Specifying and Verifying Programs (Part 2)

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Specification by State Predicates

- Hoare calculus and predicate transformers use state predicates.
  - Formulas that talk about a single (pre/post-)state.
  - In such a formula, a reference x means "the value of program variable x in the given state".
- Relationship between pre/post-state is not directly expressible.
  - Requires uninterpreted mathematical constants.

    $\{x = a\}x := x + 1\{x = a + 1\}$
- Unchanged variables have to be explicitly specified.

    $\{x = a \wedge y = b\}x := x + 1\{x = a + 1 \wedge y = b\}$
- The semantics of a command c is only implicitly specified.
  - Specifications depend on auxiliary state conditions P, Q.

    $\{P\}c\{Q\}$

    $\mathrm{wp}(c, Q) = P$

Let us turn our focus from individual states to pairs of states.

Specification by State Relations

- We introduce formulas that denote state relations.
  - Talk about a pair of states (the pre-state and the post-state).
  - old x: "the value of program variable x in the pre-state".
  - var x: "the value of program variable x in the post-state".
- We introduce the logical judgment $c : [F]^{x,\ldots}$
  - If the execution of c terminates normally, the resulting post-state is related to the pre-state as described by F.
  - Every variable y not listed in the set of variables $x, \ldots$ has the same value in the pre-state and in the post-state.

    $c : F \wedge \text{var } y = \text{old } y \wedge \ldots$

$x := x + 1 : [\text{var } x = \text{old } x + 1]^x$

$x := x + 1 : \text{var } x = \text{old } x + 1 \wedge \text{var } y = \text{old } y \wedge \text{var } z = \text{old } z \wedge \ldots$

We will discuss the termination of commands later.

State Relation Rules

$$\frac{c : [F]^{xs} \qquad y \notin xs}{c : [F \wedge \text{var } y = \text{old } y]^{xs \cup \{y\}}}$$

$$\textbf{skip} : [\text{true}]^{\emptyset} \qquad \textbf{abort} : [\text{true}]^{\emptyset} \qquad x = e : [\text{var } x = e']^{x}$$

$$\frac{c_1 : [F_1]^{xs} \quad c_2 : [F_2]^{xs}}{c_1; c_2 : [\exists ys : F_1[ys/\text{var } xs] \wedge F_2[ys/\text{old } xs]]^{xs}}$$

$$\frac{c : [F]^{xs}}{\textbf{if } e \textbf{ then } c : [\text{if } e' \text{ then } F \text{ else var } xs = \text{old } xs]^{xs}}$$

$$\frac{c_1 : [F_1]^{xs} \quad c_2 : [F_2]^{xs}}{\textbf{if } e \textbf{ then } c_1 \textbf{ else } c_2 : [\text{if } e' \text{ then } F_1 \text{ else } F_2]^{xs}}$$

$$\frac{\begin{array}{c} c : [F]^{xs} \\ \vdash \forall xs, ys, zs : I[xs/\text{old } xs, ys/\text{var } xs] \wedge e[ys/xs] \wedge F[ys/\text{old } xs, zs/\text{var } xs] \Rightarrow \\ I[xs/\text{old } xs, zs/\text{var } xs] \end{array}}{\textbf{while } e \textbf{ do } \{I, t\} \ c : [\neg e'' \wedge (I[\text{old } xs/\text{var } xs] \Rightarrow I)]^{xs}}$$

if e then F_1 else $F_2 :\Leftrightarrow (e \Rightarrow F_1) \wedge (\neg e \Rightarrow F_2)$
$e' := e[\text{old } xs/xs]$, $e'' := e[\text{var } xs/xs]$ (for all program variables xs)

Example

$c_1 = y := y + 1;$
$c_2 = x := x + y$

$c_1 : [\text{var } y = \text{old } y + 1]^{y}$
$c_2 : [\text{var } x = \text{old } x + \text{old } y]^{x}$

$c_1 : [\text{var } y = \text{old } y + 1 \wedge \text{var } x = \text{old } x]^{x,y}$
$c_2 : [\text{var } x = \text{old } x + \text{old } y \wedge \text{var } y = \text{old } y]^{x,y}$

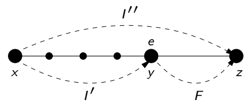
$$\begin{aligned} c_1; c_2 : [\exists x_0, y_0 :\ & \\ & y_0 = \text{old } y + 1 \wedge x_0 = \text{old } x \wedge \\ & \text{var } x = x_0 + y_0 \wedge \text{var } y = y_0]^{x,y} \end{aligned}$$

$c_1; c_2 : [\text{var } x = \text{old } x + \text{old } y + 1 \wedge \text{var } y = \text{old } y + 1]^{x,y}$

Mechanical translation and logical simplification.

