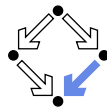


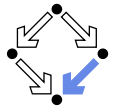
Specifying and Verifying Programs (Part 1)

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Specifying and Verifying Programs

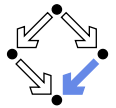


We will discuss three (closely interrelated) calculi.

- **Hoare Calculus:** $\{P\} c \{Q\}$
 - If command c is executed in a pre-state with property P and terminates, it yields a post-state with property Q .
 $\{x = a \wedge y = b\} x := x + y \{x = a + y \wedge y = b\}$
- **Predicate Transformers:** $wp(c, Q) = P$
 - If the execution of command c shall yield a post-state with property Q , it must be executed in a pre-state with property P .
 $wp(x := x + y, x = a + y \wedge y = b) = (x + y = a + y \wedge y = b)$
- **State Relations:** $c : [P \Rightarrow Q]^x, \dots$
 - The post-state generated by the execution of command c is related to the pre-state by $P \Rightarrow Q$ (where only variables x, \dots have changed).
 $x = x + y : [\text{var } x = \text{old } x + \text{old } y]^x$

1. The Hoare Calculus
2. Predicate Transformers
3. Proving Verification Conditions
4. Termination
5. Abortion
6. Procedures

The Hoare Calculus

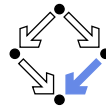


First and best-known calculus for program reasoning (C.A.R. Hoare).

- **“Hoare triple”:** $\{P\} c \{Q\}$
 - Logical propositions P and Q , program command c .
 - The Hoare triple is itself a logical proposition.
 - The Hoare calculus gives rules for constructing true Hoare triples.
- **Partial correctness** interpretation of $\{P\} c \{Q\}$:
“If c is executed in a state in which P holds, then it terminates in a state in which Q holds **unless it aborts or runs forever.**”
 - Program does not produce wrong result.
 - But program also need not produce **any** result.
 - Abortion and non-termination are not (yet) ruled out.
- **Total correctness** interpretation of $\{P\} c \{Q\}$:
“If c is executed in a state in which P holds, then it terminates in a state in which Q holds.”
 - Program produces the correct result.

We will use the partial correctness interpretation for the moment.

Weakening and Strengthening

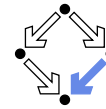


$$\frac{P \Rightarrow P' \quad \{P'\} c \{Q'\} \quad Q' \Rightarrow Q}{\{P\} c \{Q\}}$$

- **Logical derivation:** $\frac{A_1 A_2}{B}$
 - Forward: If we have shown A_1 and A_2 , then we have also shown B .
 - Backward: To show B , it suffices to show A_1 and A_2 .
- **Interpretation of above sentence:**
 - To show that, if P holds, then Q holds after executing c , it suffices to show this for a P' weaker than P and a Q' stronger than Q .

Precondition may be weakened, postcondition may be strengthened.

Special Commands

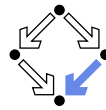


$$\{P\} \text{ skip } \{P\} \quad \{\text{true}\} \text{ abort } \{\text{false}\}$$

- The **skip** command does not change the state; if P holds before its execution, then P thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
 - Axiom implies $\{P\} \text{ abort } \{Q\}$ for arbitrary P, Q .

Useful commands for reasoning and program transformations.

Scalar Assignments

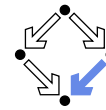


$$\{Q[e/x]\} x := e \{Q\}$$

- **Syntax**
 - Variable x , expression e .
 - $Q[e/x] \dots Q$ where every free occurrence of x is replaced by e .
- **Interpretation**
 - To make sure that Q holds for x after the assignment of e to x , it suffices to make sure that Q holds for e before the assignment.
- **Partial correctness**
 - Evaluation of e may abort.

$$\begin{array}{l} \{x + 3 < 5\} \quad x := x + 3 \quad \{x < 5\} \\ \{x < 2\} \quad x := x + 3 \quad \{x < 5\} \end{array}$$

Array Assignments



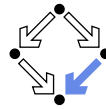
$$\{Q[a[i \mapsto e]/a]\} a[i] := e \{Q\}$$

- An array is modelled as a **function** $a : I \rightarrow V$.
 - Index set I , value set V .
 - $a[i] = e \dots$ array a contains at index i the value e .
- **Term** $a[i \mapsto e]$ (“array a updated by assigning value e to index i ”)
 - A new array that contains at index i the value e .
 - All other elements of the array are the same as in a .
- **Thus array assignment becomes a special case of scalar assignment.**
 - Think of “ $a[i] := e$ ” as “ $a := a[i \mapsto e]$ ”.

$$\{\underline{a[i \mapsto x][1]} > 0\} \quad a[i] := x \quad \{a[1] > 0\}$$

Arrays are here considered as basic values (no pointer semantics).

