Turing Machines

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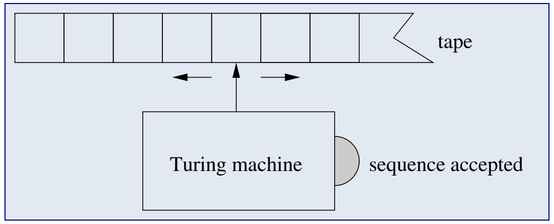
Turing Machine Model



- The machine is always in one of a finite set of states.
  - The machine starts its execution in a fixed start state.
- An infinite tape holds at its beginning the input word.
  - Tape is read and written and arbitrarily moved by the machine.
- The machine proceeds in a sequence of state transitions.
  - Machine reads symbol, overwrites it, and moves tape head left or right.
  - The symbol read and the current state determine the symbol written, the move direction, and the next state.
- If the machine cannot make another transition, it terminates.
  - The machine signals whether it is in an accepting state.

If the machine terminates in an accepting state, the word is *accepted*.

Turing Machines

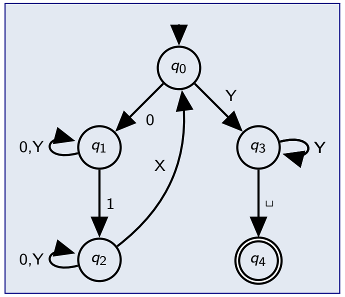
Turing Machine $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$:

- The state set Q, a fine set of states.
- A tape alphabet Γ, a finite set of tape symbols.
- The blank symbol $\sqcup \in \Gamma$.
- An input alphabet $\Sigma \subseteq \Gamma \backslash \{\sqcup\}$.
- The (partial) transition function $\delta : Q \times \Gamma \to_p Q \times \Gamma \times \{\text{'L'}, \text{'R'}\}$,
  - $\delta(q, x) = (q', x', \text{'L'}/\text{'R'}) \ldots M$ reads in state q symbol x, goes to state q', writes symbol x', and moves the tape head left/right.
- The start state $q_0 \in Q$
- A set of accepting states (final states) $F \subseteq Q$.

The crucial difference to an automaton is the infinite tape that can be arbitrarily moved and written.

Example

$$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$$
$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\Gamma = \{\sqcup, 0, 1, X, Y\}$$
$$\Sigma = \{0, 1\}$$
$$F = \{q_4\}$$

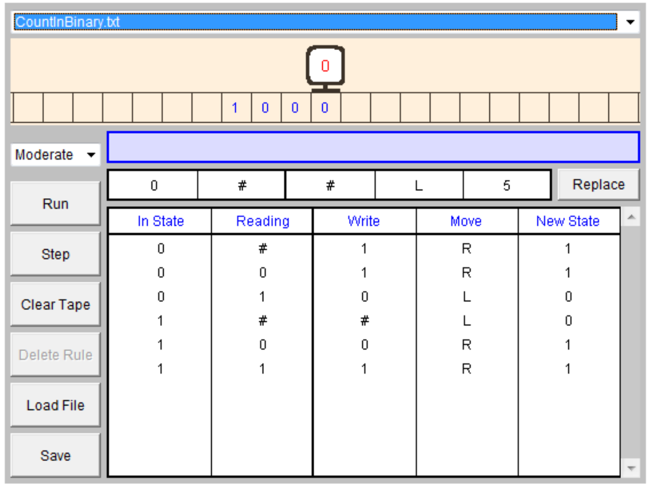


δ	$\sqcup$	0	1	X	Y
q_0	–	(q_1, X, R)	–	–	(q_3, Y, R)
q_1	–	$(q_1, 0, R)$	(q_2, Y, L)	–	(q_1, Y, R)
q_2	–	$(q_2, 0, L)$	–	(q_0, X, R)	(q_2, Y, L)
q_3	$(q_4, \sqcup, R)$	–	–	–	(q_3, Y, R)
q_4	–	–	–	–	–

Machine accepts every word of form 0^n1^n (replacing it by X^nY^n).

Turing Machine Simulators

For instance, `http://math.hws.edu/TMCM/java/xTuringMachine`.