Loops

$$\frac{\begin{array}{c} c : [F]^{xs} \\ \vdash \forall xs, ys, zs : I[xs/\text{old } xs, ys/\text{var } xs] \wedge e[ys/xs] \wedge F[ys/\text{old } xs, zs/\text{var } xs] \Rightarrow \\ I[xs/\text{old } xs, zs/\text{var } xs] \end{array}}{\textbf{while } e \textbf{ do } \{I, t\} \ c : [\neg e'' \wedge (I[\text{old } xs/\text{var } xs] \Rightarrow I)]^{xs}}$$



$w = \textbf{while } i < n \textbf{ do } \{I, t\} \ (s := s + i; i := i + 1)$
$I \Leftrightarrow 0 \le \text{var } i \le \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{var } i-1} j$

$$\frac{\begin{array}{l} (s := s + i; i := i + 1) : [\text{var } s = \text{old } s + \text{old } i \wedge \text{var } i = \text{old } i + 1]^{s,i} \\ \vdash \forall s_x, s_y, s_z, i_x, i_y, i_z : \\ \quad (0 \le i_y \le \text{old } n \wedge s_y = \sum_{j=0}^{i_y-1} j) \wedge i_y < \text{old } n \wedge (s_z = s_y + i_y \wedge i_z = i_y + 1) \Rightarrow \\ \qquad 0 \le i_z \le \text{old } n \wedge s_z = \sum_{j=0}^{i_z-1} j \end{array}}{w : [\neg(\text{var } i < \text{var } n) \wedge (0 \le \text{old } i \le \text{old } n \wedge \text{old } s = \sum_{j=0}^{\text{old } i-1} j \Rightarrow I)]^{s,i}}$$

The loop relation is derived from the invariant (not the loop body); we have to prove the preservation of the loop invariant.

Example

$c =$
  if $n < 0$
    $s := -1$
  else
    $s := 0$
    $i := 0$
    while $i < n$ **do** {I,t}
      $s := s + i$
      $i := i + 1$

$I \Leftrightarrow 0 \le \text{var } i \le \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{var } i-1} j$
$t = \text{old } n - \text{old } i$

$c : [\text{if old } n < 0$
  $\text{then var } i = \text{old } i \wedge \text{var } s = -1$
  $\text{else var } i = \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{old } n-1} j)]^{s,i}$

Let us calculate this "semantic essence" of the program.

Example

$c = \textbf{if } n < 0 \textbf{ then } s := -1 \textbf{ else } b$
$b = (s := 0; i := 0; w)$
$w = \textbf{while } i < n \textbf{ do } \{I, t\}\ (s := s + i; i = i + 1)$

$s := 0 : [\text{var } s = 0]^{s}$
$s := 0 : [\text{var } s = 0 \wedge \text{var } i = \text{old } i]^{s,i}$

$i := 0 : [\text{var } i = 0]^{i}$
$i := 0 : [\text{var } i = 0 \wedge \text{var } s = \text{old } s]^{s,i}$

$s := 0; i := 0 : [\exists s_0, i_0 : s_0 = 0 \wedge i_0 = \text{old } i \wedge\ \text{var } i = 0 \wedge \text{var } s = s_0]^{s,i}$
$s := 0; i := 0 : [\text{var } s = 0 \wedge \text{var } i = 0]^{s,i}$

$w : [\neg(\text{var } i < \text{var } n) \wedge (0 \le \text{old } i \le \text{old } n \wedge \text{old } s = \sum_{j=0}^{\text{old } i-1} j \Rightarrow I)]^{s,i}$
$w : [\text{var } i \ge \text{old } n \wedge (0 \le \text{old } i \le \text{old } n \wedge \text{old } s = \sum_{j=0}^{\text{old } i-1} j \Rightarrow I)]^{s,i}$

Example

$c = \textbf{if } n < 0 \textbf{ then } s := -1 \textbf{ else } b$
$b = (s := 0; i := 0; w)$
$w = \textbf{while } i < n \textbf{ do } \{I, t\}\ (s := s + i; i = i + 1)$

