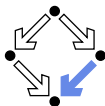


# Loose Specifications

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## 1. General Remarks

## 2. Loose Specifications

## 3. Loose Specifications with Constructors

## 4. Loose Specifications with Free Constructors

## 5. Summary

# Specifications



We will introduce various flavors of specifications of ADTs.

- **Specification semantics:**  $sp \rightarrow \mathcal{M}(sp)$ .
  - Specification  $sp$ .
  - Its meaning  $\mathcal{M}(sp)$  (an abstract datatype).
- $sp$  is an **adequate specification of an ADT  $\mathcal{C}$** :
  - $\mathcal{C} \subseteq \mathcal{M}(sp)$ .
- $sp$  is a **strictly adequate specification of an ADT  $\mathcal{C}$** :
  - $\mathcal{C} = \mathcal{M}(sp)$ .
- $sp$  is a **(strictly) adequate specification of an algebra  $A$** :
  - $sp$  is (strictly) adequate specification of the monomorphic ADT  $[A]$ .
- $sp$  is **polymorphic (monomorphic)**:
  - $sp$  defines a polymorphic (monomorphic) ADT.

**General notions independent of the kind of specification.**

# Properties of Specifications



- Is the specification inconsistent?
  - Is the specified ADT empty (i.e. does not contain any algebras)?
- Is the specification monomorphic?
  - Are all algebras of the specified ADT isomorphic?
- Are two specifications equivalent?
  - Do they specify the same ADT?
- Does the specification (strictly) adequately describe a given ADT?
  - Assumes that the ADT is mathematically defined by other means.
    - But specification itself is typically the *only* definition of the ADT.
    - Then no mathematical proof of adequacy is possible.
    - Nevertheless, by “executing the specifications” (mechanically evaluating ground terms), we may investigate the properties of the specified ADT to increase our confidence in its adequacy.

All these questions now have a precise meaning.



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1. General Remarks
  - 2. Loose Specifications**
  3. Loose Specifications with Constructors
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# Loose Specifications

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Take logic  $L$ .

- **Loose specification**  $sp = (\Sigma, \Phi)$  in  $L$ :
  - Signature  $\Sigma$ , set of formulas  $\Phi \subseteq L(\Sigma)$ .
- **Semantics**  $\mathcal{M}(sp) = Mod_{\Sigma}(\Phi)$ .
  - All  $\Sigma$ -algebras are candidates for the specified ADT.
    - $Mod_{\Sigma}(\Phi) = Mod_{Alg(\Sigma), \Sigma}(\Phi)$ .

A loose specification specifies as the abstract datatype the class of all models of its formula set.

# Concrete Syntax



**loose spec**

**sorts** *sort* ...

**opns** *operation* ...

**vars** *variable*: *sort* ...

**axioms** *formula* ...

**endspec**

- Signature  $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$ .
- Set of formulas  $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$ .

We will only use the concrete syntax to define specifications.

# Example



**loose spec**

**sorts** *el, bool, list*

**opns**

*True* :→ *bool*

*False* :→ *bool*

[ ] :→ *list*

*Add* : *el* × *list* → *list*

*\_ . \_* : *list* × *list* → *list*

**vars** *l, m* : *list*, *e* : *el*

**axioms**

[ ].*l* = *l*

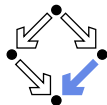
*Add*(*e, l*).*m* = *Add*(*e, l.m*)

**endspec**

Adequate specification of the “classical” list algebra in *EL*.



# Strict Adequacy



Not a *strictly* adequate specification of the “classical” list algebra.

- Carrier for *bool* may collapse (“confusion” among *True* and *False*).  
PL: **axiom**  $\neg(\text{True} = \text{False})$
- Carrier for *list* may collapse (“confusion” among  $[ ]$  and  $\text{Add}(e, l)$ ).  
PL: **axiom**  $\forall e : \text{el}, l : \text{list} . \neg([ ] = \text{Add}(e, l))$
- Size of lists may be bound (“confusion” among *Add* terms).  
PL: **axiom**  $\forall e_1, e_2 : \text{elem}, l_1, l_2 : \text{list} .$   
 $\text{Add}(e_1, l_1) = \text{Add}(e_2, l_2) \Rightarrow e_1 = e_2 \wedge l_1 = l_2$
- Carriers may contain extra values (“junk”).
  - There may a *bool* value different from *True* and *False*.  
PL: **axiom**  $\forall b : \text{bool} . b = \text{True} \vee b = \text{False}$
  - There may be *list* values different from those that can be constructed by application of  $[ ]$  and *Add*.
    - No axiom can express this in *PL*, a solution will be later presented.

