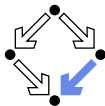


# Initial Specifications

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# Initial Specifications



We fix as our logic the equational logic  $EL$ .

- **Initial specification**  $sp = (\Sigma, \Phi)$ .
  - Signature  $\Sigma$ , set of formulas  $\Phi \subseteq EL(\Sigma)$ .
- **Semantics**  $\mathcal{M}(sp) = \{A \in Alg(\Sigma) \mid A \simeq T(\Sigma, \Phi)\}$ .
  - The class of algebras isomorphic to the quotient term algebra over  $\Phi$ .
  - Values are (isomorphic to) classes of terms that have the same value in all models of  $\Phi$ .

An initial specification specifies as the abstract datatype the class of all algebras isomorphic to the quotient term algebra over its formula set.

# Concrete Syntax



**initial spec**

**sorts** *sort* ...

**opns** *operation* ...

**vars** *variable*: *sort* ...

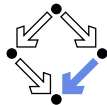
**eqns** *equation* ...

**endspec**

- Signature  $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$ .
- Set of formulas  $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{equation}), \dots\}$ .

We will only use the concrete syntax to define specifications.

# Example



## initial spec

**sorts** *nat*

**opns**

$0 : \rightarrow nat$

$Succ : nat \rightarrow nat$

$- + - : nat \times nat \rightarrow nat$

**vars**  $m, n : nat$

**eqns**

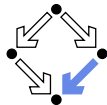
$n + 0 = n$

$n + Succ(m) = Succ(n + m)$

## endspec

- $T(\Sigma, \Phi)(nat) = \{[0], [Succ(0)], [Succ(Succ(0))], \dots\}$ 
  - $[0] = [0 + 0] = [0 + (0 + 0)] = \dots$
  - $[Succ(0)] = [Succ(0) + 0] = [0 + Succ(0)] = [Succ(0 + 0)] = \dots$
  - $\dots$

# Initiality



Take signature  $\Sigma$ , set of formulas  $\Phi \in EL(\Phi)$ .

- **Theorem:**  $T(\Sigma, \Phi) \in Mod_{\Sigma}(\Phi)$ .
  - Quotient term algebra of  $\Phi$  is itself a model of  $\Phi$ .
  - Not true for formula set  $\Phi$  from every logic  $L$ .
- **Corollary:**  $T(\Sigma, \Phi)$  is initial in  $Mod_{\Sigma}(\Phi)$ .
  - Consequence of the final theorem of the previous section.
- **Corollary:** Every algebra of  $\mathcal{M}(sp)$  is initial in  $Mod_{\Sigma}(\Phi)$ .
  - The specified algebras distinguish most among all models of  $\Phi$ .

The specified abstract datatype is the most “distinguishing” one.

# Datatype Properties

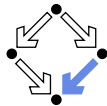


Take initial specification  $sp = (\Sigma, \Phi)$ .

- $\mathcal{M}(sp)$  is **not empty**.
  - It contains  $T(\Sigma, \Phi)$ .
  - An initial specification is *consistent*.
- $\mathcal{M}(sp)$  is **monomorphic**.
  - By definition, all algebras in  $\mathcal{M}(sp)$  are isomorphic.
  - An initial specification describes a *single datatype*.
- $\mathcal{M}(sp)$  only contains **generated** algebras.
  - Every value is isomorphic to (the congruence class of) a ground term.
  - The specified datatype *does not contain junk*.

An initial specification is much more specific than a loose specification.

# Logical Properties



Take initial specification  $(\Sigma, \Phi)$ , ground equation  $t = u$  and equation  $\forall X.v = w$  in  $EL(\Phi)$ .