$s := 0; i := 0 : [\text{var } s = 0 \wedge \text{var } i = 0]^{s,i}$
$w : [\text{var } i \ge \text{old } n \wedge (0 \le \text{old } i \le \text{old } n \wedge \text{old } s = \sum_{j=0}^{\text{old } i-1} j \Rightarrow I)]^{s,i}$

$b : [\exists s_0, i_0 : s_0 = 0 \wedge i_0 = 0 \wedge$
$\quad \text{var } i \ge \text{old } n \wedge (0 \le i_0 \le \text{old } n \wedge s_0 = \sum_{j=0}^{i_0-1} j \Rightarrow I)]^{s,i}$
$b : [\exists s_0, i_0 : s_0 = 0 \wedge i_0 = 0\ \wedge$
$\quad \text{var } i \ge \text{old } n \wedge (0 \le \text{old } n \Rightarrow I)]^{s,i}$
$b : [\text{var } i \ge \text{old } n\ \wedge$
$\quad (0 \le \text{old } n \Rightarrow 0 \le \text{var } i \le \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{var } i-1} j)]^{s,i}$
$b : [\text{var } i \ge \text{old } n\ \wedge$
$\quad (0 \le \text{old } n \Rightarrow \text{var } i = \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{old } n-1} j)]^{s,i}$

Example

$c = \textbf{if } n < 0 \textbf{ then } s := -1 \textbf{ else } b$
$b = (s := 0; i := 0; w)$
$w = \textbf{while } i < n \textbf{ do } \{I, t\}\ (s := s + i; i = i + 1)$

$s := -1 : [\text{var } s = -1]^{s}$
$s := -1 : [\text{var } i = \text{old } i \wedge \text{var } s = -1]^{s,i}$

$b : [\text{var } i \ge \text{old } n\ \wedge$
$\quad (0 \le \text{old } n \Rightarrow \text{var } i = \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{old } n-1} j)]^{s,i}$

$c : [\text{if old } n < 0$
$\quad \text{then var } i = \text{old } i \wedge \text{var } s = -1$
$\quad \text{else var } i \ge \text{old } n\ \wedge$
$\qquad (0 \le \text{old } n \Rightarrow \text{var } i = \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{old } n-1} j)]^{s,i}$
$c : [\text{if old } n < 0$
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$\quad \text{else var } i = \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{old } n-1} j)]^{s,i}$

Partial Correctness

- Specification (xs, P, Q)
  - Set of program variables xs (which may be modified).
  - Precondition P (a formula with "old xs" but no "var xs").
  - Postcondition Q (a formula with both "old xs" and "var xs").
- Partial correctness of implementation c
  1. Derive $c : [F]^{xs}$.
  2. Prove $F \Rightarrow (P \Rightarrow Q)$
     - Or: $P \Rightarrow (F \Rightarrow Q)$
     - Or: $(P \wedge F) \Rightarrow Q$

Verification of partial correctness leads to the proof of an implication.

Relationship to Other Calculi

Let all state conditions refer via "*old* xs" to program variables xs.

- Hoare Calculus
  - For proving $\{P\}c\{Q\}$,
  - it suffices to derive $c : [F]^{xs}$
  - and prove $P \wedge F \Rightarrow Q[\text{var } xs/\text{old } xs]$.
- Predicate Transformers
  - Assume we can derive $c : [F]^{xs}$.
  - If c does not contain loops, then

$$\text{wp}(c, Q) = \forall xs : F[xs/\text{var } xs] \Rightarrow Q[xs/\text{old } xs]$$
$$\text{sp}(c, P) = \exists xs : P[xs/\text{old } xs] \wedge F[xs/\text{old } xs, \text{old } xs/\text{var } xs]$$

  - If c contains loops, the result is still a valid pre/post-condition but not necessarily the weakest/strongest one.

A generalization of the previously presented calculi.

Termination

- We introduce a judgment $c \downarrow T$.

  State condition T (a formula with "old xs" but no "var xs").

  - Starting with a pre-state that satisfies condition T the execution of command c terminates.
- Total correctness of implementation c.

  Specification (xs, P, Q).

  - Derive $c \downarrow T$.
  - Prove $P \Rightarrow T$.

Also verification of termination leads to the proof of an implication.