In *PL* (not *EL* or *CEL*), additional axioms may solve *some* problems of “junk” and “confusion”.

# Example



**loose spec**

**sorts**  $el, bool, list$

**opns**

$True : \rightarrow bool$

$False : \rightarrow bool$

$[ ] : \rightarrow list$

$Add : el \times list \rightarrow list$

$.. : list \times list \rightarrow list$

**vars**  $l, l_1, l_2 : list, e, e_1, e_2 : el, b : bool$

**axioms**

$\neg(True = False)$

$b = True \vee b = False$

$\neg([ ] = Add(e, l))$

$Add(e_1, l_1) = Add(e_2, l_2) \Rightarrow$   
 $e_1 = e_2 \wedge l_1 = l_2$

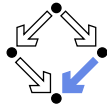
$[ ].l = l$

$Add(e, l_1).l_2 = Add(e, l_1.l_2)$

**endspec**

More (but still not strictly) adequate specification of the “classical” list algebra in *PL*.

# Example



**loose spec**

**sorts** *bool, nat*

**opns**

*True* :→ *bool*

*False* :→ *bool*

*0* :→ *nat*

*Succ* : *nat* → *nat*

- + - : *nat* × *nat* → *nat*

- \* - : *nat* × *nat* → *nat*

- ≤ - : *nat* × *nat* → *bool*

**vars** *m, n : nat, b : bool*

**axioms**

$\neg(\text{True} = \text{False})$

$b = \text{True} \vee b = \text{False}$

$\neg(0 = \text{Succ}(n))$

$\text{Succ}(n) = \text{Succ}(m) \Rightarrow n = m$

$(0 \leq n) = \text{True}$

$(\text{Succ}(n) \leq 0) = \text{False}$

$(\text{Succ}(n) \leq \text{Succ}(m)) = (n \leq m)$

$n + 0 = n$

$n + \text{Succ}(m) = \text{Succ}(n + m)$

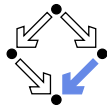
$n * 0 = 0$

$n * \text{Succ}(m) = n + (n * m)$

**endspec**

Adequate specification of Peano arithmetic in *PL* (not strictly adequate because *nat* may contain junk).

# Proving Strategies for Loose Specifications



Take loose specification  $sp = (\Sigma, \Phi)$  in logic  $L$  with inference calculus  $\vdash$ .

- Prove:  $\mathcal{M}(sp) \models \varphi$ .
  - Every implementation of the specification  $sp$  has the property expressed by formula  $\varphi$ .
  - It suffices to prove  $\Phi \vdash \varphi$ .
    - Formula  $\varphi$  can be derived from the specification axioms  $\Phi$ .
- Prove:  $\mathcal{M}(sp) \subseteq \mathcal{M}(sp')$ .
  - Loose specification  $sp' = (\Sigma, \Psi)$ .
  - Every implementation of the specification  $sp$  is also an implementation of the specification  $sp'$ .
  - It suffices to prove  $\Phi \vdash \Psi$ .
    - Every axiom  $\psi \in \Psi$  can be derived from the axioms  $\Phi$ .

**Straight-forward reduction of semantic questions to proving.**

# Expressive Power of Loose Specifications

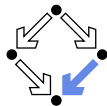


Take loose specification  $sp = (\Sigma, \Phi)$  with  $\Phi \subseteq L(\Sigma)$ .

- **Theorem 1:**  $\mathcal{M}(sp) = Mod_{\Sigma}(Th_L(\mathcal{M}(sp)))$ .
  - $Th_L(\mathcal{C}) = \{\phi \in L(\Sigma) \mid \forall A \in \mathcal{C} : A \models_{\Sigma} \phi\}$ .
    - The theory of a class of algebras w.r.t. a given logic is the set of all formulas of that logic that are satisfied by every algebra of the class.
    - Thus  $sp$  can specify an ADT  $\mathcal{C}$  only if  $\mathcal{C} = Mod_{\Sigma}(Th_L(\mathcal{C}))$ .
- **Example:**
  - Signature  $NAT = (\{nat\}, \{0 : \rightarrow nat, s : nat \rightarrow nat\})$ .
  - NAT-algebra  $N = (\{\mathbb{N}\}, \{0_{\mathbb{N}}, (\lambda x . x + 1)\})$ .
  - $[N]$  cannot be specified by any specification  $sp$  in  $EL(NAT)$ .
    - Assume specification  $sp$  with  $\mathcal{M}(sp) = [N]$ .
    - $Th_{EL}([N]) = \{0 = 0, s(0) = s(0), s(s(0)) = s(s(0)), \dots\}$ .
    - Take NAT-algebra  $A = (\{0, 1\}, 0, \lambda x . 1 - x)$
    - Clearly  $A \not\cong N$ , thus  $A \notin \mathcal{M}(sp)$ .
    - But, since  $A \models Th_{EL}(\{N\})$ ,  $A \in Mod_{\Sigma}(Th_{EL}([N]))$ , and thus, by Theorem 1,  $A \in \mathcal{M}(sp)$ .