Generalized Turing Machines

- Infinite tape in both directions.
  - Can be simulated by a machine whose tape is infinite in one direction.
- Multiple tapes.
  - Can be simulated by a machine with a single tape.
- Nondeterministic transitions.
  - We can simulate a nondeterministic M by a deterministic M'.
    - Let r be the maximum number of "choices" that M can make.
  - M' operates with 3 tapes.
    - Tape 1 holds the input (tape is only read).
  - M' writes to tape 2 all finite sequences of numbers $1, \ldots, r$.
    - First all sequences of length 1, then all of length 2, etc.
  - After writing sequence $s_1 s_2 \ldots s_n$ to tape 2, M' simulates M on tape 3.
    - M' copies the input to tape 3 and performs at most n transitions.
    - In transition i, M attempts to perform choice s_i.
    - If choice i is not possible or M terminates after n transitions in a non-accepting state, M' continues with next sequence.
    - If M terminates in accepting state, M' accepts the input.

Every generalized Turing machine can be simulated by the core form.

Turing Machine Configurations

- Configuration $a_1 \dots a_k\ q\ a_{k+1} \dots a_m$:
  - q: the current state of M.
  - a_{k+1}: the symbol currently under the tape head.
  - $a_1 \dots a_k$: the portion of the tape left to the tape head.
  - $a_{k+2} \dots a_m$: the portion right to the head (followed by $\sqcup \dots$).
- Move relation: $a_1 \dots a_k\ q\ a_{k+1} \dots a_m \vdash b_1 \dots b_l\ p\ b_{l+1} \dots b_m$

  If M is a situation described by the left configuration, it can make a transition to the situation described by the right configuration.
  - $a_i = b_i$ for all $i \neq k+1$ and one of the following:
    - $l = k+1$ and $\delta(q, a_{k+1}) = (p, b_l, R)$,
    - $l = k-1$ and $\delta(q, a_{k+1}) = (p, b_{l+2}, L)$.
- Extended move relation: $c_1 \vdash^* c_2$

  M can make in an arbitrary number of moves a transition from the situation described by configuration c_1 to the one described by c_2.

$$c_1 \vdash^0 c_2 :\Leftrightarrow c_1 = c_2$$

$$c_1 \vdash^{i+1} c_2 :\Leftrightarrow \exists c : c_1 \vdash^i c \wedge c \vdash c_2$$

$$c_1 \vdash^* c_2 :\Leftrightarrow \exists i \in \mathbb{N} : c_1 \vdash^i c_2$$

The Language of a Turing Machine

- The language $L(M)$ of Turing machine $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$:

  The set of all inputs that drive M from its initial configuration to a configuration with an accepting state such that from this configuration no further move is possible:

$$L(M) := \left\{ w \in \Sigma^* \;\middle|\; \begin{array}{l} \exists a, b \in \Gamma^*, q \in Q : q_0\ w \vdash^* a\ q\ b \wedge q \in F \\ \wedge\ \neg \exists a', b' \in \Gamma^*, q' \in Q : a\ q\ b\ \vdash a'\ q'\ b' \end{array} \right\}$$

- L is a recursively enumerable language:
  - There exists a Turing machine M such that $L = L(M)$.
- L is a recursive language:
  - There exists a Turing machine M such that $L = L(M)$ and M terminates for every possible input.

Every recursive language is recursively enumerable; as we will see, the converse does not hold.

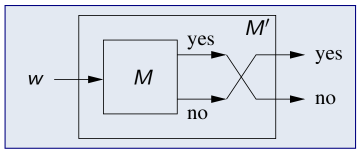
Recursiv. Enumerable/Recursive Languages

Theorem: L is recursive, if and only if both L and its complement $\overline{L}$ are recursively enumerable.

Proof $\Rightarrow$: Let L be a recursive. Since by definition L is recursively enumerable, it remains to be shown that also $\overline{L}$ is recursively enumerable.

