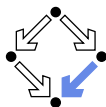


Logic and Proving

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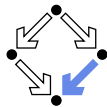


1. The Language of Logic

2. The Art of Proving

3. The RISC ProofNavigator

The Language of Logic



Two kinds of syntactic phrases.

- **Term** T denoting an **object**.
 - Variable x
 - Object constant c
 - Function application $f(T_1, \dots, T_n)$ (may be written infix)
 n -ary function constant f
- **Formula** F denoting a **truth value**.
 - Atomic formula $p(T_1, \dots, T_n)$ (may be written infix)
 n -ary predicate constant p .
 - Negation $\neg F$ ("not F ")
 - Conjunction $F_1 \wedge F_2$ (" F_1 and F_2 ")
 - Disjunction $F_1 \vee F_2$ (" F_1 or F_2 ")
 - Implication $F_1 \Rightarrow F_2$ ("if F_1 , then F_2 ")
 - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1 , then F_2 , and vice versa")
 - Universal quantification $\forall x : F$ ("for all x , F ")
 - Existential quantification $\exists x : F$ ("for some x , F ")



Syntactic Shortcuts

- $\forall x_1, \dots, x_n : F$
 - $\forall x_1 : \dots : \forall x_n : F$
- $\exists x_1, \dots, x_n : F$
 - $\exists x_1 : \dots : \exists x_n : F$
- $\forall x \in S : F$
 - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
 - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

Examples



Terms and formulas may appear in various syntactic forms.

- **Terms:**

$$\exp(x)$$

$$a \cdot b + 1$$

$$a[i] \cdot b$$

$$\sqrt{\frac{x^2 + 2x + 1}{(y+1)^2}}$$

- **Formulas:**

$$a^2 + b^2 = c^2$$

$$n \mid 2n$$

$$\forall x \in \mathbb{N} : x \geq 0$$

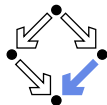
$$\forall x \in \mathbb{N} : 2 \mid x \vee 2 \mid (x + 1)$$

$$\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow$$

$$\exists z \in \mathbb{N} : x + z = y$$

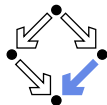
Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas



- Atomic formula $p(T_1, \dots, T_n)$
 - True if the predicate denoted by p holds for the values of T_1, \dots, T_n .
- Negation $\neg F$
 - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ (“ F_1 and F_2 ”)
 - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ (“ F_1 or F_2 ”)
 - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ (“if F_1 , then F_2 ”)
 - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ (“if F_1 , then F_2 , and vice versa”)
 - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ (“for all x , F ”)
 - True if and only if F is true for every possible value assignment of x .
- Existential quantification $\exists x : F$ (“for some x , F ”)
 - True if and only if F is true for at least one value assignment of x .

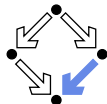
Example



We assume the domain of natural numbers and the “classical” interpretation of constants $1, 2, +, =, <$.

- $1 + 1 = 2$
 - True.
- $1 + 1 = 2 \vee 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \wedge 2 + 2 = 2$
 - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
 - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
 - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
 - True.

Example



- $x + 1 = 1 + x$
 - True, for every assignment of a number a to variable x .
- $\forall x : x + 1 = 1 + x$
 - True (because for every assignment a to x , $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
 - If x is assigned “one”, the formula is true.
 - If x is assigned “two”, the formula is false.
- $\exists x : x + 1 = 2$
 - True (because $x + 1 = 2$ is true for assignment “one” to x).
- $\forall x : x + 1 = 2$
 - False (because $x + 1 = 2$ is false for assignment “two” to x).
- $\forall x : \exists y : x < y$
 - True (because for every assignment a to x , there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
 - False (because for every assignment a to y , there is the assignment $a + 1$ to x which makes $x < y$ false).

