Problems Solved:

21	22	23	24	25

Name:

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Problem 21. Let $f : \mathbb{N} \to \mathbb{N}$ be the (partial) function

$$f(x) = \begin{cases} y & \text{such that } x = y^2 \text{ if such a } y \text{ exists,} \\ \text{undefined} & \text{if there is no } y \text{ with } x = y^2. \end{cases}$$

1. Is f loop computable? (Justify your answer.)
2. Is f a primitive recursive function? (Justify your answer.)
3. Define f by using the base functions, composition, the primitive recursion scheme, and μ-recursion (Definitions 29 and 30 from the lecture notes). Additionally you are allowed to use the (primitive recursive) functions

   $$m : \mathbb{N}^2 \to \mathbb{N}, \quad (x, y) \mapsto x \cdot y$$

   and $u : \mathbb{N}^2 \to \mathbb{N}$,

   $$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

   Other functions or rules are forbidden.
4. Why do you need the μ-recursion in your construction?
5. Is your construction in Kleene's normal form? If it is not, describe an (informal) procedure how one can turn it into Kleene's normal form.

Problem 22. Let $f : \mathbb{N} \to \mathbb{N}$ be the function

$$f(x) = \begin{cases} y & \text{such that } x = y^2 \text{ if such a } y \text{ exists,} \\ 0 & \text{if there is no } y \text{ with } x = y^2. \end{cases}$$

1. Is f loop computable? (Justify your answer.)
2. Is f a primitive recursive function? (Justify your answer.)
3. Define f by using the base functions, composition and the primitive recursion scheme. Additionally you are allowed to use the (primitive recursive) functions

   $$m : \mathbb{N}^2 \to \mathbb{N}, \quad (x, y) \mapsto x \cdot y$$

   $u : \mathbb{N}^2 \to \mathbb{N}$,

   $$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

   and $IF : \mathbb{N}^3 \to \mathbb{N}$,

   $$IF(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}$$

Problem 23. Let P be the following program.

```
x0 := x1 + 1
LOOP x1 DO
    LOOP x0 DO
        x0 := x0 + 1;
    END;
END;
```

Similar to the construction in the lecture notes, let $f_P : \mathbb{N}^2 \to \mathbb{N}^2$ be the function that maps the given $(0, x_1)$ at the start of P to the values (x_0, x_1) after the execution of the program P. Show that f_P is primitive recursive by translating the loop program into a primitive recursive definition for f_P. Follow the steps given in the lecture notes.
Compute $f_P(0, 1)$ via your primitive recursive definition and compare it with the result you get from executing P with input $x_1 = 1$.

Problem 24. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) := 2n + 1$.

1. Show that f is loop computable by giving a loop program that computes f.
2. Show that f is primitive recursive by giving a primitive recursive definition of f.

Problem 25. Let $q : \mathbb{N}^2 \to \mathbb{N}$, $(x, y) \mapsto x \cdot x$ (sic!) and $u : \mathbb{N}^2 \to \mathbb{N}$,

$$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

be given primitive recursive functions.
Let $r : \mathbb{N}^2 \to \mathbb{N}$ be definied by

$$\begin{aligned} r(x) &= (\mu p)(x) && \text{minimization} \\ p(y, x) &= u(q(y, x), \mathrm{proj}_2^2(y, x)) && \text{composition} \end{aligned}$$

Informally we have

$$r(x) = \min_y \{ y \in \mathbb{N} \mid u(q(y, x), x)) = 0 \}$$

Similar to the treatise in the lecture notes, construct a while program that computes r. For simplicity, you are allowed to write statements such as $x_k = q(x_i, x_j)$ and $x_k = u(x_i, x_j)$ into your program. What will your program compute if it is started with input $x_1 = 2$?