Algebras can be discriminated only by the expressible formulas.

# Expressive Power of Loose Specifications

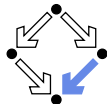


Take loose specification  $sp = (\Sigma, \Phi)$  with  $\Phi \subseteq L(\Sigma)$ .

- **Theorem 2:** If  $L$  has a sound and complete calculus and if  $\Phi$  is recursively enumerable, then  $\mathcal{M}(sp)$  is axiomatizable in  $L$ .
  - Set  $S$  is recursively enumerable, if there is an algorithm that lists all of its elements (running forever, if necessary).
  - A class  $\mathcal{C}$  of  $\Sigma$ -algebras is axiomatizable in  $L$ , if  $Th_L(\mathcal{C})$  is recursively enumerable.
- An ADT whose theory is not recursively enumerable in the given logic, may not be specifiable by a loose specification.
  - Example: Peano arithmetic (natural numbers with addition and multiplication).
  - The theory of peano arithmetic is not recursively enumerable in first-order predicate logic.
  - Gödel's second incompleteness theorem: Peano arithmetic is not axiomatizable in first-order predicate logic.

**Not every ADT can be specified by a loose specification.**

# Expressive Power of Loose Specifications



Take loose specification  $sp = (\Sigma, \Phi)$  with  $\Phi \subseteq L(\Sigma)$ .

- **Theorem 3:** If  $L$  is  $EL$  or  $CEL$ , then  $M(sp)$  also contains algebras whose carriers are singletons (i.e., whose terms are “confused”).
  - **Consequence:** No ADT with non-singleton carriers can be strictly adequately described by a loose specification in  $EL$  or  $CEL$ .
    - Cannot prevent “collapse” of the carrier.
- **Theorem 4:** If  $L$  is  $EL$ ,  $CEL$ , or  $PL$  and  $\mathcal{M}(sp)$  contains an algebra with an infinite carrier, then  $M(sp)$  also contains algebras whose corresponding carriers contain “junk”.
  - **Consequence:** No ADT with an infinite carrier can be strictly adequately described by a loose specification in  $EL$ ,  $CEL$ , or  $PL$ .
    - Cannot rule out “extra” values in addition to the desired ones.

We need some more mechanisms for strictly adequate specifications.



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# Generated Algebras



Take signature  $\Sigma = (S, \Omega)$ ,  $\Sigma$ -algebra  $A$ .

- Define set of operations  $\Omega_c \subseteq \Omega$  (the **constructors**).
  - Restricted signature  $\Sigma_c = (S, \Omega_c)$ .
- $A$  is **generated by**  $\Omega_c$ :
  - For each sort  $s \in S$  and  $a \in A(s)$ , there exists a ground term  $t \in T_{\Sigma_c, s}$  with  $a = A(t)$ .
    - Carrier  $a$  can be described by a term  $t$  that involves only constructors.
  - $A$  is **generated** if it is generated by  $\Omega$ .
- $Gen(\Sigma, \Omega_c) := \{A \in Alg(\Sigma) \mid A \text{ is generated by } \Omega_c\}$ .
  - The set of all  $\Sigma$ -algebras generated by constructors  $\Omega_c$ .
  - $Gen(\Sigma) := Gen(\Sigma, \Omega)$ .

Generated algebra does not contain “junk” in the carriers.



## Example

Take signature

$\text{NAT} = (\{\text{nat}\}, \Omega = \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}, + : \text{nat} \times \text{nat} \rightarrow \text{nat}\})$ .