Termination Condition Rules

$$\textbf{skip} \downarrow \text{true} \qquad \textbf{abort} \downarrow \text{true} \qquad x := e \downarrow \text{true}$$

$$\frac{c_1 \downarrow T_1 \quad c_2 \downarrow T_2}{c_1; c_2 \downarrow T_1 \wedge \text{wp}(c_1, T_2)}$$

$$\frac{c \downarrow T}{\textbf{if } e \textbf{ then } c \downarrow e' \Rightarrow T}$$

$$\frac{c_1 \downarrow T_1 \quad c_2 \downarrow T_2}{\textbf{if } e \textbf{ then } c_1 \textbf{ else } c_2 \downarrow \text{if } e' \text{ then } T_1 \text{ else } T_2}$$

$$\frac{\begin{array}{c} c : [F]^{xs} \qquad c \downarrow T \\ \vdash \forall xs, ys, zs : \\ I[xs/\text{old } xs, ys/\text{var } xs] \wedge e[ys/xs] \wedge F[ys/\text{old } xs, zs/\text{var } xs] \wedge t[ys/\text{old } xs] \geq 0 \Rightarrow \\ T[ys/\text{old } xs] \wedge 0 \leq t[zs/\text{old } xs] < t[ys/\text{old } xs] \end{array}}{\textbf{while } e \textbf{ do } \{I, t\}\ c \downarrow t \geq 0}$$

In every iteration of a loop, the loop body must terminate and the termination term must decrease (but not become negative).

Example

$c =$
  if $n < 0$
    $s := -1$
  else
    $s := 0$
    $i := 0$
    while $i < n$ **do** {I,t}
      $s := s + i$
      $i := i + 1$

$I \Leftrightarrow 0 \leq \text{var } i \leq \text{old } n \wedge \text{var } s = \sum_{j=0}^{\text{var } i-1} j]$
$t = \text{old } n \ - \ \text{old } i$

$c \downarrow \text{if old } n < 0 \text{ then true else } \ldots$
$c \downarrow \text{if old } n < 0 \text{ then true else old } n \geq 0$
$c \downarrow \text{true}$

We still have to prove the constraint on the loop iteration.

Example

$s := s + i; i := i + 1 \downarrow \text{true}$

$\forall s_x, s_y, s_z, i_x, i_y, i_z :$
$(0 \leq i_y \leq \text{old } n \wedge s_y = \sum_{j=0}^{i_y-1} j) \wedge$
$i_y < \text{old } n \wedge$
$(s_z = s_y + i_y \wedge i_z = i_y + 1) \wedge$
$\text{old } n - i_y \geq 0 \Rightarrow$
$\text{true} \wedge$
$0 \leq \text{old } n - i_z < \text{old } n - i_y$

Also this constraint is simple to prove.

Abortion

Also abortion can be ruled out by proving side conditions in the usual way.

Wolfgang Schreiner. *Computer-Assisted Program Reasoning Based on a Relational Semantics of Programs.* Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria, 2011.

See the report for the full calculus.

The RISC ProgramExplorer

- An integrated environment for program reasoning.
  - Research Institute for Symbolic Computation (RISC), 2008–.
    http://www.risc.jku.at/research/formal/software/ProgramExplorer
    - Integrates the RISC ProofNavigator for computer-assisted proving.
    - Written in Java, runs under Linux (only), freely available (GPL).
- Programs written in "MiniJava".
  - Subset of Java with full support of control flow interruptions.
  - Value (not pointer) semantics for arrays and objects.
- Theories and specifications written in a formula language.
  - Derived from the language of the RISC ProofNavigator.
- Semantic analysis and verification.
  - Program methods are translated into their "semantic essence".
    - Open for human inspection.
  - From the semantics, the verification tasks are generated.
    - Solved by automatic decision procedure or interactive proof.

Tight integration of executable programs, declarative specifications, mathematical semantics, and verification tasks.

Using the Software

See "The RISC ProgramExplorer: Tutorial and Manual".

- Develop a theory.
  - File "*Theory*.theory" with a `theory` *`Theory`* of mathematical types, constants, functions, predicates, axioms, and theorems.
  - Can be also added to a program file.
- Develop a program.
  - File "*Class*.java" with a `class` *`Class`* that contains class (static) and object (non-static) variables, methods and constructors.
  - Class may be annotated by a theory (and an object invariant).
  - Methods may be annotated by method specifications.
  - Loops may be annotated by invariants and termination terms.
- Analyze method semantics.
  - Transition relations, termination conditions, ... of the method body and its individual commands.
- Perform verification tasks.
  - Frame, postcondition, termination, preconditions, loop-related tasks, type-checking conditions.

