Logic and Proving

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1. **The Language of Logic**

2. The Art of Proving

3. The RISC ProofNavigator

The Language of Logic

Two kinds of syntactic phrases.

- Term T denoting an object.
  - Variable x
  - Object constant c
  - Function application $f(T_1, \ldots, T_n)$ (may be written infix)

    n-ary function constant f
- Formula F denoting a truth value.
  - Atomic formula $p(T_1, \ldots, T_n)$ (may be written infix)

    n-ary predicate constant p.
  - Negation $\neg F$ ("not F")
  - Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - Implication $F_1 \Rightarrow F_2$ ("if F_1, then F_2")
  - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1, then F_2, and vice versa")
  - Universal quantification $\forall x : F$ ("for all x, F")
  - Existential quantification $\exists x : F$ ("for some x, F")

Syntactic Shortcuts

- $\forall x_1, \ldots, x_n : F$
  - $\forall x_1 : \ldots : \forall x_n : F$
- $\exists x_1, \ldots, x_n : F$
  - $\exists x_1 : \ldots : \exists x_n : F$
- $\forall x \in S : F$
  - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
  - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

Examples

Terms and formulas may appear in various syntactic forms.

- Terms:

$$\exp(x)$$
$$a \cdot b + 1$$
$$a[i] \cdot b$$
$$\sqrt{\frac{x^2+2x+1}{(y+1)^2}}$$

- Formulas:

$$a^2 + b^2 = c^2$$
$$n \mid 2n$$
$$\forall x \in \mathbb{N} : x \geq 0$$
$$\forall x \in \mathbb{N} : 2|x \vee 2|(x+1)$$
$$\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow \\ \exists z \in \mathbb{N} : x + z = y$$

Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas

- Atomic formula $p(T_1, \ldots, T_n)$
  - True if the predicate denoted by p holds for the values of $T_1, \ldots, T_n$.
- Negation $\neg F$
  - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ ("if F_1, then F_2")
  - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1, then F_2, and vice versa")
  - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ ("for all x, F")
  - True if and only if F is true for every possible value assignment of x.
- Existential quantification $\exists x : F$ ("for some x, F")
  - True if and only if F is true for at least one value assignment of x.

Example

We assume the domain of natural numbers and the "classical" interpretation of constants 1, 2, $+$, $=$, $<$.

- $1 + 1 = 2$
  - True.
- $1 + 1 = 2 \vee 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \wedge 2 + 2 = 2$
  - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
  - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
  - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
  - True.

Example

- $x + 1 = 1 + x$
  - True, for every assignment of a number a to variable x.
- $\forall x : x + 1 = 1 + x$
  - True (because for every assignment a to x, $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
  - If x is assigned "one", the formula is true.
  - If x is assigned "two", the formula is false.
- $\exists x : x + 1 = 2$
  - True (because $x + 1 = 2$ is true for assignment "one" to x).
- $\forall x : x + 1 = 2$
  - False (because $x + 1 = 2$ is false for assignment "two" to x).
- $\forall x : \exists y : x < y$
  - True (because for every assignment a to x, there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
  - False (because for every assignment a to y, there is the assignment $a + 1$ to x which makes $x < y$ false).

Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \leftrightsquigarrow F_1$
- $\neg(F_1 \wedge F_2) \leftrightsquigarrow \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \leftrightsquigarrow \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \leftrightsquigarrow F_1 \wedge \neg F_2$
- $\neg\forall x : F \leftrightsquigarrow \exists x : \neg F$
- $\neg\exists x : F \leftrightsquigarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \leftrightsquigarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.

Example

- "All swans are white or black."
  - $\forall x : \mathit{swan}(x) \Rightarrow \mathit{white}(x) \vee \mathit{black}(x)$
- "There exists a black swan."
  - $\exists x : \mathit{swan}(x) \wedge \mathit{black}(x)$.
- "A swan is white, unless it is black."
  - $\forall x : \mathit{swan}(x) \wedge \neg \mathit{black}(x) \Rightarrow \mathit{white}(x)$
  - $\forall x : \mathit{swan}(x) \wedge \neg \mathit{white}(x) \Rightarrow \mathit{black}(x)$
  - $\forall x : \mathit{swan}(x) \Rightarrow \mathit{white}(x) \vee \mathit{black}(x)$
- "Not everything that is white or black is a swan."
  - $\neg \forall x : \mathit{white}(x) \vee \mathit{black}(x) \Rightarrow \mathit{swan}(x)$.
  - $\exists x : (\mathit{white}(x) \vee \mathit{black}(x)) \wedge \neg \mathit{swan}(x)$.
- "Black swans have at least one black parent".
  - $\forall x : \mathit{swan}(x) \wedge \mathit{black}(x) \Rightarrow \exists y : \mathit{swan}(y) \wedge \mathit{black}(y) \wedge \mathit{parent}(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.

