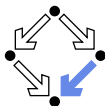


# Term Algebras

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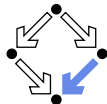
# Term Algebra

Take signature  $\Sigma = (S, \Omega)$ .

- **Term algebra**  $T(\Sigma)$ :
  - $\Sigma$ -algebra whose values are  $\Sigma$ -terms.
  - $T(\Sigma)(s) = T_{\Sigma, s}$ , for every  $s \in S$ .
  - $T(\Sigma)(\omega) = n$ 
    - for every  $\omega = (n : \rightarrow s) \in \Omega$ .
  - $T(\Sigma)(\omega)(t_1, \dots, t_k) = n(t_1, \dots, t_k)$ 
    - for every  $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega, t_i \in T(\Sigma)(s_i)$ .

$T(\Sigma)$  is the algebra of (well-typed) ground terms of  $\Sigma$ .

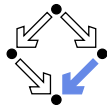
# Term Algebras



- Example:  $\text{NAT} = (\{\text{nat}\}, \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}\})$ .
  - $T(\text{NAT})(\text{nat}) = \{0, \text{Succ}(0), \text{Succ}(\text{Succ}(0)), \dots\}$ .
  - $T(\text{NAT})(0) = 0$ .
  - $T(\text{NAT})(\text{Succ})(t) = \text{Succ}(t)$ , for every  $t \in T(\text{NAT})(\text{nat})$ .
- Term value  $T(\Sigma)(t) = t$ , for every ground term  $t \in T(\Sigma)$ .
  - A ground term denotes itself.
- $T(\Sigma)$  is freely generated.
  - Generated: every value is denoted by itself.
  - Free: two different ground terms denote two different values.

In a term algebra, a ground term and its interpretation coincide.

# Initiality



Take signature  $\Sigma$ , class  $\mathcal{C} \subseteq \text{Alg}(\Sigma)$  of  $\Sigma$ -algebras, and  $\Sigma$ -algebra  $A \in \mathcal{C}$ .

- $A$  is **initial** in  $\mathcal{C}$  if
  - for every  $B \in \mathcal{C}$ , there exists exactly one homomorphism  $h : A \rightarrow B$ .
  - $A$  distinguishes most among all algebras of  $\mathcal{C}$ .
- Initial algebras are unique up to isomorphism:
  - If  $A$  is initial in  $\mathcal{C}$ , then  $B$  is initial in  $\mathcal{C}$  iff  $A \simeq B$ .
- **Theorem:**  $T(\Sigma)$  is initial in  $\text{Alg}(\Sigma)$ .
  - For every  $A \in \text{Alg}(\Sigma)$ , there exists the unique **evaluation homomorphism**:

$$h : T(\Sigma) \rightarrow A$$

$$h(t) := A(t), \text{ for every ground term } t \in T_{\Sigma}.$$

The term algebra  $T(\Sigma)$  distinguishes most among all  $\Sigma$ -algebras.



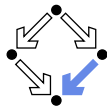
# Congruence Relation

Take signature  $\Sigma = (S, \Omega)$ ,  $\Sigma$ -algebra  $A$ .

- **Congruence relation**  $Q = (Q_s)_{s \in S}$  on  $A$ :
  - $Q_s$  is an equivalence relation on  $A(s)$  for every  $s \in S$ .
  - $(a_1, a'_1) \in Q_{s_1} \wedge \dots \wedge (a_k, a'_k) \in Q_{s_k} \Rightarrow$   
 $(A(\omega)(a_1, \dots, a_k), A(\omega)(a'_1, \dots, a'_k)) \in Q_s$ 
    - for every  $w = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega$ , and
    - for every  $a_1, a'_1 \in A(s_1), \dots, a_k, a'_k \in A(s_k)$ .
  - Equivalent arguments yield equivalent results.

**A congruence relation preserves equivalence across function applications.**

# Example



- BOOL-algebra  $D$ :

$$D(\text{bool}) = \mathbb{N}$$

$$D(\neg)(n) = \begin{cases} n + 1, & \text{if } n \text{ is even} \\ n - 1, & \text{otherwise} \end{cases}$$

$$D(\wedge)(n, m) = n * m$$

- $Q$  is a congruence relation on  $D$ .

$$(m, n) \in Q_{\text{bool}} :\Leftrightarrow m + n \text{ is even.}$$

- Take  $\omega = \neg : \text{bool} \rightarrow \text{bool}$ :

- Take  $n, n' \in D(\text{bool})$  with  $(n, n') \in Q_{\text{bool}}$ .

- We have to show  $(D(\neg)(n), D(\neg)(n')) \in Q_{\text{bool}}$ .

- $n + n'$  is even. Thus  $n$  and  $n'$  are either both even or both odd.

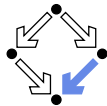
- Case 1: we have to show  $(n + 1, n' + 1) \in Q_{\text{bool}}$ , i.e.,  
 $(n + 1) + (n' + 1) = (n + n') + 2$  is even. ...

- Case 2: we have to show  $(n - 1, n' - 1) \in Q_{\text{bool}}$ , i.e.,  
 $(n - 1) + (n' - 1) = (n + n') - 2$  is even. ...

- Take  $\omega = \wedge : \text{bool} \times \text{bool} \rightarrow \text{bool}$ :

- ...

# Quotient Algebra

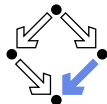


Take signature  $\Sigma = (S, \Omega)$ ,  $\Sigma$ -algebra  $A$ , congruence relation  $Q$  on  $A$ .