- Classical NAT-algebra  $A = (\mathbb{N}, 0_{\mathbb{N}}, +_{\mathbb{N}})$ .
- Constructors  $\Omega_c := \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}\}$ .
- $A$  is generated by  $\Omega_c$ :
  - For every  $n \in \mathbb{N}$ ,  $n = A(\underbrace{s(s(s(\dots(s(0))))}_{n \text{ times}}))$ .
- $A$  is also generated by  $\Omega$ .
  - Any superset of a set of constructors is also a set of constructors.

Usually one looks for the minimal set of constructors.



# Algebras Generated in Some Sorts

Take signature  $\Sigma = (S, \Omega)$ ,  $\Sigma$ -algebra  $A$ .

- Define set of sorts  $S_c \subseteq S$  and set of operations  $\Omega_c \subseteq \Omega$  (the **constructors**) with target sorts in  $S_c$ .
  - Restricted signature  $\Sigma_c = (S, \Omega_c)$ .
- $A$  is **generated by**  $\Omega_c$  in  $S_c$ :
  - For each sort  $s \in S_c$  and  $a \in A(s)$ , there exists
    - a set  $X$  of variables in  $\Sigma$  with  $X_s = \emptyset$  for every  $s$  in  $S_c$ ,
    - an assignment  $\alpha : X \rightarrow A$ ,
    - and a term  $t \in T_{\Sigma_c(X), s}$with  $a = A(\alpha)(t)$ .
    - Value  $a$  can be described by a term  $t$  that involves only constructors in the generated sorts and variables in the non-generated sorts.
- $A$  is **generated in**  $S_c$  if it is generated in  $S_c$  by  $\Omega$ .

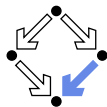
**Algebra does not contain “junk” in the carriers of the generated sorts.**

# Example



- Signature  $LIST = (S, \Omega)$ :
  - $S = \{el, list\}$ .
  - $\Omega = \{[ ] : \rightarrow list, Add : el \times list \rightarrow list, \_ \cdot \_ : list \times list \rightarrow list\}$ .
- LIST-algebra  $A$ :
  - $A(el)$  ... a set of "elements".
  - $A(list)$  ... the set of finite lists of elements.
  - $A([ ])$  ... the empty list.
  - $A(Add)$  adds an element at the front of the list.
  - $A(\cdot)$  concatenates two lists.
- $A$  is generated by  $\Omega_c = \{[ ], Add\}$  in  $S_c = \{list\}$ :
  - Take arbitrary  $l = [e_1, e_2, \dots, e_n] \in A(list)$ .
  - Define  $X_{el} := \{x_1, x_2, \dots, x_n\}$ .
  - Define  $\alpha_{el} := [x_1 \mapsto e_1, x_2 \mapsto e_2, \dots, x_n \mapsto e_n]$ .
  - Then  $l = A(\alpha)(Add(x_1, Add(x_2, \dots, Add(x_n, [ ]))))$ .

# Proofs by Induction



In generated sorts, the principle of structural induction can be applied.

- Take the LIST-algebra  $A$  of the previous example.
  - Notation:  $c_A$  for  $A(c)$ .
  - Knowledge: (1)  $\forall l \in list_A : [ ]_A \cdot_A l = l$ .  
(2)  $\forall e \in el_A, l, r \in list_A :$   
 $Add_A(e, l) \cdot_A r = Add_A(e, l \cdot_A r)$ .
- Prove:  $\forall l \in list_A : l \cdot_A [ ]_A = l$ .
- Induction base  $l = [ ]_A$ :
  - $l \cdot_A [ ]_A = [ ]_A \cdot_A [ ]_A \stackrel{(1)}{=} [ ]_A = l$ .
- Induction step  $l = Add_A(e, r)$  (for some  $e \in el_A, r \in list_A$ ).
  - Induction Hypothesis (H):  $r \cdot_A [ ]_A = r$ .
  - $l \cdot_A [ ]_A = Add_A(e, r) \cdot_A [ ]_A$   
 $\stackrel{(2)}{=} Add_A(e, r \cdot_A [ ]_A)$   
 $\stackrel{(H)}{=} Add_A(e, r) = l$ .

# Loose Specifications with Constructors



Take logic  $L$ .