Array Assignments



How to reason about $a[i \mapsto e]$?

$$\frac{Q[a[i \mapsto e][j]]}{(i = j \Rightarrow Q[e]) \wedge (i \neq j \Rightarrow Q[a[j]])}$$

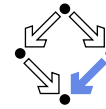
Array Axioms

$$\begin{aligned} i = j &\Rightarrow a[i \mapsto e][j] = e \\ i \neq j &\Rightarrow a[i \mapsto e][j] = a[j] \end{aligned}$$

$$\frac{\{a[i \mapsto x][1] > 0\} \quad a[i] := x \quad \{a[1] > 0\}}{\{(i = 1 \Rightarrow x > 0) \wedge (i \neq 1 \Rightarrow a[1] > 0)\} \quad a[i] := x \quad \{a[1] > 0\}}$$

Get rid of “array update terms” when applied to indices.

Command Sequences



$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

Interpretation

- To show that, if P holds before the execution of $c_1; c_2$, then Q holds afterwards, it suffices to show for some R that
 - if P holds before c_1 , that R holds afterwards, and that
 - if R holds before c_2 , then Q holds afterwards.

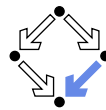
Problem: find suitable R .

- Easy in many cases (see later).

$$\frac{\{x + y - 1 > 0\} y := y - 1 \{x + y > 0\} \quad \{x + y > 0\} x := x + y \{x > 0\}}{\{x + y - 1 > 0\} y := y - 1; x := x + y \{x > 0\}}$$

The calculus itself does not indicate how to find intermediate property.

Conditionals



$$\frac{\{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{if } b \text{ then } c_1 \text{ else } c_2 \{Q\}}$$

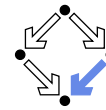
$$\frac{\{P \wedge b\} c \{Q\} \quad (P \wedge \neg b) \Rightarrow Q}{\{P\} \text{if } b \text{ then } c \{Q\}}$$

Interpretation

- To show that, if P holds before the execution of the conditional, then Q holds afterwards,
- it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

$$\frac{\{x \neq 0 \wedge x \geq 0\} y := x \{y > 0\} \quad \{x \neq 0 \wedge x \not\geq 0\} y := -x \{y > 0\}}{\{x \neq 0\} \text{if } x \geq 0 \text{ then } y := x \text{ else } y := -x \{y > 0\}}$$

Loops



$$\{\text{true}\} \text{loop} \{\text{false}\} \quad \frac{\{I \wedge b\} c \{I\}}{\{I\} \text{while } b \text{ do } c \{I \wedge \neg b\}}$$

Interpretation:

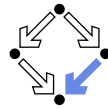
- The **loop** command does not terminate and thus trivially satisfies partial correctness.
 - Axiom implies $\{P\} \text{loop} \{Q\}$ for arbitrary P, Q .
- If it is the case that
 - I holds before the execution of the **while**-loop and
 - I also holds after every iteration of the loop body,
 then I holds also after the execution of the loop (together with the negation of the loop condition b).
 - I is a **loop invariant**.

Problem:

- Rule for **while**-loop does not have arbitrary pre/post-conditions P, Q .

In practice, we combine this rule with the strengthening/weakening-rule.

Loops (Generalized)



$$\frac{P \Rightarrow I \quad \{I \wedge b\} c \quad \{I\} (I \wedge \neg b) \Rightarrow Q}{\{P\} \text{ while } b \text{ do } c \quad \{Q\}}$$

Interpretation:

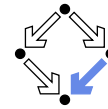
- To show that, if before the execution of a **while**-loop the property P holds, after its termination the property Q holds, it suffices to show for some property I (the **loop invariant**) that
 - I holds before the loop is executed (i.e. that P implies I),
 - if I holds when the loop body is entered (i.e. if also b holds), that after the execution of the loop body I still holds,
 - when the loop terminates (i.e. if b does not hold), I implies Q .

Problem: find appropriate loop invariant I .

- Strongest relationship between all variables modified in loop body.

The calculus itself does not indicate how to find suitable loop invariant.

Example



$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n + 1$$

$$\begin{aligned} (n \geq 0 \wedge i = 1 \wedge s = 0) &\Rightarrow I \\ \{I \wedge i \leq n\} s := s + i; i := i + 1 &\{I\} \\ (I \wedge i \not\leq n) &\Rightarrow s = \sum_{j=1}^n j \end{aligned}$$

$$\frac{\{n \geq 0 \wedge i = 1 \wedge s = 0\} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \quad \{s = \sum_{j=1}^n j\}}$$

The invariant captures the “essence” of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.

1. The Hoare Calculus

2. Predicate Transformers

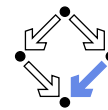
3. Proving Verification Conditions

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Backward Reasoning



Implication of rule for command sequences and rule for assignments:

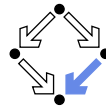
$$\frac{\{P\} c \quad \{Q[e/x]\}}{\{P\} c; x := e \quad \{Q\}}$$

Interpretation

- If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
- By multiple application, assignment sequences can be removed from the back to the front.