- $T(\Sigma, \Phi) \models t = u$  iff  $\Phi \models t = u$ .
  - $\Phi \models t = u$  iff  $A \models t = u$  for every  $\Sigma$ -algebra  $A$  that is a model of  $\Phi$ .
  - The equation holds in specified datatype, iff the equation is a logical consequence of the specification equations.
- $T(\Sigma, \Phi) \models \forall X.v = w$  iff  $\Phi \models_{Ind} \forall X.v = w$ .
  - $\Phi \models_{Ind} t = u$  iff  $A \models t = u$  for every model  $A$  of  $\Phi$  that is generated.
  - The equation holds in the specified datatype, iff the equation is a logical consequence of the specification equations; for proving this, we may apply the principle of structural induction.

In the specified abstract datatype, no other equalities hold than those that are consequences of  $\Phi$ .

# Example



**initial spec**

**sorts**  $nat, set$

**opns**

$0 : \rightarrow nat$

$Succ : nat \rightarrow nat$

$\emptyset : \rightarrow set$

$Insert : set \times nat \rightarrow set$

**vars**  $m, n : nat, s : set$

**eqns**

$Insert(Insert(s, n), n) = Insert(s, n)$

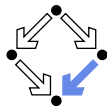
$Insert(Insert(s, n), m) = Insert(Insert(s, m), n)$

**endspec**

Strictly adequate specification of “classical” set algebra (compare with a corresponding loose specification with constructors).



# Example



## loose spec

sorts freely generated  $nat$

opns

**constr**  $0 : \rightarrow nat$

**constr**  $Succ : nat \rightarrow nat$

$Pred : nat \rightarrow nat$

**vars**  $n : nat$

**axioms**  $Pred(Succ(n)) = n$

endspec

## initial spec

sorts  $nat$

opns

$0 : \rightarrow nat$

$Succ : nat \rightarrow nat$

$Pred : nat \rightarrow nat$

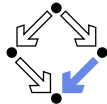
**vars**  $n : nat$

**eqns**  $Pred(Succ(n)) = n$

endspec

- Loose specification is **polymorphic**:
  - Algebra  $A$  may define any value  $A(Pred(0)) \in A(nat)$ .
  - Same is true for the “classical” algebra of natural numbers.
- Initial specification is **monomorphic**:
  - $T(\Sigma, \Phi)(nat)$  has  $[Pred(0)], [Pred(Pred(0))], [Succ(Pred(0))], \dots$
  - Certainly not isomorphic to the “classical” algebra of natural numbers.

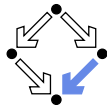
Initial specifications may yield carriers that are larger than expected.



# Possible Solution Attempts

- Add equation  $Pred(0) = 0$ .
  - Ambiguity is resolved by fixing the value of  $Pred(0)$ .
  - Problem: specification is less abstract than possible.
- Add equation  $Succ(Pred(n)) = n$ .
  - Unsatisfactory: carriers  $[Pred^i(0)]$ ,  $i \geq 1$  are not removed.
- Add constant  $Error : \rightarrow nat$ .
  - Additional equations:
$$Pred(0) = Error$$
$$Succ(Error) = Error$$
$$Pred(Error) = Error$$
    - Problem: additional carrier  $[Error]$  has to be considered.
- Combine last two solutions:
  - $[Succ^{i+1}(Pred(0))] = [Succ^i(0)]$ .
  - $[Succ^{i+1}(Pred(0))] = [Succ^{i+1}(Error)] = [Error]$ .
  - Effect:  $T(\Sigma, \Phi)(nat) = \{[Error]\}$ .

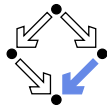
# Loose vs Initial Specifications



- Adding an additional formula to a specification.
  - Loose specification: some models are removed.
    - (Not strictly) adequate specification becomes “more adequate”.
  - Initial specification: some values are identified.
    - Carrier becomes smaller.
    - Adequate specification becomes inadequate.
- Adding an incompatible formula to a specification.
  - Loose specification:  $\mathcal{M}(sp) = \emptyset$ .
    - Specification becomes inconsistent; all models are “killed”.
  - Initial spec:  $\mathcal{M}(sp)$  consists of algebras with singleton carriers.
    - Specification becomes too constrained; some carriers “collapse”.