- **Loose specification with constructors**  $sp = (\Sigma, \Phi, S_c, \Omega_c)$  in  $L$ :
  - Signature  $\Sigma = (S, \Omega)$ , set of formulas  $\Phi \subseteq L(\Sigma)$ , **generated sorts**  $S_c \subseteq S$ , **constructors**  $\Omega_c \subseteq \Omega$  with target sorts in  $S_c$ .
  - **Semantics**  $\mathcal{M}(sp) = Mod_{\mathcal{U}, \Sigma}(\Phi)$  where  $\mathcal{U} = \{A \in Alg(\Sigma) \mid A \text{ is generated in } S_c \text{ by } \Omega_c\}$ .
  - Only generated  $\Sigma$ -algebras are candidates for the specified ADT.

A loose specification with constructors specifies as the ADT the class of all models of its formula set that are generated by the constructors.



**loose spec**

**sorts** [ **generated** ] *sort* ...

**opns** [ **constr** ] *operation* ...

**vars** *variable: sort* ...

**axioms** *formula* ...

**endspec**

- Signature  $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$ .
- Set of formulas  $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$ .
- Generated sorts  $\mathcal{S}_c = \{\mathbf{generated} \textit{ sort}, \dots\}$ .
- Constructors  $\Omega_c = \{\mathbf{constr} \textit{ operation}, \dots\}$ .

We will only use the concrete syntax to define specifications.

# Example



loose spec

sorts *el*

generated *bool*

generated *list*

opns

constr *True* :  $\rightarrow$  *bool*

constr *False* :  $\rightarrow$  *bool*

constr *[ ]* :  $\rightarrow$  *list*

constr *Add* : *el*  $\times$  *list*  $\rightarrow$  *list*

*\_ . \_* : *list*  $\times$  *list*  $\rightarrow$  *list*

vars *l, m* : *list*, *e, e<sub>1</sub>, e<sub>2</sub>* : *el*

axioms

$\neg(\text{True} = \text{False})$

$\neg([\ ] = \text{Add}(e, l))$

$\text{Add}(e_1, l_1) = \text{Add}(e_2, l_2) \Rightarrow e_1 = e_2$

$[\ ].l = l$

$\text{Add}(e, l).m = \text{Add}(e, l.m)$

endspec

Strictly adequate specification of the “classical” list algebra in *PL*.



# Example



**loose spec**

**sorts**

**generated** *bool*

**generated** *nat*

**opns**

**constr** *True* :→ *bool*

**constr** *False* :→ *bool*

**constr** *0* :→ *nat*

**constr** *Succ* : *nat* → *nat*

- *+* *\_* : *nat* × *nat* → *nat*

- *\** *\_* : *nat* × *nat* → *nat*

- *≤* *\_* : *nat* × *nat* → *bool*

**vars** *m, n* : *nat*

**axioms**

$\neg(\text{True} = \text{False})$

$\neg(0 = \text{Succ}(n))$

$\text{Succ}(n) = \text{Succ}(m) \Rightarrow n = m$

$(0 \leq n) = \text{True}$

$(\text{Succ}(n) \leq 0) = \text{False}$

$(\text{Succ}(n) \leq \text{Succ}(m)) = (n \leq m)$

$n + 0 = n$

$n + \text{Succ}(m) = \text{Succ}(n + m)$

$n * 0 = 0$

$n * \text{Succ}(m) = n + (n * m)$

**endspec**

Strictly adequate specification of Peano arithmetic in *PL*.



# Specified ADT is Strictly Adequate

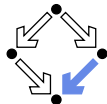
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Proof requires two parts.

- Peano arithmetic satisfies the specified axioms.
  - Can be easily checked.
- Specified ADT is monomorphic:  $\forall B, C \in \mathcal{M}(sp) : B \simeq C$ .
  - There is an isomorphism  $h : B \rightarrow C$ .
    - A bijective homomorphism.
  - Definition of unique term representation for every value.
    - Simplifies the remainder of the proof.
  - Definition of bijective mapping  $h$ :
    - By pattern matching on term representation.
  - Proof that  $h$  is a homomorphism:
    - By using properties expressed with the help of the term representation.

Term representation essential for this kind of proofs.

# Values have Unique Term Representations



Take arbitrary  $A \in \mathcal{M}(sp)$ .