$\{P\}$	$\{P\}$	$\{P\}$	$\{P\}$	$P \Rightarrow x = 4$
$x := x+1;$	$x := x+1;$	$x := x+1;$	$\{x + 1 = 5\}$	
$y := 2*x;$	$y := 2*x;$	$\{x + 2x = 15\}$	$(\Leftrightarrow x = 4)$	
$z := x+y$	$\{x + y = 15\}$	$(\Leftrightarrow 3x = 15)$		
$\{z = 15\}$		$(\Leftrightarrow x = 5)$		

Weakest Preconditions

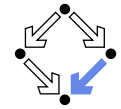


A calculus for “backward reasoning” (E.W. Dijkstra).

- **Predicate transformer wp**
 - Function “wp” that takes a command c and a postcondition Q and returns a precondition.
 - Read $\text{wp}(c, Q)$ as “the weakest precondition of c w.r.t. Q ”.
- $\text{wp}(c, Q)$ is a **precondition** for c that ensures Q as a postcondition.
 - Must satisfy $\{ \text{wp}(c, Q) \} c \{ Q \}$.
- $\text{wp}(c, Q)$ is the **weakest** such precondition.
 - Take any P such that $\{ P \} c \{ Q \}$.
 - Then $P \Rightarrow \text{wp}(c, Q)$.
- Consequence: $\{ P \} c \{ Q \}$ iff $(P \Rightarrow \text{wp}(c, Q))$
 - We want to prove $\{ P \} c \{ Q \}$.
 - We may prove $P \Rightarrow \text{wp}(c, Q)$ instead.

Verification is reduced to the calculation of weakest preconditions.

Weakest Preconditions

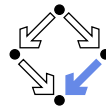


The weakest precondition of each program construct.

$\text{wp}(\text{skip}, Q) = Q$
 $\text{wp}(\text{abort}, Q) = \text{true}$
 $\text{wp}(x := e, Q) = Q[e/x]$
 $\text{wp}(c_1; c_2, Q) = \text{wp}(c_1, \text{wp}(c_2, Q))$
 $\text{wp}(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) = (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q))$
 $\text{wp}(\text{if } b \text{ then } c, Q) \Leftrightarrow (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)$
 $\text{wp}(\text{while } b \text{ do } c, Q) = \dots$

Loops represent a special problem (see later).

Forward Reasoning



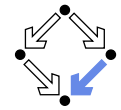
Sometimes, we want to derive a postcondition from a given precondition.

$$\{ P \} x := e \{ \exists x_0 : P[x_0/x] \wedge x = e[x_0/x] \}$$

- **Forward Reasoning**
 - What is the maximum we know about the post-state of an assignment $x := e$, if the pre-state satisfies P ?
 - We know that P holds for some value x_0 (the value of x in the pre-state) and that x equals $e[x_0/x]$.

$$\begin{aligned} & \{ x \geq 0 \wedge y = a \} \\ & \quad x := x + 1 \\ & \{ \exists x_0 : x_0 \geq 0 \wedge y = a \wedge x = x_0 + 1 \} \\ & (\Leftrightarrow (\exists x_0 : x_0 \geq 0 \wedge x = x_0 + 1) \wedge y = a) \\ & (\Leftrightarrow x > 0 \wedge y = a) \end{aligned}$$

Strongest Postcondition

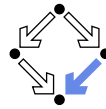


A calculus for forward reasoning.

- **Predicate transformer sp**
 - Function “sp” that takes a precondition P and a command c and returns a postcondition.
 - Read $\text{sp}(c, P)$ as “the strongest postcondition of c w.r.t. P ”.
- $\text{sp}(c, P)$ is a **postcondition** for c that is ensured by precondition P .
 - Must satisfy $\{ P \} c \{ \text{sp}(c, P) \}$.
- $\text{sp}(c, P)$ is the **strongest** such postcondition.
 - Take any P, Q such that $\{ P \} c \{ Q \}$.
 - Then $\text{sp}(c, P) \Rightarrow Q$.
- Consequence: $\{ P \} c \{ Q \}$ iff $(\text{sp}(c, P) \Rightarrow Q)$.
 - We want to prove $\{ P \} c \{ Q \}$.
 - We may prove $\text{sp}(c, P) \Rightarrow Q$ instead.

Verification is reduced to the calculation of strongest postconditions.