The Usage of Formulas

Precise formulation of statements describing object relationships.

- Statement:

  If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y.

- Formula:

  $$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow$$
  $$\quad q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$$

- Problem specification:

  Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y.

  - Inputs: x, y
  - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
  - Output: q
  - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$

Problem Specifications

- The specification of a computation problem:
  - Input: variables $x_1 \in S_1, \ldots, x_n \in S_n$
  - Input condition: formula $I(x_1, \ldots, x_n)$.
  - Output: variables $y_1 \in T_1, \ldots, y_m \in T_n$
  - Output condition: formula $O(x_1, \ldots, x_n, y_1, \ldots, y_m)$.
    - $F(x_1, \ldots, x_n)$: only $x_1, \ldots, x_n$ are free in F.
    - x is *free* in F, if not every occurrence of x is inside the scope of a quantifier (such as $\forall$ or $\exists$) that binds x.
- An implementation of the specification:
  - A function (program) $f : S_1 \times \ldots \times S_n \to T_1 \times \ldots \times T_m$ such that

$$\begin{array}{l}\forall x_1 \in S_1, \ldots, x_n \in S_n : I(x_1, \ldots, x_n) \Rightarrow \\ \qquad \textbf{let } (y_1, \ldots, y_m) = f(x_1, \ldots, x_n) \textbf{ in} \\ \qquad O(x_1, \ldots, x_n, y_1, \ldots, y_m)\end{array}$$

  - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.

Example: A Problem Specification

Given an integer array a, a position p in a, and a length l, return the array b derived from a by removing $a[p], \ldots, a[p+l]$.

- Input: $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- Input condition:

  $$p + l \leq \text{length}_{\mathbb{Z}}(a)$$

- Output: $b \in \mathbb{Z}^*$
- Output condition:

  $$\begin{array}{l} \textbf{let } n = \text{length}_{\mathbb{Z}}(a) \textbf{ in} \\ \text{length}_{\mathbb{Z}}(b) = n - l \land \\ (\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \land \\ (\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i+l]) \end{array}$$

Mathematical theory:

$$\begin{array}{l} T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \to T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\} \\ \text{length}_T : T^* \to \mathbb{N}, \text{length}_T(a) = \textbf{such } i \in \mathbb{N} : a \in T^i \end{array}$$

Validating Problem Specifications

Given a problem specification with input condition $I(x)$ and output condition $O(x, y)$.

- Correctness: take some legal input(s) a with legal output(s) b.
  - Check that $I(a)$ and $O(a, b)$ indeed hold.
- Falseness: take some legal input(s) a with illegal output(s) b.
  - Check that $I(a)$ holds and $O(a, b)$ does not hold.
- Satisfiability: every legal input should have some legal output.
  - Check $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- Non-triviality: for every legal input not every output should be legal.
  - Check $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.

A formal specification does not necessarily capture our intention!

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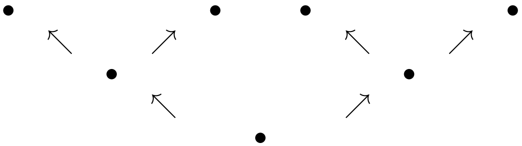
Proofs

A proof is a structured argument that a formula is true.

- A tree whose nodes represent proof situations (states).



- Each proof situation consists of knowledge and a goal.
  - $K_1, \ldots, K_n \vdash G$
  - Knowledge $K_1, \ldots, K_n$: formulas assumed to be true.
  - Goal G: formula to be proved relative to knowledge.
- The root of the tree is the initial proof situation.
  - $K_1, \ldots, K_n$: axioms of mathematical background theories.
  - G: formula to be proved.

Proof Rules

A proof rules describes how a proof situation can be reduced to zero, one, or more "subsituations".

$$\frac{\ldots \vdash \ldots \qquad\qquad \ldots \vdash \ldots}{K_1, \ldots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
  - Zero subsituations: G has been proved, (sub)proof is closed.
  - One or more subsituations: G is proved, if all subgoals are proved.
- Top-down rules: focus on G.
  - G is decomposed into simpler goals $G_1, G_2, \ldots$
- Bottom-up rules: focus on $K_1, \ldots, K_n$.
  - Knowledge is extended to $K_1, \ldots, K_n, K_{n+1}$.

In each proof situation, we aim at showing that the goal is "apparently" true with respect to the given knowledge.

Conjunction $F_1 \wedge F_2$

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2} \qquad \frac{\ldots, K_1 \wedge K_2, K_1, K_2 \vdash G}{\ldots, K_1 \wedge K_2 \vdash G}$$

- Goal $G_1 \wedge G_2$.
  - Create two subsituations with goals G_1 and G_2.

    We have to show $G_1 \wedge G_2$.