Since L is recursive, there exists a Turing machine M such that M halts for every input w: if $w \in L$, then M accepts w; if $w \notin L$, then M does not accept w. With the help of M, we can construct the following M' with $L(M') = \overline{L}$:

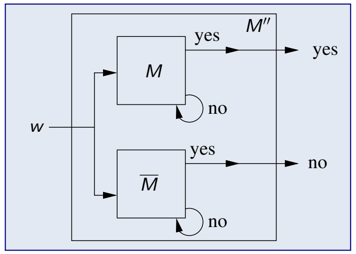
```
function M'(w):
  case M(w) of
    yes: return no
    no: return yes
  end case
end function
```



Recursiv. Enumerable/Recursive Languages

Proof $\Leftarrow$: Let L be such that both L and $\overline{L}$ are recursively enumerable. We show that L is recursive. Since L is r.e., there exists M such that $L = L(M)$ and M halts for $w \in L$ with $M(w) = \text{yes}$. Since $\overline{L}$ is r.e., there exists $\overline{M}$ with $\overline{L} = L(\overline{M})$ and $\overline{M}$ halts for $w \in \overline{L}$ with $\overline{M}(w) = \text{yes}$. We can thus construct M'' with $L(M'') = L$ that always halts:

```
function M''(w):
  parallel
    begin
      if M(w) = yes then
        return yes
      end if
      loop forever
    end
    begin
      if M̄(w) = yes then
        return no
      end if
      loop forever
    end
  end parallel
end function
```



Closure of Recursive Languages

Let L, L_1, L_2 be recursive languages. Then also

- the complement $\overline{L}$,
- the union $L_1 \cup L_2$,
- the intersection $L_1 \cap L_2$

are recursive languages.

Proof by construction of the corresponding Turing machines.

Enumerators

Turing machine $M = (Q, \Gamma, \sqcup, \emptyset, \delta, q_0, F)$ with special symbol $\# \in \Gamma$.

- M is an enumerator, if M has an additional output tape on which
  - M moves its tape head only to the right, and
  - M writes only symbols different from $\sqcup$.
- The generated language $Gen(M)$ of enumerator M is the set of all words that M eventually writes on its output tape.
  - The end of each word is marked by a trailing $\#$.

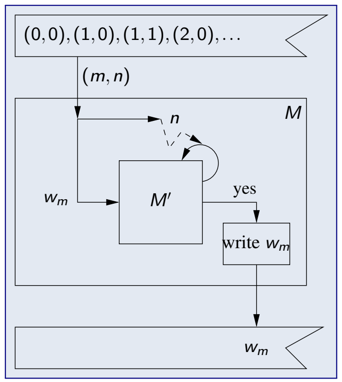
M may run forever and thus $Gen(M)$ may be infinite.

Recognizing versus Generating Languages

Theorem: L is recursively enumerable, if and only if there exists some enumerator M such that $L = Gen(M)$.

Proof $\Rightarrow$: Let L be recursively enumerable, i.e., $L = L(M')$ for some M'. We construct enumerator M such that $L = Gen(M)$.

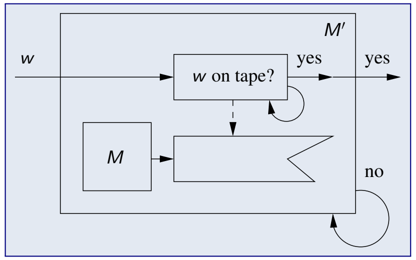
```
procedure M:
  loop
    produce next (m, n) on working tape
    if M'(w_m) = yes in at most n steps then
      write w_m to output tape
    end if
  end loop
end procedure
```



Recognizing versus Generating Languages

Proof $\Leftarrow$: Let L be such that $L = Gen(M)$ for some enumerator M. We show that there exists some Turing machine M' such that $L = L(M')$.

```
function M'(w):
  while M is not terminated do
    M writes next word w'
    if w = w' then
      return yes
    end if
  end while
  return no
end function
```



Recognizing is possible, if and only if generating is possible.

Functions

Take binary relation $f \subseteq A \times B$.

- $f : A \to B$: f is a total function from A to B.
  - For every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.
- $f : A \to_p B$: f is a partial function from A to B.
  - For every $a \in A$, there is at most one $b \in B$ such that $(a, b) \in f$.
- Auxiliary notions:

$$domain(f) := \{a \mid \exists b : (a, b) \in f\}$$
$$range(f) := \{b \mid \exists a : (a, b) \in f\}$$
$$f(a) := \text{ such } b : (a, b) \in f$$

Every total function $f : A \to B$ is a partial function $f : A \to_p B$; every partial function $f : A \to_p B$ is a total function $f : domain(f) \to B$.