Strongest Postconditions

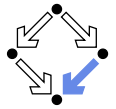


The strongest postcondition of each program construct.

$$\begin{aligned} \text{sp}(\text{skip}, P) &= P \\ \text{sp}(\text{abort}, P) &= \text{false} \\ \text{sp}(x := e, P) &= \exists x_0 : P[x_0/x] \wedge x = e[x_0/x] \\ \text{sp}(c_1; c_2, P) &= \text{sp}(c_2, \text{sp}(c_1, P)) \\ \text{sp}(\text{if } b \text{ then } c_1 \text{ else } c_2, P) &\Leftrightarrow \text{sp}(c_1, P \wedge b) \vee \text{sp}(c_2, P \wedge \neg b) \\ \text{sp}(\text{if } b \text{ then } c, P) &= \text{sp}(c, P \wedge b) \vee (P \wedge \neg b) \\ \text{sp}(\text{while } b \text{ do } c, P) &= \dots \end{aligned}$$

Forward reasoning as a (less-known) alternative to backward-reasoning.

Hoare Calculus and Predicate Transformers



In practice, often a combination of the calculi is applied.

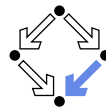
$$\{P\} c_1; \text{while } b \text{ do } c; c_2 \{Q\}$$

- Assume c_1 and c_2 do not contain loop commands.
- It suffices to prove

$$\{\text{sp}(P, c_1)\} \text{while } b \text{ do } c \{\text{wp}(c_2, Q)\}$$

Predicate transformers are applied to reduce the verification of a program to the Hoare-style verification of loops.

Weakest Liberal Preconditions for Loops



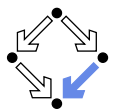
Why not apply predicate transformers to loops?

$$\begin{aligned} \text{wp}(\text{loop}, Q) &= \text{true} \\ \text{wp}(\text{while } b \text{ do } c, Q) &= L_0(Q) \wedge L_1(Q) \wedge L_2(Q) \wedge \dots \end{aligned}$$

$$\begin{aligned} L_0(Q) &= \text{true} \\ L_{i+1}(Q) &= (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q))) \end{aligned}$$

- Interpretation**
 - Weakest precondition that ensures that loops stops in a state satisfying Q , unless it aborts or runs forever.
- Infinite sequence of predicates $L_i(Q)$:**
 - Weakest precondition that ensures that **after less than i iterations** the state satisfies Q , unless the loop aborts or does not yet terminate.
- Alternative view: $L_i(Q) = \text{wp}(\text{if}_i, Q)$**
 - $\text{if}_0 = \text{loop}$
 - $\text{if}_{i+1} = \text{if } b \text{ then } (c; \text{if}_i)$

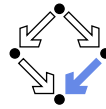
Example



$$\text{wp}(\text{while } i < n \text{ do } i := i + 1, Q)$$

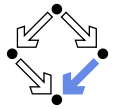
$$\begin{aligned} L_0(Q) &= \text{true} \\ L_1(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, \text{true})) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{true}) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \\ L_2(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, i \not< n \Rightarrow Q)) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\ &\quad (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i])) \\ L_3(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, \\ &\quad (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i]))) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\ &\quad (i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge \\ &\quad (i + 1 < n \Rightarrow (i + 2 \not< n \Rightarrow Q[i + 2/i]))) \end{aligned}$$

Weakest Liberal Preconditions for Loops



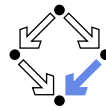
- Sequence $L_i(Q)$ is monotonically increasing in strength:
 - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$.
- The weakest precondition is the “lowest upper bound”:
 - $\forall i \in \mathbb{N} : \text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow L_i(Q)$.
 - $\forall P : (\forall i \in \mathbb{N} : P \Rightarrow L_i(Q)) \Rightarrow (P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q))$.
- We can only compute weaker **approximation** $L_i(Q)$.
 - $\text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow L_i(Q)$.
- We want to prove $\{P\} \text{ while } b \text{ do } c \{Q\}$.
 - This is equivalent to proving $P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q)$.
 - Thus $P \Rightarrow L_i(Q)$ must hold as well.
- If we can prove $\neg(P \Rightarrow L_i(Q))$, ...
 - $\{P\} \text{ while } b \text{ do } c \{Q\}$ does **not** hold.
 - If we fail, we may try the easier proof $\neg(P \Rightarrow L_{i+1}(Q))$.

Falsification is possible by use of approximation L_i , but verification is not.



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A Constructive Definition of Arrays



```
% constructive array definition
newcontext "arrays2";

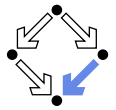
% the types
INDEX: TYPE = NAT;
ELEM:  TYPE;
ARR:   TYPE =
  [INDEX, ARRAY INDEX OF ELEM];

% error constants
any:   ARRAY INDEX OF ELEM;
anyelem: ELEM;
anyarray: ARR;

% a selector operation
content:
  ARR -> (ARRAY INDEX OF ELEM) =
    LAMBDA(a:ARR): a.1;

% the array operations
length: ARR -> INDEX =
  LAMBDA(a:ARR): a.0;
new: INDEX -> ARR =
  LAMBDA(n:INDEX): (n, any);
put: (ARR, INDEX, ELEM) -> ARR =
  LAMBDA(a:ARR, i:INDEX, e:ELEM):
    IF i < length(a)
      THEN (length(a),
            content(a) WITH [i]:=e)
      ELSE anyarray
ENDIF;
get: (ARR, INDEX) -> ELEM =
  LAMBDA(a:ARR, i:INDEX):
    IF i < length(a)
      THEN content(a)[i]
      ELSE anyelem
ENDIF;
```