In an initial specification, adding an “incompatible” equation lets a carrier collapse to a singleton.

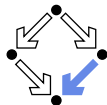
# Expressive Power of Initial Specifications



- **Theorem:** Any generated algebra may be specified by an initial specification consisting of ground equations only.
  - Catch: number of equations may be infinite and even not recursively enumerable (e.g. Peano arithmetic).
- **Theorem:** There exist algebras that can be specified by an initial specification with a finite number of equations but cannot be specified with a finite number of ground equations.
  - Universally quantified variables really add expressive power.
- **Theorem:** There exist generated algebras that cannot be specified by an initial specification consisting of a finite number of equations.
  - Universally quantified variables do not suffice.

Seems to impose fundamental limitations on initial specifications.

# Example



■  $\Sigma = (\{nat\}, \{0 : \rightarrow nat, Succ : nat \rightarrow nat, Square : nat \rightarrow nat\})$ .

■ Classical  $\Sigma$ -algebra  $A$ :

$$A(nat) = \mathbb{N}, A(0) = 0, A(Succ)(n) = n + 1,$$

$$A(Square)(n) = n^2.$$

■ Attempt to an adequate specification of  $A$ :

$$Square(0) = 0$$

$$Square(Succ(n)) = \dots?$$

■  $\Sigma' = (\dots, \{\dots, + : nat \times nat \rightarrow nat\})$ .

■ Classical  $\Sigma'$ -algebra  $A$ :

$$\dots, A(+)(n, m) = n + m$$

■ Adequate specification of  $A$ :

...

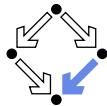
$$n + 0 = n$$

$$n + Succ(m) = Succ(n + m)$$

$$Square(Succ(n)) = Succ(Square(n) + (n + n))$$

Additional operations may be needed for an adequate initial specification.

# Expressive Power of Initial Specifications



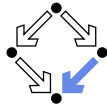
Take finite signature  $\Sigma$ .

- **Theorem:** For every generated  $\Sigma$ -algebra  $A$  with computable functions, there exists a signature  $\Sigma' \supseteq \Sigma$  and an initial specification  $(\Sigma', \Phi)$  with a finite set of formulas  $\Phi \subseteq EL(\Sigma')$  such that

$$A \simeq T(\Sigma', \Phi)|\Sigma$$

- $C|\Sigma$  is the  $\Sigma$ -**reduct** of class  $C$ .
  - From every algebra of  $C$ , all carriers and functions are removed that correspond to sorts and operations not mentioned in  $\Sigma$ .

Any computable function can be defined by recursive equations, thus any generated algebra with computable functions can be adequately specified by an initial specification (after extending the signature appropriately).



# Other Specification Logics

Not in every logic there exists an initial model of a set of formulas.

- Specification  $(\Sigma, \Phi)$ :
  - $\Sigma = (\{S\}, \{a, b, c, d : \rightarrow S\})$ .
  - $\Phi = \{(a \neq b \wedge c = d) \vee (a = b \wedge c \neq d)\}$ .
- $\Sigma$ -algebra  $B \in \text{Mod}_{\Sigma}(\Phi)$ :  
 $B(a) \neq B(b), B(c) = B(d)$ .
- $\Sigma$ -algebra  $C \in \text{Mod}_{\Sigma}(\Phi)$ :  
 $C(a) = C(b), C(c) \neq C(d)$ .
- Assume that some  $\Sigma$ -algebra  $A$  is initial in  $\text{Mod}_{\Sigma}(\Phi)$ :
  - Since  $A$  is initial, an equation that does not hold for some algebra in  $\text{Mod}_{\Sigma}(\Phi)$ , does also not hold for  $A$ .
  - Since  $B(a) \neq B(b)$ , also  $A(a) \neq A(b)$ .
  - Since  $C(c) \neq C(d)$ , also  $A(c) \neq A(d)$ .
  - Since  $A(a) \neq A(b)$  and  $A(c) \neq A(d)$ ,  $A \notin \text{Mod}_{\Sigma}(\Phi)$ .