Functions

- Let $f : \Sigma^* \to_p \Gamma^*$ where $\sqcup \notin \Sigma \cup \Gamma$.
  - f is a function over words in some alphabets.
- f is Turing computable, if there exists a Turing machine M such that
  - for input w (i.e. initial tape content $w\sqcup\ldots$), M terminates in an accepting state, if and only if $w \in domain(f)$;
  - for input w, M terminates in an accepting state with output w' (i.e. final tape content $w'\sqcup\ldots$), if and only if $w' = f(w)$.
- Not every function $f : \Sigma^* \to_p \Gamma^*$ is Turing computable:
  - The set of all Turing machines is countably infinite: all machines can be ordered in a single list (in the alphabetic order of their definitions).
  - The set of all functions $\Sigma^* \to_p \Gamma^*$ is more than countably infinite (Cantor's diagonalization argument).
  - Consequently, there are more functions than Turing machines.

M computes f, if M terminates for arguments in the domain of f with output $f(a)$ and does not terminate for arguments outside the domain.

Example

We show that natural number subtraction is Turing computable.

- Subtraction $\ominus$ on $\mathbb{N}$:

$$m \ominus n := \begin{cases} m-n & \text{if } m \geq n \\ 0 & \text{else} \end{cases}$$

- Unary representation of $n \in \mathbb{N}$:

$$\underbrace{000\ldots0}_{n \text{ times}} \in L(0^*)$$

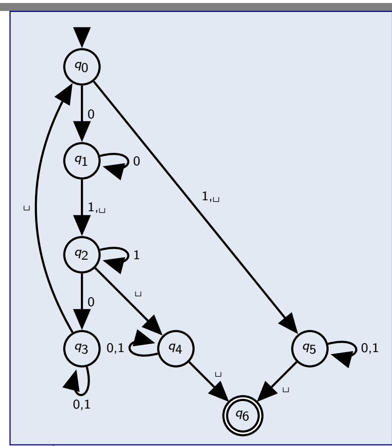
- Input $00\sqcup0$ shall lead to output 0.
  - $2 \ominus 1 = 1$.

Idea: replace every pair of 0 in m and n by $\sqcup$.

Example (Contd)

$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$
$Q = \{q_0, \ldots, q_6\}$
$\Sigma = \{0\}, \Gamma = \{0, 1, \sqcup\}, F = \{q_6\}$

δ	0	1	$\sqcup$
q_0	$(q_1, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$
q_1	$(q_1, 0, R)$	$(q_2, 1, R)$	$(q_2, 1, R)$
q_2	$(q_3, 1, L)$	$(q_2, 1, R)$	$(q_4, \sqcup, L)$
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	$(q_0, \sqcup, R)$
q_4	$(q_4, 0, L)$	$(q_4, \sqcup, L)$	$(q_6, 0, R)$
q_5	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_6, \sqcup, R)$
q_6	−	−	−



- In q_0, the leading 0 is replaced by $\sqcup$.
- In q_1, M searches for the next $\sqcup$ and replaces it by a 1.
- In q_2, M searches for the next 0 and replaces it by 1, then moves left.
- In q_3, M searches for previous $\sqcup$, moves right and starts from begin.
- In q_4, M has found a $\sqcup$ instead of 0 and replaces all previous 1 by $\sqcup$.
- In q_5, n is (has become) 0; the rest of the tape is erased.
- In q_6, the computation successfully terminates.

Example (Contd)

- $2 \ominus 1 = 1$:

$q_0 00\sqcup0 \vdash \sqcup q_1 0\sqcup0 \vdash \sqcup 0 q_1 \sqcup 0 \vdash \sqcup 01 q_2 0$
$\vdash \sqcup 0 q_3 11 \vdash \sqcup q_3 011 \vdash q_3 \sqcup 011 \vdash \sqcup q_0 011$
$\vdash \sqcup\sqcup q_1 11 \vdash \sqcup\sqcup 1 q_2 1 \vdash \sqcup\sqcup 11 q_2 \vdash \sqcup\sqcup 1 q_4 1$
$\vdash \sqcup\sqcup q_4 1 \vdash \sqcup q_4 \vdash \sqcup 0 q_6$