Proof of Fundamental Array Properties



```
% the classical array axioms as formulas to be proved
length1: FORMULA
  FORALL(n:INDEX): length(new(n)) = n;

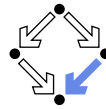
length2: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => length(put(a, i, e)) = length(a);

get1: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => get(put(a, i, e), i) = e;

get2: FORMULA
  FORALL(a:ARR, i, j:INDEX, e:ELEM):
    i < length(a) AND j < length(a) AND
    i /= j =>
      get(put(a, i, e), j) = get(a, j);
```

[adu]: expand length, get, put, content
 [c3b]: scatter
 [qid]: proved (CVCL)

Proof of a Higher-Level Array Property

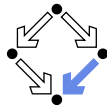


```
% extensionality on low-level arrays
extensionality: AXIOM
  FORALL(a, b:ARRAY INDEX OF ELEM):
    a=b <=> (FORALL(i:INDEX):a[i]=b[i]);

% unassigned parts hold identical values
unassigned: AXIOM
  FORALL(a:ARR, i:INT):
    (i >= length(a)) => content(a)[i]
    [adt]: expand length, get, content
    [cw2]: scatter
    [qey]: proved (CVCL)
    [rey]: assume b_0.1 = a_0.1
    [zpt]: proved (CVCL)
    [1pt]: instantiate a_0.1, b_0.1 in 1fm
    [y51]: scatter
    [ku2]: auto
    [iub]: proved (CVCL)

% extensionality on arrays to be prc
equality: FORMULA
  FORALL(a:ARR, b:ARR): a = b <=>
    length(a) = length(b) AND
    (FORALL(i:INDEX): i < length(a) => get(a,i) = get(b,i));
```

A Program Verification

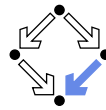


Verification of the following Hoare triple:

```
{olda = a ∧ oldx = x ∧ n = |a| ∧ i = 0 ∧ r = -1}
while i < n ∧ r = -1 do
  if a[i] = x
  then r := i
  else i := i + 1
{a = olda ∧ x = oldx ∧
((r = -1 ∧ ∀i: 0 ≤ i < |a| ⇒ a[i] ≠ x) ∨
(0 ≤ r < |a| ∧ a[r] = x ∧ ∀i: 0 ≤ i < r ⇒ a[i] ≠ x))}
```

Find the smallest index r of an occurrence of value x in array a ($r = -1$, if x does not occur in a).

The Verification Conditions



$A \Leftrightarrow \text{Input} \Rightarrow \text{Invariant}$
 $B_1 \Leftrightarrow \text{Invariant} \wedge i < n \wedge r = -1 \wedge a[i] = x \Rightarrow \text{Invariant}[i/r]$
 $B_2 \Leftrightarrow \text{Invariant} \wedge i < n \wedge r = -1 \wedge a[i] \neq x \Rightarrow \text{Invariant}[i + 1/i]$
 $C \Leftrightarrow \text{Invariant} \wedge \neg(i < n \wedge r = -1) \Rightarrow \text{Output}$

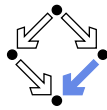
$\text{Input} \Leftrightarrow \text{olda} = a \wedge \text{oldx} = x \wedge n = \text{length}(a) \wedge i = 0 \wedge r = -1$

$\text{Output} \Leftrightarrow a = \text{olda} \wedge x = \text{oldx} \wedge$
 $((r = -1 \wedge \forall i: 0 \leq i < \text{length}(a) \Rightarrow a[i] \neq x) \vee$
 $(0 \leq r < \text{length}(a) \wedge a[r] = x \wedge \forall i: 0 \leq i < r \Rightarrow a[i] \neq x))$

$\text{Invariant} \Leftrightarrow \text{olda} = a \wedge \text{oldx} = x \wedge n = \text{length}(a) \wedge$
 $0 \leq i \leq n \wedge \forall j: 0 \leq j < i \Rightarrow a[j] \neq x \wedge$
 $(r = -1 \vee (r = i \wedge i < n \wedge a[r] = x))$

The verification conditions A, B_1, B_2, C have to be proved.