Not every logic is suitable for initial specifications.

# Other Specification Logics



Take signature  $\Sigma$ .

- **Theorem:** If  $\Phi \subseteq \text{CEL}(\Sigma)$ ,  $T(\Sigma, \Phi) \in \text{Mod}_{\Sigma}(\Phi)$ .

**initial spec**

**sorts** *bool, nat, list*

*True*  $\rightarrow$  *bool*

*False*  $\rightarrow$  *bool*

*0*  $\rightarrow$  *nat*

*Succ* : *nat*  $\rightarrow$  *nat*

*[ ]*  $\rightarrow$  *list*

*Add* : *nat*  $\times$  *list*  $\rightarrow$  *list*

*\_ . \_* : *list*  $\times$  *list*  $\rightarrow$  *list*

*Isprefix* : *list*  $\times$  *list*  $\rightarrow$  *bool*

**vars** *l, m* : *list*, *e, e'* : *nat*

**cond eqns**

*[ ].l* = *l*

*Add(e, l).m* = *Add(e, l.m)*

*Isprefix([ ], l)* = *True*

*Isprefix(Add(e, l), [ ])* = *False*

*Isprefix(Add(e, l), Add(e, m))* =

*Isprefix(l, m)*

*Isprefix(Add(e, l), Add(e', m))* = *True*

$\Rightarrow e = e'$

**endspec**

Initial specifications may also use conditional equations.



# Properties and Proofs

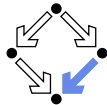


Additional proof techniques for ADTs defined by initial specifications.

- Take initial specification  $sp = (\Sigma, \Omega)$ .
  - Goal:  $\mathcal{M}(sp) \models \forall X.v = w$ .
  - Prove:  $\Omega \models v\sigma = w\sigma$  for each ground substitution  $\sigma : X \rightarrow T_\Sigma$ .
    - Proof by induction on the structure of the substitution.
- Example:  $sp = (\Sigma, \Phi)$ ,  $\Sigma = (\{nat\}, \{0, Succ, +\})$   
 $\Phi = \{(1) n + 0 = n, (2) n + Succ(m) = Succ(n + m), (3) (n + m) + p = n + (m + p)\}$ .
  - Goal:  $\mathcal{M}(sp) \models 0 + n = n$ .
  - Prove:  $\Phi \models 0 + t = t$ , for every ground term  $t \in T_\Sigma$ .
    - Case  $t = 0$ :  $0 + 0 \stackrel{(1)}{=} 0$ .
    - Case  $t = Succ(t')$ :  $0 + Succ(t') \stackrel{(2)}{=} Succ(0 + t') \stackrel{(Ind)}{=} Succ(t')$ .
    - Case  $t = (t' + t'')$ :  $0 + (t' + t'') \stackrel{(3)}{=} (0 + t') + t'' \stackrel{(Ind)}{=} t' + t''$ .

**Proof by induction on the structure of substitution terms.**

# Properties and Proofs



Take initial specification  $(\Sigma, \Phi)$  and  $\Sigma$ -algebra  $A$ .

- Goal:  $(\Sigma, \Phi)$  adequately specifies  $A$ .
- $A \simeq T(\Sigma, \Phi)$ .

1. Prove  $A$  is a model of  $\Phi$ .

Since  $T(\Sigma, \Phi)$  is initial in  $Mod_{\Sigma}(\Phi)$ , we have an evaluation homomorphism  $h : T(\Sigma, \Phi) \rightarrow A$ , i.e.  $h([t]) := A(t)$ .