- $bool_A = \{True_A, False_A\}$  and  $True_A \neq False_A$ .
  - $A$  is generated by  $\{True, False\}$  in  $bool$ .
  - **axiom**  $\neg(True = False)$ .
- $nat_A = \{Succ^k(0)_A : k \in \mathbb{N}\}$  and  $\forall k \neq l : Succ^k(0)_A \neq Succ^l(0)_A$ .
  - $A$  is generated by  $\{0, Succ\}$  in  $nat$ .
  - Proof by induction on  $k$ :  $\forall l \neq k : Succ^k(0)_A \neq Succ^l(0)_A$ .
    - $k = 0, l \neq 0$ :  $0_A \neq Succ^l(0)_A$  (by **axiom**  $\neg(0 = Succ(n))$ ).
    - $k \neq 0, l \neq k$ : assume  $Succ^k(0)_A = Succ^l(0)_A$ , show  $k = l$ .
      - Know  $l \neq 0$  (by **axiom**  $\neg(0 = Succ(n))$ ).
      - Thus  $k = k' + 1, l = l' + 1$ , it suffices to show  $k' = l'$ .
      - By assumption,  $Succ(Succ^{k'}(0))_A = Succ(Succ^{l'}(0))_A$ .
      - Thus  $Succ^{k'}(0)_A = Succ^{l'}(0)_A$  (**axiom**  $Succ(n) = Succ(m) \Rightarrow n = m$ ).
      - By induction hypothesis,  $k' = l'$ .

Values are uniquely described by constructor applications.

# Definition of Bijective Mapping

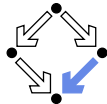


Take arbitrary  $B, C \in \mathcal{M}(sp)$ .

- $h$  is defined by **pattern matching** on constructor terms:
  - $h_{bool}(True_B) := True_C$ .
  - $h_{bool}(False_B) := False_C$ .
  - $h_{nat}(Succ^k(0)_B) = Succ^k(0)_C$ , for all  $k \geq 0$ .
- $h$  is consistently defined:
  - $True_B$  and  $False_B$  denote different values.
  - $Succ^k(0)_B$  denote different values for different  $k$ .
- $h$  is bijective:
  - $True_C$  and  $False_C$  denote different values.
  - $Succ^k(0)_C$  denote different values for different  $k$ .

**One-to-one correspondence between the carriers of  $B$  and  $C$ .**

# Homomorphism Proof



- Clear for constructors *True*, *False*, *0*, *Succ*:
  - Definition of  $h$  already expresses homomorphism condition.
- **Goal:**  $\forall m, n \in \text{nat}_B . h(\text{op}_B(m, n)) = \text{op}_C(h(m), h(n))$ .  
 $\text{op} \dots +, *, \leq$ .
  - $\forall k, l \geq 0 . h(\text{op}_B(\text{Succ}^k(0)_B, \text{Succ}^l(0)_B)) = \text{op}_C(h(\text{Succ}^k(0)_B), h(\text{Succ}^l(0)_B))$ .
    - $B$  and  $C$  are generated by  $\{0, \text{Succ}\}$  in  $\text{nat}$ .
  - $\forall k, l \geq 0 . h(\text{op}_B(\text{Succ}^k(0)_B, \text{Succ}^l(0)_B)) = \text{op}_C(\text{Succ}^k(0)_C, \text{Succ}^l(0)_C)$ .
    - By definition of  $h$ .
  - $\forall k, l \geq 0 . h(\text{op}(\text{Succ}^k(0), \text{Succ}^l(0))_B) = \text{op}(\text{Succ}^k(0), \text{Succ}^l(0))_C$ .
    - By definition of term semantics.

Proof goal is expressed with the help of constructor terms.



# Homomorphism Proof

The core of the homomorphism proof.

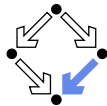
- **Goal:**  $h((\text{Succ}^k(0) + \text{Succ}^l(0))_B) = (\text{Succ}^k(0) + \text{Succ}^l(0))_C$ .
  - First simplify left and right hand side of the equation.
- **Lemma:**  $\forall A \in \mathcal{M}(sp) : (\text{Succ}^k(0) + \text{Succ}^l(0))_A = \text{Succ}^{k+l}(0)_A$ .
  - Induction base  $l = 0$ : by **axiom**  $n + 0 = n$ .
  - Induction step  $l = l' + 1$ :
$$\begin{aligned} & (\text{Succ}^k(0) + \text{Succ}^{l'+1}(0))_A \\ &= \text{Succ}(\text{Succ}^k(0) + \text{Succ}^{l'}(0))_A \\ &= \text{Succ}(\text{Succ}^{k+l'}(0))_A \\ &= \text{Succ}^{k+l'+1}(0)_A. \end{aligned}$$
- **Simplified goal:**  $h(\text{Succ}^{k+l}(0)_B) = \text{Succ}^{k+l}(0)_C$ .
  - By definition of  $h$ .