- **Quotient (algebra)  $A/Q$  of  $A$  by  $Q$ :**
  - $\Sigma$ -algebra whose values are congruence classes.
    - $[a]_Q = \{a' : (a, a') \in Q\}$ .
    - Class of  $a$  with respect to congruence relation  $Q$ .
  - $A/Q(s) = \{[a]_{Q_s} \mid a \in A(s)\}$ 
    - for every  $s \in S$ .
  - $A/Q(\omega) = [A(\omega)]_{Q_s}$ 
    - for every  $\omega = (n : \rightarrow s) \in \Omega$ .
  - $A/Q(\omega)([a_1]_{Q_{s_1}}, \dots, [a_k]_{Q_{s_k}}) = [A(\omega)(a_1, \dots, a_k)]_{Q_s}$ 
    - for every  $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega$ .

Congruent elements of  $A$  are combined to a single element of  $A/Q$ .

# Example



- BOOL-algebra  $D$  and congruence relation  $Q$  on  $D$  (as before).

$$(m, n) \in Q_{bool} :\Leftrightarrow m + n \text{ is even.}$$

- Quotient algebra  $D/Q$ :

$$[0] = \{n \in \mathbb{N} \mid 0 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is even}\}$$

$$[1] = \{n \in \mathbb{N} \mid 1 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is odd}\}$$

- $(D/Q)(bool) = \{[0], [1]\}$ .

- $(D/Q)(\neg)(n) = \begin{cases} [1] & \text{if } n = [0] \\ [0] & \text{if } n = [1] \end{cases}$

- $(D/Q)(\wedge)(n, m) = \begin{cases} [1] & \text{if } n = m = [1] \\ [0] & \text{else} \end{cases}$

- $(D/Q) \simeq C$

$$C(bool) = \{0, 1\}$$

$$C(True) = 1$$

$$C(False) = 0$$

$$C(\neg)(n) = 1 - n$$

$$C(\wedge)(n, m) = n * m$$



# Quotient Term Algebra



Take signature  $\Sigma = (S, \Omega)$  and class of algebras  $\mathcal{C} \subseteq \text{Alg}(\Sigma)$ .

- **Congruence relation**  $\equiv_{\mathcal{C}}$  of  $\mathcal{C}$ :

- $\equiv_{\mathcal{C}} := (\equiv_{\mathcal{C},s})_{s \in S}$ .
- $\equiv_{\mathcal{C},s} := \{(t, u) \in T_{\Sigma,s} \times T_{\Sigma,s} \mid \forall A \in \mathcal{C} : A(t) = A(u)\}$ .
- All ground terms are congruent that have the same value in all algebras of  $\mathcal{C}$ .

- **Quotient Term Algebra**  $T(\Sigma, \mathcal{C})$  of  $\mathcal{C}$ :

- $T(\Sigma, \mathcal{C}) := T(\Sigma) / \equiv_{\mathcal{C}}$ .
- $\Sigma$ -algebra whose values are congruence classes of ground terms of  $\Sigma$ .

- **Theorem:** If  $T(\Sigma, \mathcal{C}) \in \mathcal{C}$ , then  $T(\Sigma, \mathcal{C})$  is initial in  $\mathcal{C}$ .

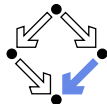
- For every  $A \in \mathcal{C}$ , there exists the unique **evaluation homomorphism**:

$$h : T(\Sigma, \mathcal{C}) \rightarrow A$$

$$h([t]) := A(t), \text{ for every ground term } t \in T_{\Sigma}.$$

$T(\Sigma, \mathcal{C})$  relates similarly to  $\mathcal{C}$  as  $T(\Sigma)$  relates to  $\text{Alg}(\Sigma)$ .

# Examples



- $T(\Sigma, Alg(\Sigma)) \simeq T(\Sigma)$ .
  - Values of  $T(\Sigma, Alg(\Sigma))$  are singletons  $[t] = \{t\}$  for every ground term  $t \in T_\Sigma$ .
- $T(\Sigma, \{A\}) \simeq A$ , for every  $\Sigma$ -algebra  $A$ .
  - Values of  $T(\Sigma, \{A\})$  are classes of all those terms that denote the same value in  $A$ .
- Let  $B$  be the “classical” NATBOOL-algebra.
  - Terms  $True$  and  $\neg False$  belong to the same value of  $T(\Sigma, \{B\})$ .
  - Terms  $0$  and  $0 + 0$  belong to the same value of  $T(\Sigma, \{B\})$ .

# Quotient Term Algebra of a Set of Formulas



Take logic  $L$ , signature  $\Sigma$ , set of formulas  $\Phi \subseteq L(\Sigma)$ .

- **Quotient term algebra**  $T(\Sigma, \Phi)$  of  $\Phi$ :
  - $T(\Sigma, \Phi) := T(\Sigma, \text{Mod}_\Sigma(\Phi)) (= T(\Sigma)/\equiv_{\text{Mod}_\Sigma(\Phi)})$ .
    - $\text{Mod}_\Sigma(\Phi) = \{A \in \text{Alg}(\Sigma) \mid A \text{ is a model of } \Phi\}$ .
    - $\equiv_{\text{Mod}_\Sigma(\Phi), s} = \{(t, u) \in T_{\Sigma, s} \times T_{\Sigma, s} \mid \forall A \in \text{Mod}_\Sigma(\Phi) : A(t) = A(u)\}$ .
  - $\Sigma$ -algebra whose values are classes of those terms that have the same value in all models of  $\Phi$ .
- **Theorem:** If  $T(\Sigma, \Phi)$  is model of  $\Phi$ ,  $T(\Sigma, \Phi)$  is initial in  $\text{Mod}_\Sigma(\Phi)$ .
  - For every model  $A$  of  $\Phi$ , there exists the unique **evaluation homomorphism**:
$$h : T(\Sigma, \Phi) \rightarrow A$$
$$h([t]) := A(t), \text{ for every ground term } t \in T_\Sigma.$$

**Basis of initial specification semantics.**