2. Prove  $A$  is generated.

Thus for every carrier  $a$  there exists a term  $t$  with  $A(t) = a$  and consequently  $h([t]) = a$ , i.e.  $h$  is surjective.

3. Prove  $h$  is injective.

Prove, for arbitrary  $t, u \in T_{\Sigma}$ ,

$h([t]) = h([u]) \Rightarrow [t] = [u]$ , i.e.

$A(t) = A(u) \Rightarrow [t] = [u]$ , i.e.

$A(t) = A(u) \Rightarrow T(\Sigma, \Phi)(t) = T(\Sigma, \Phi)(u)$ , i.e.

$A(t) = A(u) \Rightarrow \Phi \models t = u$ .



## Example

- $sp = (\Sigma, \Phi)$ ,  $\Sigma = (\{\text{nat}\}, \{0, \text{Succ}, +\})$   
 $\Phi = \{(1) n + 0 = n, (2) n + \text{Succ}(m) = \text{Succ}(n + m),$   
 $(3) (n + m) + p = n + (m + p)\}$ .
  - Goal:  $sp$  adequately specifies classical  $\Sigma$ -algebra  $A$ .
1. Prove  $A \models \Phi$ .
  2. Prove  $A$  is generated.
  3. Take terms  $t, u \in T_\Sigma$  with  $A(t) = A(u)$  and prove  $\Phi \models t = u$ .
    - 3.1 Lemma:  $A(v) = n$  implies  $\Phi \models v = \text{Succ}^n(0)$ , for  $n \in \mathbb{N}, v \in T_\Sigma$ .  
Proof by induction on the structure of  $v$
    - 3.2 Take  $n = A(t) = A(u)$ . Then, by the lemma,  $\Phi \models t = \text{Succ}^n(0)$  and  $\Phi \models u = \text{Succ}^n(0)$ . Thus  $\Phi \models t = u$ .

A canonical term representation for carriers simplifies proofs.

# Characteristic Term Algebras



Take signature  $\Sigma = (S, \Omega)$  and initial specification  $sp = (\Sigma, \Phi)$ .

- A  $\Sigma$ -algebra  $C$  with  $C(s) \in T_{\Sigma, s}$  for every sort  $s \in S$  is called a **characteristic term algebra** for  $sp$ , if
  - $C \models \Phi$
  - $\Phi \models C(t) = t$  for each ground term  $t \in T_{\Sigma}$ .
    - $C$  maps every ground term to another term with the same value.
- **Theorem:** If  $C$  is charact. term alg. for  $(\Sigma, \Phi)$ , then  $C \simeq T(\Sigma, \Phi)$ .
  - $C$  is a term algebra isomorphic to the quotient term algebra.
  - $C$  may serve in proofs as a replacement of the quotient term algebra.

By using characteristic term algebras, proofs can be simplified.

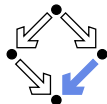


# Example

- $sp = (\Sigma, \Phi)$ ,  $\Sigma = (\{nat\}, \{0, Succ, +\})$   
 $\Phi = \{n + 0 = n, n + Succ(m) = Succ(n + m)\}$
- $\Sigma$ -algebra  $C$ :  
 $C(nat) = \{Succ^n(0) \mid n \in \mathbb{N}\}$   
 $C(0) = 0$   
 $C(Succ)(t) = Succ(t)$ , for all  $t \in C(nat)$   
 $C(+)(t, u) = Succ^{p+q}(0)$  where  $p, q \in \mathbb{N}$  such that  
 $t = Succ^p(0)$ ,  $u = Succ^q(0)$ , for all  $t, u \in C(nat)$ .

$C$  maps *nat*-terms to terms involving 0 and *Succ* only.