The Verification Conditions



```
newcontext
  "linsearch";
  Input: BOOLEAN = olda = a AND oldx = x AND
        n = length(a) AND i = 0 AND r = -1;

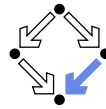
% declaration
% of arrays
...
  Output: BOOLEAN = a = olda AND
        ((r = -1 AND
          (FORALL(j:NAT): j < length(a) =>
            get(a,j) /= x)) OR
          (0 <= r AND r < length(a) AND get(a,r) = x AND
            (FORALL(j:NAT):
              j < r => get(a,j) /= x)));

a: ARR;
olda: ARR;
x: ELEM;
oldx: ELEM;
i: NAT;
n: NAT;
r: INT;

Invariant: (ARR, ELEM, NAT, NAT, INT) -> BOOLEAN =
  LAMBDA(a: ARR, x: ELEM, i: NAT, n: NAT, r: INT):
    olda = a AND oldx = x AND
    n = length(a) AND i <= n AND
    (FORALL(j:NAT): j < i => get(a,j) /= x) AND
    (r = -1 OR (r = i AND i < n AND get(a,r) = x));
...

```


The Verification Conditions (Contd)



...

A: FORMULA

Input => Invariant(a, x, i, n, r);

B1: FORMULA

Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) = x
=> Invariant(a,x,i,n,i);

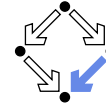
B2: FORMULA

Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) /= x
=> Invariant(a,x,i+1,n,r);

C: FORMULA

Invariant(a, x, i, n, r) AND NOT(i < n AND r = -1)
=> Output;

The Proofs



A: [bca]: expand Input, Invariant
[fuo]: scatter
[bxg]: proved (CVCL)

(2 user actions)

B1: [p1b]: expand Invariant
[lf6]: proved (CVCL)

(1 user action)

B2: [q1b]: expand Invariant in 6kv
[slx]: scatter
[a1y]: auto
[cch]: proved (CVCL)
[b1y]: proved (CVCL)
[c1y]: proved (CVCL)
[d1y]: proved (CVCL)
[e1y]: proved (CVCL)

(3 user actions)

C: [dca]: expand Invariant, Output in zfg
[tvj]: scatter
[dcu]: auto
[t4c]: proved (CVCL)
[ecu]: split pkg
[kel]: proved (CVCL)
[lel]: scatter
[lvn]: auto
[lap]: proved (CVCL)
[fcu]: auto
[blt]: proved (CVCL)
[gcu]: proved (CVCL)

(6 user actions)

1. The Hoare Calculus

2. Predicate Transformers

3. Proving Verification Conditions

4. Termination

5. Abortion

6. Procedures

Termination

Hoare rules for **loop** and **while** are replaced as follows:

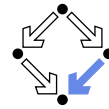
$$\frac{\{\text{false}\} \text{ loop } \{\text{false}\} \quad I \Rightarrow t \geq 0 \quad \{I \wedge b \wedge t = N\} c \quad \{I \wedge t < N\}}{\{I\} \text{ while } b \text{ do } c \quad \{I \wedge \neg b\}}$$

$$\frac{P \Rightarrow I \quad I \Rightarrow t \geq 0 \quad \{I \wedge b \wedge t = N\} c \quad \{I \wedge t < N\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\} \text{ while } b \text{ do } c \quad \{Q\}}$$

- New interpretation of $\{P\} c \{Q\}$.
 - If execution of c starts in a state where P holds, then execution **terminates** in a state where Q holds, unless it aborts.
 - Non-termination is ruled out, abortion not (yet).
 - The **loop** command thus does not satisfy total correctness.
 - **Termination term t (type-checked to denote an integer)**.
 - Becomes smaller by every iteration of the loop.
 - But does not become negative.
 - Consequently, the loop must eventually terminate.
- The initial value of t limits the number of loop iterations.

Any well-founded ordering may be used for the domain of t .

Example



$$I \Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$

$$(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I \quad I \Rightarrow n - i + 1 \geq 0$$

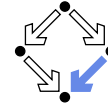
$$\{I \wedge i \leq n \wedge n - i + 1 = N\} s := s + i; i := i + 1 \{I \wedge n - i + 1 < N\}$$

$$(I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^n j$$

$$\{n \geq 0 \wedge i = 1 \wedge s = 0\} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \{s = \sum_{j=1}^n j\}$$

In practice, termination is easy to show (compared to partial correctness).

Weakest Preconditions for Loops



$$\text{wp}(\text{loop}, Q) = \text{false}$$

$$\text{wp}(\text{while } b \text{ do } c, Q) = L_0(Q) \vee L_1(Q) \vee L_2(Q) \vee \dots$$

$$L_0(Q) = \text{false}$$

$$L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$$

■ New interpretation

- Weakest precondition that ensures that the loop terminates in a state in which Q holds, unless it aborts.

■ New interpretation of $L_i(Q)$

- Weakest precondition that ensures that the loop terminates **after less than i iterations** in a state in which Q holds, unless it aborts.

■ Preserves property: $\{P\} c \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$

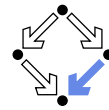
- Now for **total correctness** interpretation of Hoare calculus.