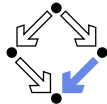


Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \iff F_1$
- $\neg(F_1 \wedge F_2) \iff \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \iff \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \iff F_1 \wedge \neg F_2$
- $\neg\forall x : F \iff \exists x : \neg F$
- $\neg\exists x : F \iff \forall x : \neg F$
- $F_1 \Rightarrow F_2 \iff \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \iff \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \iff \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.

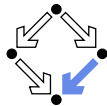


Example

- “All swans are white or black.”
 - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- “There exists a black swan.”
 - $\exists x : swan(x) \wedge black(x).$
- “A swan is white, unless it is black.”
 - $\forall x : swan(x) \wedge \neg black(x) \Rightarrow white(x)$
 - $\forall x : swan(x) \wedge \neg white(x) \Rightarrow black(x)$
 - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- “Not everything that is white or black is a swan.”
 - $\neg \forall x : white(x) \vee black(x) \Rightarrow swan(x).$
 - $\exists x : (white(x) \vee black(x)) \wedge \neg swan(x).$
- “Black swans have at least one black parent” .
 - $\forall x : swan(x) \wedge black(x) \Rightarrow \exists y : swan(y) \wedge black(y) \wedge parent(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.

The Usage of Formulas



Precise formulation of statements describing object relationships.

- **Statement:**

If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y .

- **Formula:**

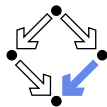
$$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow \\ q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$$

- **Problem specification:**

Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y .

- Inputs: x, y
- Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
- Output: q
- Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$

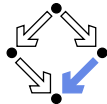
Problem Specifications



- The **specification** of a computation problem:
 - Input: variables $x_1 \in S_1, \dots, x_n \in S_n$
 - Input condition: formula $I(x_1, \dots, x_n)$.
 - Output: variables $y_1 \in T_1, \dots, y_m \in T_n$
 - Output condition: formula $O(x_1, \dots, x_n, y_1, \dots, y_m)$.
 - $F(x_1, \dots, x_n)$: only x_1, \dots, x_n are free in F .
 - x is *free* in F , if not every occurrence of x is inside the scope of a quantifier (such as \forall or \exists) that binds x .
- An **implementation** of the specification:
 - A function (program) $f : S_1 \times \dots \times S_n \rightarrow T_1 \times \dots \times T_m$ such that
$$\forall x_1 \in S_1, \dots, x_n \in S_n : I(x_1, \dots, x_n) \Rightarrow$$

let $(y_1, \dots, y_m) = f(x_1, \dots, x_n)$ **in**
 $O(x_1, \dots, x_n, y_1, \dots, y_m)$
 - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.



Example: A Problem Specification

Given an integer array a , a position p in a , and a length l , return the array b derived from a by removing $a[p], \dots, a[p + l]$.

■ **Input:** $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$

■ **Input condition:**

$$p + l \leq \text{length}_{\mathbb{Z}}(a)$$

■ **Output:** $b \in \mathbb{Z}^*$

■ **Output condition:**

let $n = \text{length}_{\mathbb{Z}}(a)$ **in**

$$\text{length}_{\mathbb{Z}}(b) = n - l \wedge$$

$$(\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge$$

$$(\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i + l])$$

Mathematical theory:

$$T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \rightarrow T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\}$$

$$\text{length}_T : T^* \rightarrow \mathbb{N}, \text{length}_T(a) = \mathbf{such}_T i \in \mathbb{N} : a \in T^i$$



Validating Problem Specifications

Given a problem specification with input condition $I(x)$ and output condition $O(x, y)$.

- **Correctness:** take some legal input(s) a with legal output(s) b .
 - Check that $I(a)$ and $O(a, b)$ indeed hold.
- **Falseness:** take some legal input(s) a with illegal output(s) b .
 - Check that $I(a)$ holds and $O(a, b)$ does not hold.
- **Satisfiability:** every legal input should have some legal output.
 - Check $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- **Non-triviality:** for every legal input not every output should be legal.
 - Check $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.