Starting the Software

- Starting the software:

  `module load ProgramExplorer` (only at RISC)

  `ProgramExplorer &`
- Command line options:

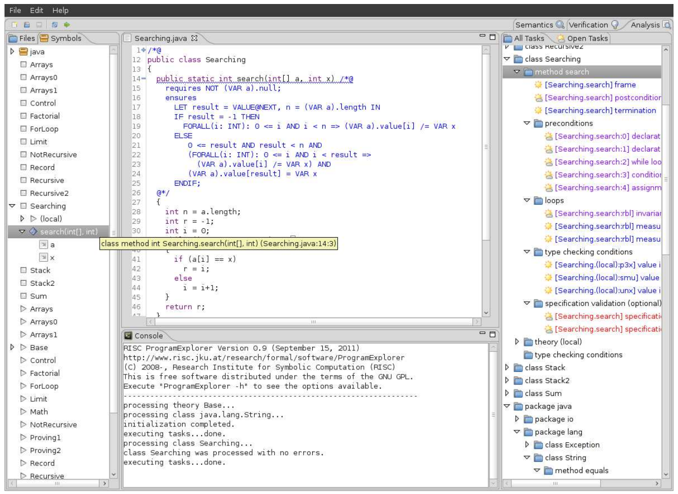
```
Usage: ProgramExplorer [OPTION]...
OPTION: one of the following options:
    -h, --help: print this message.
    -cp, --classpath [PATH]:
      directories representing top package.

Environment Variables:
    PE_CLASSPATH:
        the directories (separated by ":") representing the
        top package (default the current working directory)
    ...
```

- Task repository created/read in current working directory:

  Subdirectory `.PETASKS.`*`timestamp`* (ProgramExplorer tasks)

  Subdirectory `.ProofNavigator` (ProofNavigator legacy)

The Graphical User Interface



A Program

```
/*@..
class Sum
{
  static int sum(int n) /*@..
  {
    int s;
    if (n < 0)
      s = -1;
    else
    {
      s = 0;
      int i = 1;
      while (i <= n) /*@..
      {
        s = s+i;
        i = i+1;
      }
    }
    return s;
  }
}
```

Markers `/*@..` indicate hidden mathematical annotations.

A Theory

```
/*@
  theory {
    sum: (INT, INT) -> INT;
    sumaxiom: AXIOM
      FORALL(m: INT, n: INT):
        IF n<m THEN
          sum(m, n) = 0
        ELSE
          sum(m, n) = n+sum(m, n-1)
        ENDIF;
  }
@*/
class Sum
...
```

The introduction of a function $sum(m, n) = \sum_{j=m}^{n} j$.

A Method Specification

```
static int sum(int n) /*@
  requires VAR n < Base.MAX_INT;
  ensures
    LET result=VALUE@NEXT IN
    IF VAR n < 0
      THEN result = -1
      ELSE result = sum(1, VAR n)
    ENDIF;
@*/
...
```

For non-negative n, a call of program method `sum`(n) returns $sum(1, n)$ (and does not modify any global variable).

A Loop Annotation

```
while (i <= n) /*@
  invariant VAR n < Base.MAX_INT
        AND 1 <= VAR i AND VAR i <= VAR n+1
        AND VAR s=sum(1, VAR i-1);
  decreases VAR n - VAR i + 1;
@*/
{
  s = s+i;
  i = i+1;
}
}
```

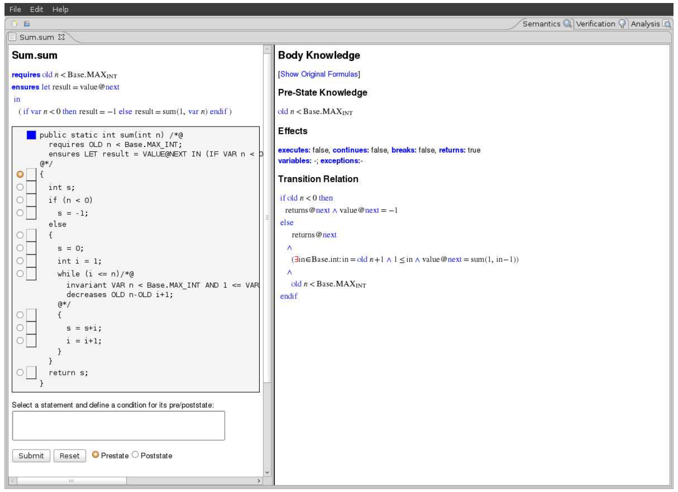
The loop invariant and termination term (measure).