## Example (Contd)



$C$  is a characteristic term algebra for  $(\Sigma, \Phi)$ :

- $C \models t + 0 = t : C(t + 0) = C(+)(C(t), C(0)) = C(+)(C(t), 0) = C(t)$ .  
 $C \models t + Succ(u) = Succ(t + u) : C(t + Succ(u)) = C(+)(C(t), C(Succ)(C(u))) = C(+)(Succ^p(0), Succ(Succ^q(0))) = Succ^{p+(1+q)}(0) = Succ(Succ^{p+q}(0)) = Succ(C(+)(Succ^p(0), Succ^q(0))) = C(Succ)(C(+)(C(t), C(u))) = C(Succ(t + u))$ .

- $\Phi \models C(t) = t$  : proof by induction on  $t$ .

$t = 0$ :  $C(0) = 0$ , thus  $\Phi \models C(0) = 0$ .

$t = Succ(u)$ :

$C(Succ(u)) = C(Succ)(C(u)) = Succ(C(u))$ . By induction hypothesis,  $\Phi \models C(u) = u$ , thus  $\Phi \models C(Succ(u)) = Succ(u)$ .

$t = u + v$ :

$C(u + v) = C(+)(C(u), C(v)) = C(+)(Succ^p(0), Succ^q(0)) = Succ^{p+q}(0)$ . By the lemma below,  $\Phi \models Succ^{p+q}(0) = C(u) + C(v)$ .

By induction hypothesis,  $\Phi \models C(u) = u$  and  $\Phi \models C(v) = v$ , thus  $\Phi \models Succ^{p+q}(0) = u + v$  and finally  $\Phi \models C(u + v) = u + v$ .

**Lemma:**  $\phi \models Succ^{p+q}(0) = Succ^p(0) + Succ^q(0)$ , for all  $p, q \in \mathbb{N}$ .

Proof by induction on  $q$ .



## Example (Contd'2)

Once it has been proved that  $C$  is a characteristic term algebra of  $sp$ , proving properties of  $sp$  becomes simple.

- Prove that  $sp$  adequately specifies classical  $A$  with  $A(nat) = \mathbb{N}$ .
  - Prove that  $A \simeq C$ .
- Find isomorphism  $h : C \rightarrow A$ .

$$h_{nat}(Succ^n(0)) := n.$$

- Clearly  $h$  is bijective.
- $h$  is a homomorphism:

$$h(C(Succ)(Succ^n(0))) = h(Succ^{n+1}(0)) = n + 1 = A(Succ)(n) = A(Succ)(h(Succ^n(0))).$$

$$h(C(+)(Succ^n(0), Succ^m(0))) = h(Succ^{n+m}(0)) = n + m = A(+)(n, m) = A(+)(h(Succ^n(0)), h(Succ^m(0))).$$

Proof can make use of the structure of characteristic terms.



## Example (Contd'3)

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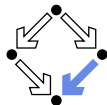
- Prove that  $\mathcal{M}(sp) \models 0 + t = t$ .
- Prove that  $C \models 0 + t = t$ .

$$C(0 + t) = C(+)(C(0), C(t)) = C(+)(0, Succ^n(0)) = Succ^n(0) = C(t).$$

The work invested in proving that  $C$  is characteristic is well spent, since all other proofs become much simpler.



# Summary



- Initial specifications have some **nice properties**:
  - The specified ADT is not empty, it is monomorphic, and it only consists of generated algebras.
    - The specification already describes a concrete implementation design.
  - The quotient term algebra is the canonical representative of the ADT.
    - Values are classes of terms whose values are considered identical.
  - Only those equalities hold that are explicitly specified.
- However, there are also **potential pitfalls**:
  - Adding “incompatible” equations lets carriers collapse.
    - Still necessary to investigate the adequacy of a specification.
  - “Undefined” terms represent additional values.
    - Carriers then contain more elements than intended.
  - Rather than undefined terms, one may prefer error values.
    - However, then one has to deal with these values in all computations.
  - Only generated algebras can be specified.
    - E.g. no initial specifications of the reals.

**The necessity to be concrete had advantages and disadvantages.**