A formal specification does not necessarily capture our intention!



1. The Language of Logic

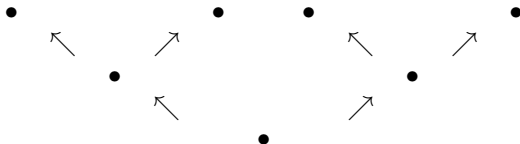
2. The Art of Proving

3. The RISC ProofNavigator



A **proof** is a structured argument that a formula is true.

- A tree whose nodes represent **proof situations (states)**.



- Each proof situation consists of **knowledge** and a **goal**.
 - $K_1, \dots, K_n \vdash G$
 - Knowledge K_1, \dots, K_n : formulas assumed to be true.
 - Goal G : formula to be proved relative to knowledge.
- The **root** of the tree is the initial proof situation.
 - K_1, \dots, K_n : axioms of mathematical background theories.
 - G : formula to be proved.



Proof Rules

A **proof rules** describes how a proof situation can be reduced to zero, one, or more “subsituations”.

$$\frac{\dots \vdash \dots \quad \dots \vdash \dots}{K_1, \dots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
 - Zero subsituations: G has been proved, (sub)proof is closed.
 - One or more subsituations: G is proved, if all subgoals are proved.
- **Top-down rules:** focus on G .
 - G is decomposed into simpler goals G_1, G_2, \dots
- **Bottom-up rules:** focus on K_1, \dots, K_n .
 - Knowledge is extended to K_1, \dots, K_n, K_{n+1} .

In each proof situation, we aim at showing that the goal is “apparently” true with respect to the given knowledge.



Conjunction $F_1 \wedge F_2$

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{\dots, K_1 \wedge K_2, K_1, K_2 \vdash G}{\dots, K_1 \wedge K_2 \vdash G}$$

■ Goal $G_1 \wedge G_2$.

- Create two subsituations with goals G_1 and G_2 .

We have to show $G_1 \wedge G_2$.

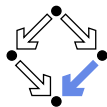
- *We show G_1 : ... (proof continues with goal G_1)*
- *We show G_2 : ... (proof continues with goal G_2)*

■ Knowledge $K_1 \wedge K_2$.

- Create one subsituation with K_1 and K_2 in knowledge.

*We know $K_1 \wedge K_2$. We thus also know K_1 and K_2 .
(proof continues with current goal and additional knowledge K_1 and K_2)*

Disjunction $F_1 \vee F_2$



$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{\dots, K_1 \vdash G \quad \dots, K_2 \vdash G}{\dots, K_1 \vee K_2 \vdash G}$$

■ Goal $G_1 \vee G_2$.

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

*We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 .
(proof continues with goal G_2 and additional knowledge $\neg G_1$)*

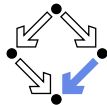
■ Knowledge $K_1 \vee K_2$.

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We know $K_1 \vee K_2$. We thus proceed by case distinction:

- *Case K_1 : ... (proof continues with current goal and additional knowledge K_1).*
- *Case K_2 : ... (proof continues with current goal and additional knowledge K_2).*

Implication $F_1 \Rightarrow F_2$



$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

■ Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

*We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 .
(proof continues with goal G_2 and additional knowledge G_1)*

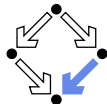
■ Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We know $K_1 \Rightarrow K_2$.

- *We show K_1 : ... (proof continues with goal K_1)*
- *We know K_2 : ... (proof continues with current goal and additional knowledge K_2).*

Equivalence $F_1 \Leftrightarrow F_2$



$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

■ Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$.

- *We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$)*
- *We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)*

■ Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Leftrightarrow K_2$.

- *We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$)*
- *We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)*

Universal Quantification $\forall x : F$



$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G} \quad (x_0 \text{ new for } K, G) \qquad \frac{\dots, \forall x : K, K[T/x] \vdash G}{\dots, \forall x : K \vdash G}$$

■ Goal $\forall x : G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x_0/x]$.