■ Preserves alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

$$\text{if}_0 = \text{loop}$$

$$\text{if}_{i+1} = \text{if } b \text{ then } (c; \text{if}_i)$$

Example



$$\text{wp}(\text{while } i < n \text{ do } i := i + 1, Q)$$

$$L_0(Q) = \text{false}$$

$$L_1(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, L_0(Q)))$$

$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{false})$$

$$\Leftrightarrow i \not< n \wedge Q$$

$$L_2(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, L_1(Q)))$$

$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge$$

$$i < n \Rightarrow (i + 1 \not< n \wedge Q[i + 1/i])$$

$$L_3(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, L_2(Q)))$$

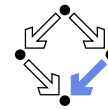
$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge$$

$$i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge$$

$$(i + 1 < n \Rightarrow (i + 2 \not< n \wedge Q[i + 2/i])))$$

...

Weakest Preconditions for Loops



■ Sequence $L_i(Q)$ is now monotonically **decreasing** in strength:

$$\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q).$$

■ The weakest precondition is the “greatest lower bound”:

$$\forall i \in \mathbb{N} : L_i(Q) \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q).$$

$$\forall P : (\forall i \in \mathbb{N} : L_i(Q) \Rightarrow P) \Rightarrow (\text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow P).$$

■ We can only compute a stronger approximation $L_i(Q)$.

$$L_i(Q) \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q).$$

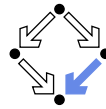
■ We want to prove $\{P\} c \{Q\}$.

$$\text{It suffices to prove } P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q).$$

$$\text{It thus also suffices to prove } P \Rightarrow L_i(Q).$$

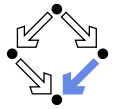
$$\text{If proof fails, we may try the easier proof } P \Rightarrow L_{i+1}(Q)$$

However, verifications are typically not successful with any finite approximation of the weakest precondition.



1. The Hoare Calculus
2. Predicate Transformers
3. Proving Verification Conditions
4. Termination
- 5. Abortion**
6. Procedures

Abortion



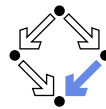
New rules to prevent abortion.

$$\frac{\{\text{false}\} \text{abort} \{\text{true}\} \quad \{Q[e/x] \wedge D(e)\} x := e \{Q\}}{\{Q[a[i \mapsto e]/a] \wedge D(e) \wedge D(i) \wedge 0 \leq i < \text{length}(a)\} a[i] := e \{Q\}}$$

- New interpretation of $\{P\} c \{Q\}$.
 - If execution of c starts in a state, in which property P holds, then it does not abort and eventually terminates in a state in which Q holds.
- Sources of abortion.
 - Division by zero.
 - Index out of bounds exception.

$D(e)$ makes sure that every subexpression of e is well defined.

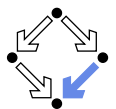
Definedness of Expressions



$$\begin{aligned} D(0) &= \text{true}. \\ D(1) &= \text{true}. \\ D(x) &= \text{true}. \\ D(a[i]) &= D(i) \wedge 0 \leq i < \text{length}(a). \\ D(e_1 + e_2) &= D(e_1) \wedge D(e_2). \\ D(e_1 * e_2) &= D(e_1) \wedge D(e_2). \\ D(e_1/e_2) &= D(e_1) \wedge D(e_2) \wedge e_2 \neq 0. \\ D(\text{true}) &= \text{true}. \\ D(\text{false}) &= \text{true}. \\ D(\neg b) &= D(b). \\ D(b_1 \wedge b_2) &= D(b_1) \wedge D(b_2). \\ D(b_1 \vee b_2) &= D(b_1) \wedge D(b_2). \\ D(e_1 < e_2) &= D(e_1) \wedge D(e_2). \\ D(e_1 \leq e_2) &= D(e_1) \wedge D(e_2). \\ D(e_1 > e_2) &= D(e_1) \wedge D(e_2). \\ D(e_1 \geq e_2) &= D(e_1) \wedge D(e_2). \end{aligned}$$

Assumes that expressions have already been type-checked.

Abortion



Slight modification of existing rules.

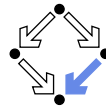
$$\frac{P \Rightarrow D(b) \quad \{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{if } b \text{ then } c_1 \text{ else } c_2 \{Q\}}$$

$$\frac{P \Rightarrow D(b) \quad \{P \wedge b\} c \{Q\} \quad (P \wedge \neg b) \Rightarrow Q}{\{P\} \text{if } b \text{ then } c \{Q\}}$$

$$\frac{I \Rightarrow (t \geq 0 \wedge D(b)) \quad \{I \wedge b \wedge t = N\} c \{I \wedge t < N\}}{\{I\} \text{while } b \text{ do } c \{I \wedge \neg b\}}$$

Expressions must be defined in any context.

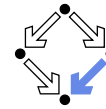
Abortion



Similar modifications of weakest preconditions.