Similar for the homomorphism proofs of the other operations.



- 
1. General Remarks
  2. Loose Specifications
  3. Loose Specifications with Constructors
  - 4. Loose Specifications with Free Constructors**
  5. Summary

# Freely Generated Algebras



Take signature  $\Sigma = (S, \Omega)$ ,  $\Sigma$ -algebra  $A$ .

- Define set of operations  $\Omega_c \subseteq \Omega$  (the **constructors**).
  - Restricted signature  $\Sigma_c = (S, \Omega_c)$ .
- $A$  is **freely generated by**  $\Omega_c$ :
  - For each sort  $s \in S$  and  $a \in A(s)$ , there exists **exactly one** ground term  $t \in T_{\Sigma_c, s}$  with  $a = A(t)$ .
    - Value  $a$  can be described by a **unique** term  $t$  that involves only constructors.
  - $A$  is **freely generated** if it is generated by  $\Omega$ .
- $A$  is **freely generated by**  $\Omega_c$  **in**  $S_c$ :
  - Analogous definition as for **generated by ... in ...**.

Freely generated algebras have unique constructor term representations for the values of the freely generated sorts (no “junk” in carriers and no “confusion” among constructor terms).





# Example

- The “classical” BOOL-algebra  $(\{true, false\}, \dots)$ :
  - Freely generated by  $\{True, False\}$ .
  - Not freely generated by  $\{True, False, \neg\}$ .
- The “one-element” BOOL-algebra  $(\{\#\}, \dots)$ .
  - Freely generated by  $\{True\}$  and by  $\{False\}$ .
  - Not freely generated by  $\{True, False\}$ .
- The “classical” NAT-algebra  $(\mathbb{N}, \dots)$ :
  - Freely generated by  $\{0, Succ\}$ .
  - Not freely generated by  $\{0, Succ, +\}$ .
- The “classical” INT-algebra  $(\mathbb{Z}, \dots)$ :
  - $INT = (int, \{0 : \rightarrow int, Succ : int \rightarrow int, Pred : int \rightarrow int\})$ .
  - Not freely generated by (any subset of) operations.

A set of free constructors cannot be extended.



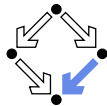
# Inductive Function Definitions

Freely generated algebras allow inductive function definitions.

- Signature  $\text{LIST} = (S, \Omega)$ :
  - $S = \{el, list\}$ .
  - $\Omega = \{[ ] : \rightarrow list, Add : el \times list \rightarrow list, \_ \cdot \_ : list \times list \rightarrow list\}$ .
- Classical LIST-algebra  $A$  as in the previous example.
  - $A$  is freely generated by  $\Omega_c = \{[ ], Add\}$  in  $S_c = \{list\}$ :
- Inductive definition of function  $g : A(list) \rightarrow \mathbb{N}$ .
  - $g([ ]_A) = 0$ .
  - $g(Add(x, t)_A) = g(t_A) + 1$  for all  $x \in X, t \in T_{\Sigma_c(X), list}$ .

Inductive definition by “pattern matching” on constructor terms (independent of the nature of the carrier).

# Loose Specifications with Free Constructors



Take logic  $L$ .

- **Loose specification with free constructors**  $sp = (\Sigma, \Phi, S_c, \Omega_c)$  in  $L$ :
  - Signature  $\Sigma = (S, \Omega)$ , set of formulas  $\Phi \subseteq L(\Sigma)$ , **freely generated sorts**  $S_c \subseteq S$ , **constructors**  $\Omega_c \subseteq \Omega$  with target sorts in  $S_c$ .
  - **Semantics**  $\mathcal{M}(sp) = \text{Mod}_{\mathcal{U}, \Sigma}(\Phi)$  where
$$\mathcal{U} = \{A \in \text{Alg}(\Sigma) \mid A \text{ is freely generated in } S_c \text{ by } \Omega_c\}.$$
    - Only freely generated  $\Sigma$ -algebras are candidates for the specified ADT.