We have to show $\forall x : G$. Take arbitrary x_0 .

We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)

■ Knowledge $\forall x : K$

- Choose term T to create one subsituation with formula $K[T/x]$ added to the knowledge.

We know $\forall x : K$ and thus also $K[T/x]$.

(proof continues with current goal and additional knowledge $K[T/x]$)

Existential Quantification $\exists x : F$



$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \quad \frac{\dots, K[x_0/x] \vdash G}{\dots, \exists x : K \vdash G} \quad (x_0 \text{ new for } K, G)$$

■ Goal $\exists x : G$

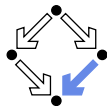
- Choose term T to create one subsituation with goal $G[T/x]$.

*We have to show $\exists x : G$. It suffices to show $G[T/x]$.
(proof continues with goal $G[T/x]$)*

■ Knowledge $\exists x : K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K[x_0/x]$.

*We know $\exists x : K$. Let x_0 be such that $K[x_0/x]$.
(proof continues with current goal and additional knowledge $K[x_0/x]$)*



Example

We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

$$(b) \forall y : \exists x : P(x, y)$$

Take arbitrary y_0 . We show

$$(c) \exists x : P(x, y_0)$$

From (1) we know for some x_0

$$(2) \forall y : P(x_0, y)$$

From (2) we know

$$(3) P(x_0, y_0)$$

From (3), we know (c). QED.



Example

We show

$$(a) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y)) \Rightarrow (\exists x, y : q(x, y))$$

We assume

$$(1) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y))$$

and show

$$(b) \exists x, y : q(x, y)$$

From (1), we know

$$(2) \exists x : p(x)$$

$$(3) \forall x : p(x) \Rightarrow \exists y : q(x, y)$$

From (2) we know for some x_0

$$(4) p(x_0)$$

...

Example (Contd)



...

From (3), we know

$$(5) p(x_0) \Rightarrow \exists y : q(x_0, y)$$

From (4) and (5), we know

$$(6) \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7) q(x_0, y_0)$$

From (7), we know (b). QED.

Indirect Proofs



$$\frac{K, \neg G \vdash \text{false}}{K \vdash G} \quad \frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G} \quad \frac{\dots, \neg G \vdash \neg K}{\dots, K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.
 - Prove that “false” is true.
 - Prove that a formula F is true and also prove that it is false.
 - Prove that some knowledge K is false, i.e. that $\neg K$ is true.
 - Switches goal G and knowledge K (negating both).

Sometimes simpler than a direct proof.



Example

We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

$$(b) \forall y : \exists x : P(x, y)$$

We assume

$$(2) \neg \forall y : \exists x : P(x, y)$$

and show a contradiction.

...



Example

...

From (2), we know

$$(3) \exists y : \forall x : \neg P(x, y)$$

Let y_0 be such that

$$(4) \forall x : \neg P(x, y_0)$$

From (1) we know for some x_0

$$(5) \forall y : P(x_0, y)$$

From (5) we know

$$(6) P(x_0, y_0)$$

From (4), we know

$$(7) \neg P(x_0, y_0)$$

From (6) and (7), we have a contradiction. QED.

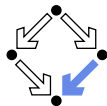


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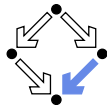
The RISC ProofNavigator



- **An interactive proving assistant for program verification.**
 - Research Institute for Symbolic Computation (RISC), 2005–:
<http://www.risc.jku.at/research/formal/software/ProofNavigator>.
 - Development based on prior experience with PVS (SRI, 1993–).
 - Kernel and GUI implemented in Java.
 - Uses external SMT (satisfiability modulo theories) solver.
 - CVCL (Cooperating Validity Checker Lite) 2.0, CVC3.
 - Runs under Linux (only); freely available as open source (GPL).
- **A language for the definition of logical theories.**
 - Based on a strongly typed higher-order logic (with subtypes).
 - Introduction of types, constants, functions, predicates.
- **Computer support for the construction of proofs.**
 - Commands for basic inference rules and combinations of such rules.
 - Applied interactively within a sequent calculus framework.
 - Top-down elaboration of proof trees.

