Logic

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Term Syntax

Take signature $\Sigma = (S, \Omega)$.

- Variables:
  - Family $V = (V_s)_{s \in S}$ of infinite sets disjoint with Ω and each other.
    - V_s ... the set of variables of sort s.
  - Any family $X \subseteq V$ is called a set of variables for Σ.
- Terms:
  - Family $T_{\Sigma(X)} = (T_{\Sigma(X),s})_{s \in S}$ of terms with set of variables X for Σ.
    - Variables are terms: $X_s \subseteq T_{\Sigma(X),s}$.
    - Constants are terms: if $n : \to s \in \Omega$, then $n \in T_{\Sigma(X),s}$.
    - Applications are terms: if $n : s_1 \times \ldots \times s_k \to s \in \Omega$ and, for $1 \leq i \leq k$, $t_i \in T_{\Sigma(X),s_i}$, then $n(t_1, \ldots, t_k) \in T_{\Sigma(X),s}$.
- $Var(t) \subseteq X$:
  - The set of variables occuring in term $t \in T_{\Sigma(X)}$.
- Ground terms:
  - Term t is a ground term, if $Var(t) = \emptyset$.
  - The set of ground terms T_Σ $(= (T_{\Sigma,s})_{s \in S})$.

Example

- Signature $\mathrm{NATBOOL} = (\{nat, bool\},$
  $\{True :\to bool, False :\to bool,$
  $\neg : bool \to bool, \wedge : bool \times bool \to bool,$
  $0 :\to nat, Succ : nat \to nat,$
  $+ : nat \times nat \to nat,$
  $\leq: nat \times nat \to bool\})$
- Variable set X with $X_{bool} = \{b, c\}$ and $X_{nat} = \{m, n\}$.
- Terms in $T_{\mathrm{NATBOOL}(X),bool}$:

  c
  $\wedge(\wedge(True, b), False)$
  $\leq (0, +(m, Succ(n)))$

All terms are strongly typed.

Term Semantics

Take signature $\Sigma = (S, \Omega)$, set of variables X for Σ, Σ-algebra A.

- Assignment $\alpha : X \to A$ of X in A:
  - Family $\alpha = (\alpha_s)_{s \in S}$ of functions $\alpha_s : X_s \to A(s)$.
    - Every variable is mapped to an A-value of the corresponding sort.
- Value $A(\alpha)(t)$ of term t for assignment α:
  - If $t = x$ with $x \in X_s$, then $\alpha_s(x)$.
  - If $t = n$ with $\omega = n :\to s \in \Omega$, then $A(\omega)$.
  - If $t = n(t_1, \ldots, t_k)$ with $\omega = n : s_1 \times \ldots \times s_k \to s \in \Omega$ and, for $1 \leq i \leq k$, $t_i \in T_{\Sigma(X),s_i}$, then $A(\omega)(A(\alpha)(t_1), \ldots, A(\alpha)(t_k))$.
- Value $A(t)$ of ground term t.
  - $A(\alpha)(t)$ for any assignment α.
    - Value of ground term does not depend on assignment.

Semantics maps terms to algebra values.

Algebra Logic

General logical framework for specifying ADTs.

- (Algebra) Logic L: for each signature Σ,
  - a set $L(\Sigma)$ of Σ-formulas.
  - a relation $\models_\Sigma \subseteq Alg(\Sigma) \times L(\Sigma)$ between Σ-algebras and Σ-formulas (the satisfaction relation for Σ).
    - If $A \models_\Sigma \varphi$, we say "φ is valid in A" or "A satisfies φ".
- L must satisfy the isomorphism condition:
  - If $A \simeq B$, then ($A \models_\Sigma \varphi$ iff $B \models_\Sigma \varphi$).
    - For any signature Σ, Σ-formula φ, Σ-algebras A and B.
  - *L cannot distinguish between isomorphic algebras.*
    - L has no more information about A and B than visible in Σ.

We will investigate three specific logics.

Equational Logic *EL*

- Formulas $EL(\Sigma)$:
  - $EL(\Sigma) = \{\forall X . t = u \mid X$ is a set of variables for Σ,
    $t, u \in T_{\Sigma(X),s}$ for some sort s of $\Sigma\}$.
    - May drop "$\forall X$", if $X = Var(t) \cup Var(u)$.
- Satisfaction Relation $\models_\Sigma$:
  - $A \models_\Sigma \forall X . t = u$ iff
    for all assignments $\alpha : X \to A$:
    $A(\alpha)(t) = A(\alpha)(u)$
    - For each Σ-algebra A and equation $\forall X . t = u \in EL(\Sigma)$.

The logic of universally quantified equations.

Example

Take "classical" $\mathrm{NATBOOL}$-algebra A (with $A(\mathit{nat}) = \mathbb{N}$).

$$
\begin{array}{l}
A \models x + 1 = 1 + x \\
A \models (x \leq 0 \land \neg x \leq 0) = \mathit{False} \\
A \models x = x \\
A \not\models x = y
\end{array}
$$

Note: predicate $\leq$ is operation of sort *bool*.

Conditional Equational Logic *CEL*

- Formulas $CEL(\Sigma)$:
  - $CEL(\Sigma) = \{\forall X.t_1 = u_1 \wedge \ldots \wedge t_k = u_k \Rightarrow t_{k+1} = u_{k+1} \mid$
    X is a set of variables for Σ,
    $t_i, u_i \in T_{\Sigma(X),s_i}$, for some sort $s_i\}$.
    - Drop "$\forall X$", if $X = Var(t_1) \cup Var(u_1) \cup \ldots \cup Var(t_{k+1}) \cup Var(u_{k+1})$.
- Satisfaction Relation $\models_\Sigma$:
  - $A \models_\Sigma \forall X.t_1 = u_1 \wedge \ldots \wedge t_k = u_k \Rightarrow t_{k+1} = u_{k+1}$ iff
    for all assignments $\alpha : X \rightarrow A$:
    if $A(\alpha)(t_1) = A(\alpha)(u_1)$ and ... and $A(\alpha)(t_k) = A(\alpha)(u_k)$ then
    $A(\alpha)(t_{k+1}) = A(\alpha)(u_{k+1})$.