**A loose specification with free constructors specifies the class of all models of its formula set that are freely generated by the constructors.**



**loose spec**

**sorts** [ **freely generated** ] *sort* ...

**opns** [ **constr** ] *operation* ...

**vars** *variable: sort* ...

**axioms** *formula* ...

**endspec**

- Signature  $\Sigma = (\{\textit{sort}, \dots\}, \{\textit{operation}, \dots\})$ .
- Set of formulas  $\Phi = \{(\forall \textit{variable} : \textit{sort}, \dots . \textit{formula}), \dots\}$ .
- Generated sorts  $\mathcal{S}_c = \{\mathbf{freely\ generated\ sort}, \dots\}$ .
- Constructors  $\Omega_c = \{\mathbf{constr\ operation}, \dots\}$ .

Also mixing of generated sorts with freely generated sorts possible.

# Example



**loose spec**

**sorts** *el*

**freely generated** *bool*

**freely generated** *list*

**opns**

**constr** *True* :→ *bool*

**constr** *False* :→ *bool*

**constr** *[ ]* :→ *list*

**constr** *Add* : *el* × *list* → *list*

*\_ . \_* : *list* × *list* → *list*

**vars** *l, m* : *list*, *e, e<sub>1</sub>, e<sub>2</sub>* : *el*

**axioms**

*[ ].l* = *l*

*Add(e, l).m* = *Add(e, l.m)*

**endspec**

Strictly adequate specification of the “classical” list algebra in EL; the non-constructor operation is inductively defined.

# Example



**loose spec**

**sorts**

**freely generated** *bool*

**freely generated** *nat*

**opns**

**constr** *True* :→ *bool*

**constr** *False* :→ *bool*

**constr** *0* :→ *nat*

**constr** *Succ* : *nat* → *nat*

- *+* *\_* : *nat* × *nat* → *nat*

- *\** *\_* : *nat* × *nat* → *nat*

- *≤* *\_* : *nat* × *nat* → *bool*

**vars** *m, n* : *nat*

**axioms**

$(0 \leq n) = \text{True}$

$(\text{Succ}(n) \leq 0) = \text{False}$

$(\text{Succ}(n) \leq \text{Succ}(m)) = (n \leq m)$

$n + 0 = n$

$n + \text{Succ}(m) = \text{Succ}(n + m)$

$n * 0 = 0$

$n * \text{Succ}(m) = n + (n * m)$

**endspec**

Strictly adequate specification of the “classical” list algebra in EL; the non-constructor operations are inductively defined.



- 
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# Summary



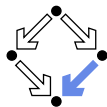
A couple of core messages. . .

- A loose specification describes a class of models as an ADT.
  - To check whether a given algebra implements the specification (i.e., whether it is an element of the specified ADT):
    - Check whether the algebra satisfies the specification axioms.
  - There may exist “confusion” among terms.
    - Carriers may collapse to singletons (or be too “small”).
    - In *PL*, additional axioms can prevent this.
    - Non-equalities of operation results (injectiveness of operations).
  - Carriers may contain “junk”.
    - In *PL*, an additional axiom can prevent this for a finite carrier.
    - Axiom enumerates constants that denote all *s* of the sort.

Without constructors, loose specifications are generally clumsy because many “boring” axioms are needed.



# Summary (Contd)



- Loose specifications with **constructors**.
  - Every value is denoted by **some** constructor term.
  - Thus junk is removed from (also infinite) carriers.
  - Induction proofs on term representation of  $s$  become possible.
  - Problem: not all carriers allow term representations.
    - ADT “real” (carrier is not countable).
- Loose specifications with **free constructors**.
  - Every value is denoted by **exactly one** constructor term.
  - Thus there is no “confusion” among constructor terms and the collapse of carriers is prevented.
  - Inductive function definitions by pattern matching on term representations of  $s$  become possible.
  - Problem: not all carriers have unique term representations.
    - ADT “set” (no unique representation at all).
    - ADT “integer” (unique representation is unconvient).

With constructors, loose specifications become easy to use.

# Summary (Contd)



So what is the role of loose specifications. . .

- Loose specifications are **good** for specifying **requirements**.
  - May specify zero, one, many datatypes (polymorphic ADTs).
  - Thus allow arbitrarily many implementations.
    - A loose specification may not have any model (implementation) at all!
  - Specification axioms can (should) be abstract.
    - Later verification that concrete implementation satisfies the axioms.
- Loose specifications are **not good** for specifying **designs**.
  - Not descriptions of concrete algorithms/implementations.
- Loose specifications are generally **not executable**.
  - No engines to execute loose specifications for rapid prototyping.

Loose specifications are for *reasoning*, not for *executing*; they are the basis of program specification languages such as Larch/C++.