- $1 \ominus 2 = 0$:

$q_0 0\sqcup00 \vdash \sqcup q_1 \sqcup 00 \vdash \sqcup 1 q_2 00 \vdash \sqcup q_3 110$
$\vdash q_3 \sqcup 110 \vdash \sqcup q_0 110 \vdash \sqcup\sqcup q_5 10 \vdash \sqcup\sqcup\sqcup q_5 0$
$\vdash \sqcup\sqcup\sqcup\sqcup q_5 \vdash \sqcup\sqcup\sqcup\sqcup\sqcup q_6.$

For $m > n$, leading blanks still have to be removed.

Turing Computability

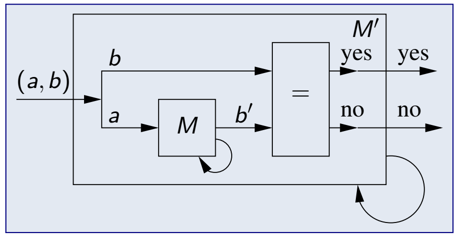
Theorem: $f : \Sigma^* \to_p \Gamma^*$ is Turing computable, if and only if

$$L_f := \{(a,b) \in \Sigma^* \times \Gamma^* \mid a \in domain(f) \wedge b = f(a)\}$$

is recursively enumerable.

Proof $\Rightarrow$: Since $f : \Sigma^* \to_p \Gamma^*$ is Turing computable, there exists a Turing machine M which computes f. To show that L_f is r.e., we construct M' with $L(M') = L_f$:

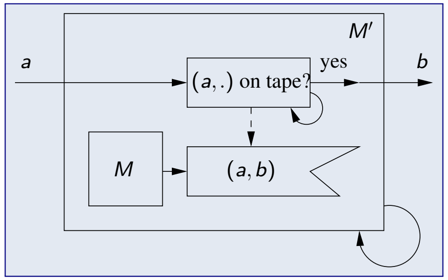
```
function M'(a, b):
  b' ← M(a)
  if b' = b then
    return yes
  else
    return no
  end if
end function
```



Turing Computability

Proof $\Leftarrow$: Since L_f is recursively enumerable, there exists an enumerator M with $Gen(M) = L_f$. We construct the following Turing machine M' which computes f:

function $M'(a)$:
  while M is not terminated **do**
    M writes (a', b') to tape
    if $a = a'$ **then**
      return b'
    end if
  end while
  loop forever
end function



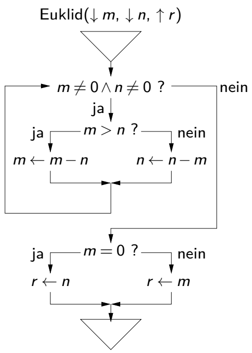
Computing is possible, if and only if recognizing is possible.

Algorithms

Computer science is based on algorithms.

Compute as follows the greatest common divisor of two natural numbers m, n that are not both 0:

1. If $m = 0$, the result is n.
2. If $n = 0$, the result is m.
3. If $m > n$, subtract n from m and continue with step 1.
4. Otherwise subtract m from n and continue with step 1.



Euklid($\downarrow m$, $\downarrow n$, $\uparrow r$):
  while $m \neq 0 \wedge n \neq 0$ **do**
    if $m > n$
      then $m \leftarrow m - n$
      else $n \leftarrow n - m$
  if $m = 0$
    then $r \leftarrow n$
    else $r \leftarrow m$
end Euklid.

What is an "algorithm" and what is computable by an algorithm?

The Church-Turing Thesis

Church-Turing Thesis: Every problem that is solvable by an algorithm (in an intuitive sense) is solvable by a Turing machine. Thus the set of intuitively computable functions is identical with the set of Turing computable functions.

- Replaces fuzzy notion "algorithm" by precise notion "Turing machine".
- Unprovable thesis, exactly because the notion "algorithm" is fuzzy.
- Substantially validated, because many different computational models have no more computational power than Turing machines.
  - Random access machines, loop programs, recursive functions, goto programs, λ-calculus, rewriting systems, grammars, ...

Turing machines represent the most powerful computational model known, but there are many other equally powerful ("Turing complete") models.