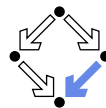
$$\begin{aligned} \text{wp}(\text{abort}, Q) &= \text{false} \\ \text{wp}(x := e, Q) &= Q[e/x] \wedge D(e) \\ \text{wp}(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) &= \\ & D(b) \wedge (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q)) \\ \text{wp}(\text{if } b \text{ then } c, Q) &= D(b) \wedge (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q) \\ \text{wp}(\text{while } b \text{ do } c, Q) &= (L_0(Q) \vee L_1(Q) \vee L_2(Q) \vee \dots) \end{aligned}$$
$$\begin{aligned} L_0(Q) &= \text{false} \\ L_{i+1}(Q) &= D(b) \wedge (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q))) \end{aligned}$$

$\text{wp}(c, Q)$ now makes sure that the execution of c does not abort but eventually terminates in a state in which Q holds.



1. The Hoare Calculus
2. Predicate Transformers
3. Proving Verification Conditions
4. Termination
5. Abortion
6. Procedures

Procedure Specifications

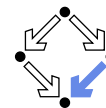


```
global g;  
requires Pre;  
ensures Post;  
o := p(i) { c }
```

- Specification of a procedure p implemented by a command c .
 - Input parameter i , output parameter o , global variable g .
 - Command c may read/write i , o , and g .
 - Precondition Pre (may refer to i, g).
 - Postcondition $Post$ (may refer to i, o, g, g_0).
 - g_0 denotes the value of g before the execution of p .
- Proof obligation
$$\{Pre \wedge i_0 = i \wedge g_0 = g\} c \{Post[i_0/i]\}$$

Proof of the correctness of the implementation of a procedure with respect to its specification.

Example

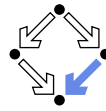


- Procedure specification:

```
global g  
requires  $g \geq 0 \wedge i > 0$   
ensures  $g_0 = g \cdot i + o \wedge 0 \leq o < i$   
 $o := p(i) \{ o := g \% i; g := g / i \}$ 
```
- Proof obligation:
$$\begin{aligned} &\{g \geq 0 \wedge i > 0 \wedge i_0 = i \wedge g_0 = g\} \\ &o := g \% i; g := g / i \\ &\{g_0 = g \cdot i_0 + o \wedge 0 \leq o < i_0\} \end{aligned}$$

A procedure that divides g by i and returns the remainder.

Procedure Calls



A call of p provides actual input argument e and output variable x .

$$x := p(e)$$

Similar to assignment statement; we thus first give an alternative (equivalent) version of the assignment rule.

- Original:

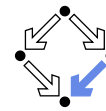
$$\frac{\{D(e) \wedge Q[e/x]\}}{x := e} \{Q\}$$

- Alternative:

$$\frac{\{D(e) \wedge \forall x' : x' = e \Rightarrow Q[x'/x]\}}{x := e} \{Q\}$$

The new value of x is given name x' in the precondition.

Procedure Calls



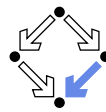
From this, we can derive a rule for the correctness of procedure calls.

$$\frac{\{D(e) \wedge Pre[e/i] \wedge \forall x', g' : Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]\}}{x := p(e)} \{Q\}$$

- $Pre[e/i]$ refers to the values of the actual argument e (rather than to the formal parameter i).
- x' and g' denote the values of the vars x and g after the call.
- $Post[...]$ refers to the argument values before and after the call.
- $Q[x'/x, g'/g]$ refers to the argument values after the call.

Modular reasoning: rule only relies on the *specification* of p , not on its implementation.

Corresponding Predicate Transformers

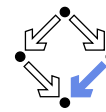


$$\begin{aligned} wp(x = p(e), Q) = & \\ & D(e) \wedge Pre[e/i] \wedge \\ & \forall x', g' : \\ & \quad Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g] \end{aligned}$$

$$\begin{aligned} sp(P, x = p(e)) = & \\ & \exists x_0, g_0 : \\ & \quad P[x_0/y, g_0/g] \wedge \\ & \quad (Pre[e[x_0/x, g_0/g]/i, g_0/g] \Rightarrow Post[e[x_0/x, g_0/g]/i, x/o]) \end{aligned}$$

Explicit naming of old/new values required.

Example



- Procedure specification:

global g
 requires $g \geq 0 \wedge i > 0$
 ensures $g_0 = g \cdot i + o \wedge 0 \leq o < i$
 $o = p(i) \{ o := g \% i; g := g / i \}$

- Procedure call:

$\{g \geq 0 \wedge g = N \wedge b \geq 0\}$
 $x = p(b + 1)$
 $\{g \cdot (b + 1) \leq N < (g + 1) \cdot (b + 1)\}$

- To be proved:

$g \geq 0 \wedge g = N \wedge b \geq 0 \Rightarrow$
 $D(b + 1) \wedge g \geq 0 \wedge b + 1 > 0 \wedge$
 $\forall x', g' :$
 $g = g' \cdot (b + 1) + x' \wedge 0 \leq x' < b + 1 \Rightarrow$
 $g' \cdot (b + 1) \leq N < (g' + 1) \cdot (b + 1)$