The logic of universally quantified conditional equations.

Example

Take "classical" $\mathrm{NATBOOL}$-algebra A (with $A(nat) = \mathbb{N}$) augmented by operation $- : nat \times nat \rightarrow nat$.

$$A \models x \leq y = True \Rightarrow (y - x) + x = y$$
$$A \models x + y = z \Rightarrow z - y = x$$
$$A \models x \leq y = False \Rightarrow y \leq x = True$$

Note: only equalities allowed as atomic predicates.

First-Order Predicate Logic PL

- Formulas $PL(\Sigma)$:
  - If $t, u \in T_{\Sigma(X),s}$ for some sort s of Σ, then $t = u \in PL(\Sigma)$.
  - If $\varphi \in PL(\Sigma)$, then $\neg\varphi \in PL(\Sigma)$.
  - If $\varphi_1, \varphi_2 \in PL(\Sigma)$, then $\varphi_1 \wedge \varphi_2 \in PL(\Sigma)$.
  - If s is a sort of Σ, x is a variable of sort s, and $\varphi \in PL(\Sigma)$, then $(\forall x : s \,.\, \varphi) \in PL(\Sigma)$.
- Value $A(\alpha)(\varphi)$ of formula φ for assignment $\alpha : \mathit{free}(\varphi) \to A$:
  ($\mathit{free}(\varphi)$ ... the set of free variables of φ)
  - $A(\alpha)(t = u) = \mathit{true}$ iff $A(\alpha)(t) = A(\alpha)(u)$.
  - $A(\alpha)(\neg\varphi) = \mathit{true}$ iff $A(\alpha)(\varphi) = \mathit{false}$.
  - $A(\alpha)(\varphi_1 \wedge \varphi_2) = \mathit{true}$ iff $A(\alpha)(\varphi_1) = A(\alpha)(\varphi_2) = \mathit{true}$.
  - $A(\alpha)(\forall x : s \,.\, \varphi) = \mathit{true}$ iff $A(\alpha[a/x])(\varphi) = \mathit{true}$ for all $a \in A(s)$.
    - $\alpha[a/x](x) = a$; $\alpha[a/x](y) = \alpha(y)$, if $x \neq y$.
- Satisfaction Relation $\models_\Sigma$:
  - $A \models_\Sigma (\varphi)$ iff $A(\alpha)(\varphi) = \mathit{true}$ for all assignments $\alpha : \mathit{free}(\varphi) \to A$.

Classical predicate logic in a typed framework.

Example

Take "classical" $\mathrm{NATBOOL}$-algebra A (with $A(\mathit{nat}) = \mathbb{N}$).

$$A \models (\forall x : \mathit{nat} \, . \, (0 \le x) = \mathit{True})$$
$$A \models \neg(\forall x : \mathit{nat} \, . \, (\forall y : \mathit{nat} \, . \, (x \le y) = \mathit{True})).$$
$$A \models (\forall x : \mathit{nat} \, .(\forall y : \mathit{nat}.(x \le y) = \mathit{True}) \Rightarrow x = 0)$$

The connectives $\vee, \Rightarrow, \Leftrightarrow$ and the quantifier $\exists$ can be introduced as abbreviations of formulas that use $\neg, \wedge, \forall$ (e.g. $a \vee b :\Leftrightarrow \neg(\neg a \wedge \neg b)$).

Models

- A model of a set of formulas $\Phi \subseteq L(\Sigma)$:
  - A Σ-algebra A is a model of Φ iff $A \models_\Sigma \Phi$.
    - $A \models_\Sigma \Phi$ iff $A \models_\Sigma \varphi$ for all $\varphi \in \Phi$.
- Domain (universe) for a signature Σ (a Σ-domain):
  - A class $\mathcal{U}$ of Σ-algebras closed under isomorphism.
    - Note: a domain is an abstract datatype.
- $Mod_{\mathcal{U},\Sigma}(\Phi) \subseteq \mathcal{U}$:
  - The class of all algebras of domain $\mathcal{U}$ that are models of Φ.
    - If Σ is clear, then we write $Mod_{\mathcal{U}}(\Phi)$.
    - If $\mathcal{U} = Alg(\Sigma)$, then we write $Mod_\Sigma(\Phi)$.
    - If both holds, then we simply write $Mod(\Phi)$.
- Theorem: $Mod_{\mathcal{U},\Sigma}(\Phi)$ is an abstract datatype.
  - Logic L, signature Σ, formula set $\Phi \subseteq L(\Sigma)$, Σ-domain $\mathcal{U}$.

A set of formulas specifies a subset of a given Σ-domain as an ADT.

Example

- $\Sigma = (\{s\}, \{0 :\rightarrow s, + : s \times s \rightarrow s\})$.
- $$\begin{aligned}\Phi = \{&x + (y + z) = (x + y) + z,\\ &x + 0 = x,\\ &0 + x = x,\\ &\forall x : s\ .\ \exists y : s\ .\ x + y = 0 \wedge y + x = 0\}.\end{aligned}$$
- $Mod_\Sigma(\Phi) = \{A \in Alg(\Sigma) \mid A(s)$ and $A(+)$ form a group with neutral element $A(0)\}$.

Specification of the abstract datatype "group" (polymorphic, because the group may or may not be commutative).