The Specification Language

Derived from the language of the RISC ProofNavigator.

- State conditions/relations, state terms.
  - State condition: method precondition (`requires`).
  - State relation: method postcondition (`ensures`), loop invariant (`invariant`).
  - State term: termination term (`decreases`).
- References to program variables.
  - `OLD` x: the value of program variable x in the pre-state.
  - `VAR` x: the value of program variable x in the post-state.
  - In state conditions/terms, both refer to the value in the current state.
  - If program variable is of the program type T, then then `OLD`/`VAR` x is of the mathematical type T'.
    - `int` $\rightarrow$ `Base.int`=[`Base.MIN_INT`,`Base.MAX_INT`].
- Function results
  - `VALUE@NEXT`: the return value of a program function.
  - The value of the function call's post-state `NEXT`.

The Semantics View



The Method Body

Body Knowledge

[Show Original Formulas]

Pre-State Knowledge

old $n < \mathrm{Base.MAX_{INT}}$

Effects

executes: false, **continues:** false, **breaks:** false, **returns:** true
variables: -; **exceptions:**-

Transition Relation

if old $n < 0$ then
  returns@next $\wedge$ value@next $= -1$
else
  returns@next
  $\wedge$
  $(\exists \mathrm{in} \in \mathrm{Base.int}: \mathrm{in} = \text{old } n+1 \wedge 1 \leq \mathrm{in} \wedge \mathrm{value@next} = \mathrm{sum}(1, \mathrm{in}-1))$
  $\wedge$
  old $n < \mathrm{Base.MAX_{INT}}$
endif

Select method symbol "sum" and menu entry "Show Semantics".

A Body Command

Statement Knowledge

[Show Original Formulas]

Pre-State Knowledge

old $n < \mathrm{Base.MAX_{INT}} \wedge$ old $n \geq 0 \wedge$ old $s = 0 \wedge$ old $i = 1$

Precondition

old $n < \mathrm{Base.MAX_{INT}} \wedge 1 \leq$ old $i \wedge$ old $i \leq$ old $n+1 \wedge$ old $s = \mathrm{sum}(1,$ old $i-1)$

Effects

executes: true, **continues:** false, **breaks:** false, **returns:** false
variables: s, i; **exceptions:**-

Transition Relation

var $i =$ old $n+1 \wedge$ old $n < \mathrm{Base.MAX_{INT}} \wedge 1 \leq$ var $i \wedge$ var $s = \mathrm{sum}(1,$ var $i-1)$

Termination Condition

executes@now $\Rightarrow$ old $n -$ old $i \geq -1$

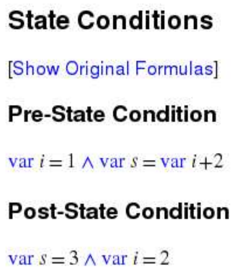
Move the mouse pointer over the box to the left of the loop.

The Semantics Elements

- Pre-State Knowledge
  What is known about the pre-state of the command.
- Precondition
  What has to be true for the pre-state of the command such that the command may be executed.
- Effects
  Which kind of effects may the command have.
  - variables: which variables may be changed.
  - exceptions: which exceptions may be thrown.
  - executes, continues, breaks, returns: may the execution terminate normally, may it be terminated by a `continue`, `break`, `return`.
- Transition Relation
  The prestate/poststate relationship of the command.
- Termination
  What has to be true for the pre-state of the command such that the command terminates.

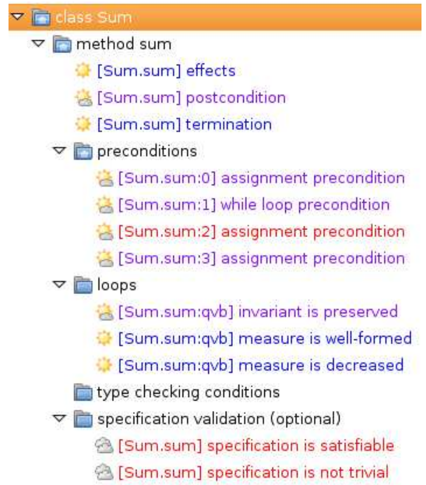
Formulas are shown after simplification (see "Show Original Formulas").

Constraining a State



Select the loop body, enter in the box the condition `VAR s=2 AND VAR i=1`, press "Submit", and move the mouse to `i=i+1`.