    - *We show G_1: ...(proof continues with goal G_1)*
    - *We show G_2: ...(proof continues with goal G_2)*
- Knowledge $K_1 \wedge K_2$.
  - Create one subsituation with K_1 and K_2 in knowledge.

    We know $K_1 \wedge K_2$. We thus also know K_1 and K_2. (proof continues with current goal and additional knowledge K_1 and K_2)

Disjunction $F_1 \vee F_2$

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots, K_1 \vdash G \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$.
  - Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

    We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2. (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- Knowledge $K_1 \vee K_2$.
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \vee K_2$. We thus proceed by case distinction:
    - *Case K_1: ... (proof continues with current goal and additional knowledge K_1).*
    - *Case K_2: ... (proof continues with current goal and additional knowledge K_2).*

Implication $F_1 \Rightarrow F_2$

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$.
    - *We show K_1: ... (proof continues with goal K_1)*
    - *We know K_2: ... (proof continues with current goal and additional knowledge K_2).*

Equivalence $F_1 \Leftrightarrow F_2$

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots, (\neg)K_2 \vdash G}{\ldots, K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$.

    - *We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$)*
    - *We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)*
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$.

    - *We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$)*
    - *We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)*

Universal Quantification $\forall x : F$

$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G}\ (x_0 \text{ new for } K, G) \qquad \frac{\ldots, \forall x : K, K[T/x] \vdash G}{\ldots, \forall x : K \vdash G}$$

- Goal $\forall x : G$
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x_0/x]$.

    We have to show $\forall x : G$. Take arbitrary x_0.
    We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)
- Knowledge $\forall x : K$
  - Choose term T to create one subsituation with formula $K[T/x]$ added to the knowledge.

    We know $\forall x : K$ and thus also $K[T/x]$.
    (proof continues with current goal and additional knowledge $K[T/x]$)

Existential Quantification $\exists x : F$

$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \qquad \frac{\ldots, K[x_0/x] \vdash G}{\ldots, \exists x : K \vdash G} \ (x_0 \text{ new for } K, G)$$

- Goal $\exists x : G$
  - Choose term T to create one subsituation with goal $G[T/x]$.

    We have to show $\exists x : G$. It suffices to show $G[T/x]$.
    (proof continues with goal $G[T/x]$)
- Knowledge $\exists x : K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K[x_0/x]$.

    We know $\exists x : K$. Let x_0 be such that $K[x_0/x]$.
    (proof continues with current goal and additional knowledge $K[x_0/x]$)

Example

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

Take arbitrary y_0. We show

(c) $\exists x : P(x, y_0)$

From (1) we know for some x_0

(2) $\forall y : P(x_0, y)$

From (2) we know

(3) $P(x_0, y_0)$

From (3), we know (c). QED.

Example

We show

(a) $(\exists x : p(x)) \land (\forall x : p(x) \Rightarrow \exists y : q(x,y)) \Rightarrow (\exists x, y : q(x,y))$

We assume

(1) $(\exists x : p(x)) \land (\forall x : p(x) \Rightarrow \exists y : q(x,y))$

and show

(b) $\exists x, y : q(x,y)$

From (1), we know

(2) $\exists x : p(x)$
(3) $\forall x : p(x) \Rightarrow \exists y : q(x,y)$

From (2) we know for some x_0

(4) $p(x_0)$

...

Example (Contd)

...

From (3), we know

$$(5)\ p(x_0) \Rightarrow \exists y : q(x_0, y)$$

From (4) and (5), we know

$$(6)\ \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7)\ q(x_0, y_0)$$

From (7), we know (b). QED.

Indirect Proofs

$$\frac{K, \neg G \vdash \text{false}}{K \vdash G} \qquad \frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G} \qquad \frac{\ldots, \neg G \vdash \neg K}{\ldots, K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.
  - Prove that “false” is true.
  - Prove that a formula F is true and also prove that it is false.
  - Prove that some knowledge K is false, i.e. that $\neg K$ is true.
    - Switches goal G and knowledge K (negating both).

Sometimes simpler than a direct proof.

Example

We show

(a) $(\exists x : \forall y : P(x,y)) \Rightarrow (\forall y : \exists x : P(x,y))$

We assume

(1) $\exists x : \forall y : P(x,y)$

and show

(b) $\forall y : \exists x : P(x,y)$

We assume

(2) $\neg\forall y : \exists x : P(x,y)$

and show a contradiction.

...

Example

...

From (2), we know

$$(3)\ \exists y : \forall x : \neg P(x, y)$$

Let y_0 be such that

$$(4)\ \forall x : \neg P(x, y_0)$$

From (1) we know for some x_0

$$(5)\ \forall y : P(x_0, y)$$

From (5) we know

$$(6)\ P(x_0, y_0)$$

From (4), we know

$$(7)\ \neg P(x_0, y_0)$$

From (6) and (7), we have a contradiction. QED.