantiations.
- Finally: simple proof states are yielded that can be automatically closed by the validity checker.
 - `auto` and `autostar` may help to close formulas by the heuristic instantiation of quantified formulas.

Appropriate combination of semi-automatic proof decomposition, critical hints given by the user, and the application of a validity checker is crucial.