The Verification Tasks



The Verification Tasks

- Effects: does the method only change those global variables indicated in the method's `assignable` clause?
- Postcondition: do the method's precondition and the body's state relation imply the method's postcondition?
- Termination: does the method's precondition imply the body's termination condition?
- Precondition: does a statement's prestate knowledge imply the statement's precondition?
- Loops: is the loop invariant preserved, the measure well-formed (does not become negative) and decreased?
- Type checking conditions: are all formulas well-typed?
- Specification validation: does for every input that satisfies a precondition exist a result that does (not) satisfy the postcondition?

Partially solved by automatic decision procedure, partially by an interactive computer-supported proof.

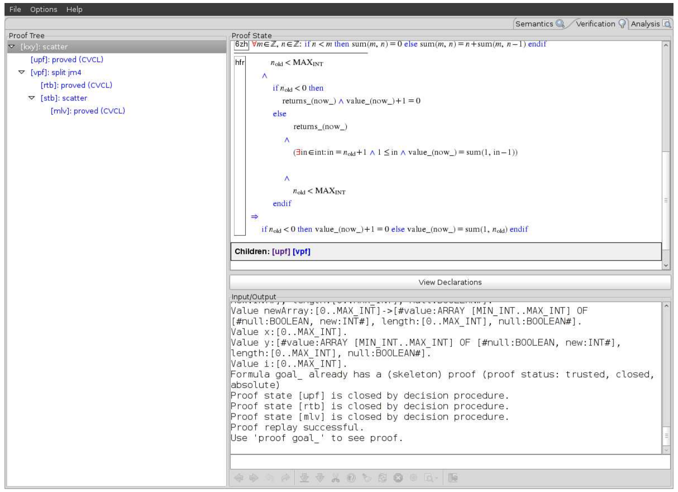
The Task States

The task status is indicated by color (icon).

- Blue (sun): the task was solved in the current execution of the RISC ProgramExplorer (automatically or by an interactive proof).
- Violet (partially clouded): the task was solved in a previous execution by an interactive proof.
  - Nothing has changed, so we need not perform the proof again.
  - However, we may replay the proof to investigate it.
- Red (partially clouded): there exists a proof but it is either not complete or cannot be trusted any more (something has changed).
- Red (fully clouded): there does not yet exist a proof.

Select "Execute Task" to start/replay a proof, "Show Proof" to display a proof, "Reset Task" to delete a proof.

A Postcondition Proof



Linear Search

```
/*@..
public class Searching
{
  public static int search(int[] a, int x) /*@..
  {
    int n = a.length;
    int r = -1;
    int i = 0;
    while (i < n && r == -1) /*@..
    {
      if (a[i] == x)
        r = i;
      else
        i = i+1;
    }
    return r;
  }
}
```


The Representation of Arrays

The program type `int[]` is mapped to the mathematical type `Base.IntArray`.

```
theory Base
{
  ...
  IntArray: TYPE =
    [#value: ARRAY int OF int, length: nat, null: BOOLEAN#];
  ...
}
```

- `(VAR a).length`: the number of elements in array *a*.
- `(VAR a).value[i]`: the element with index *i* in array *a*.
- `(VAR a).null`: *a* is the `null` pointer.

Program type *`Class`* is mapped to mathematical type *`Class.Class`*; *`Class`*`[]` is mapped to *`Class`*`.Array`.

Theory

```
/*@
theory uses Base {
  int: TYPE = Base.int;
  intArray: TYPE = Base.IntArray;
  smallestPosition: FORMULA
    FORALL(a: intArray, n: NAT, x: int):
      (EXISTS(i:int): 0 <= i AND i < n AND a.value[i] = x) =>
      (EXISTS(i:int): 0 <= i AND i < n AND a.value[i] = x AND
         (FORALL(j:int): 0 <= j AND j < n AND a.value[j] = x =>
           j >= i));
}
@*/
public class Searching
...
```

Method Specification

```
public static int search(int[] a, int x) /*@
  requires (VAR a).null = FALSE;
  ensures
    LET result = VALUE@NEXT, n = (VAR a).length IN
    IF result = -1 THEN
      FORALL(i: INT): 0 <= i AND i < n =>
        (VAR a).value[i] /= VAR x
    ELSE
       0 <= result AND result < n AND
       (FORALL(i: INT): 0 <= i AND i < result =>
         (VAR a).value[i] /= VAR x) AND
       (VAR a).value[result] = VAR x
    ENDIF;
@*/
...
```


Loop Annotation

```
while (i < n && r == -1) /*@
  invariant (VAR a).null = FALSE AND VAR n = (VAR a).length
        AND 0 <= VAR i AND VAR i <= VAR n
        AND (FORALL(i: INT): 0 <= i AND i < VAR i =>
              (VAR a).value[i] /= VAR x)
        AND (VAR r = -1 OR (VAR r = VAR i AND VAR i < VAR n AND
               (VAR a).value[VAR r] = VAR x));
  decreases IF VAR r = -1 THEN VAR n - VAR i ELSE 0 ENDIF;
@*/
{
  if (a[i] == x)
    r = i;
  else
    i = i+1;
}
```


Method Semantics

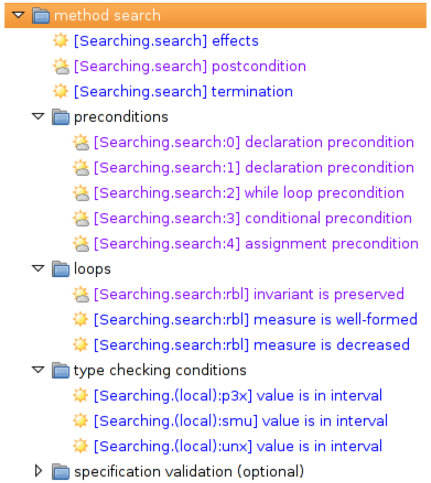
Transition Relation

$$
\begin{aligned}
&(\exists \mathrm{in} \in \mathrm{Base.int},\ n \in \mathrm{Base.int}:\\
&\quad n = \mathrm{old}\ a.\mathrm{length} \wedge (\mathrm{in} \geq n \vee \mathrm{value@next} \neq -1) \wedge 0 \leq \mathrm{in} \wedge \mathrm{in} \leq n\\
&\quad \wedge\\
&\quad (\forall i \in \mathbb{Z}: 0 \leq i \wedge i < \mathrm{in} \Rightarrow \mathrm{old}\ a.\mathrm{value}[i] \neq \mathrm{old}\ x)\\
&\quad \wedge\\
&\quad (\ \mathrm{value@next} = -1\\
&\quad\ \vee\\
&\quad\ \ \mathrm{value@next} = \mathrm{in} \wedge \mathrm{in} < n \wedge \mathrm{old}\ a.\mathrm{value}[\mathrm{value@next}] = \mathrm{old}\ x)) \wedge \neg\, \mathrm{old}\ a.\mathrm{null}\\
&\wedge\\
&\ \mathrm{returns@next}
\end{aligned}
$$

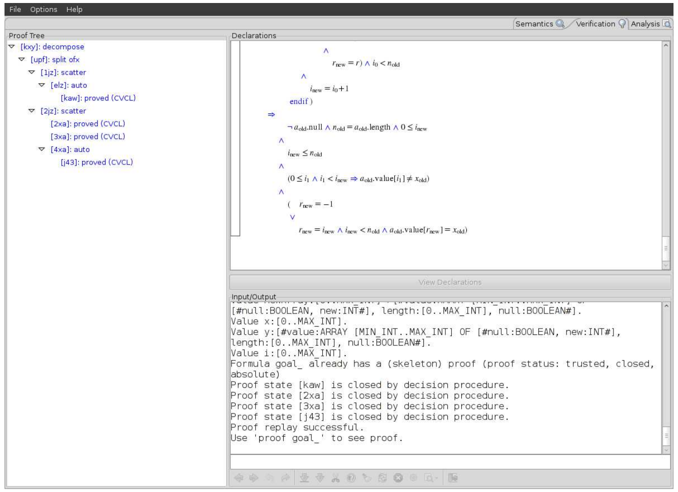
Termination Condition

$$\mathrm{executes@now} \Rightarrow \mathrm{old}\ a.\mathrm{length} \geq 0$$

Verification Tasks



Invariant Proof



Working Strategy

- Develop theory.
  - Introduce interesting theorems that may be used in verifications.
- Develop specifications.
  - Validate specifications, e.g. by showing satisfiability and non-triviality.
- Develop program with annotations.
  - Validate programs/annotations by investigating program semantics.
- Prove postcondition and termination.
  - Partial and total correctness.
  - By proofs necessity of additional theorems may be detected.
- Prove precondition tasks and loop tasks.
  - By proofs necessity of additional theorems may be detected.
- Prove mathematical theorems.
  - Validation of auxiliary knowledge used in verifications.

The integrated development of theories, specifications, programs, annotations is crucial for the design of provably correct programs.