Designed for simplicity of use; applied to non-trivial verifications.

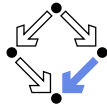
Using the Software



For survey, see “Program Verification with the RISC ProofNavigator”.
For details, see “The RISC ProofNavigator: Tutorial and Manual”.

- **Develop a theory.**
 - Text file with declarations of types, constants, functions, predicates.
 - Axioms (propositions assumed true) and formulas (to be proved).
- **Load the theory.**
 - File is read; declarations are parsed and type-checked.
 - Type-checking conditions are generated and proved.
- **Prove the formulas in the theory.**
 - Human-guided top-down elaboration of proof tree.
 - Steps are recorded for later replay of proof.
 - Proof status is recorded as “open” or “completed”.
- **Modify theory and repeat above steps.**
 - Software maintains dependencies of declarations and proofs.
 - Proofs whose dependencies have changed are tagged as “untrusted”.

Starting the Software



■ Starting the software:

ProofNavigator & (32 bit machines at RISC)
ProofNavigator64 & (64 bit machines at RISC)

■ Command line options:

Usage: ProofNavigator [OPTION]... [FILE]

FILE: name of file to be read on startup.

OPTION: one of the following options:

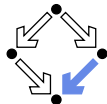
- n, --nogui: use command line interface.
- c, --context NAME: use subdir NAME to store context.
- cvcl PATH: PATH refers to executable "cvcl".
- s, --silent: omit startup message.
- h, --help: print this message.

■ Repository stored in subdirectory of current working directory:

ProofNavigator/

- Option -c *dir* or command newcontext "*dir*" :
 - Switches to repository in directory *dir*.

A Theory



```
% switch repository to "sum"
newcontext "sum";

% the recursive definition of the sum from 0 to n
sum: NAT->NAT;
S1: AXIOM sum(0)=0;
S2: AXIOM FORALL(n:NAT): n>0 => sum(n)=n+sum(n-1);

% proof that explicit form is equivalent to recursive definition
S: FORMULA FORALL(n:NAT): sum(n) = (n+1)*n/2;
```

Declarations written with an external editor in a text file.

Proving a Formula



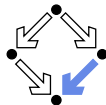
When the file is loaded, the declarations are pretty-printed:

```
sum ∈ ℕ → ℕ
axiom S1 ≡ sum(0) = 0
axiom S2 ≡ ∀ n ∈ ℕ : n > 0 ⇒ sum(n) = n + sum(n-1)
S ≡ ∀ n ∈ ℕ : sum(n) =  $\frac{(n+1) \cdot n}{2}$ 
```

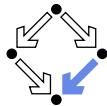
The proof of a formula is started by the `prove` command.

	Formula S
<code>prove S:</code>	Construct Proof
<code>proof S:</code>	Show Proof
<code>formula S:</code>	Print Formula

Proving a Formula



Proving a Formula



- Proof of formula F is represented as a **tree**.
 - Each tree node denotes a **proof state (goal)**.
 - Logical sequent:
 $A_1, A_2, \dots \vdash B_1, B_2, \dots$
 - Interpretation:
 $(A_1 \wedge A_2 \wedge \dots) \Rightarrow (B_1 \vee B_2 \vee \dots)$
 - Initially single node $Axioms \vdash F$.
- The **tree must be expanded to completion**.
 - Every leaf must denote an obviously valid formula.
 - Some A_i is false or some B_j is true.
- A proof step consists of the **application of a proving rule to a goal**.
 - Either the goal is recognized as true.
 - Or the goal becomes the parent of a number of children (subgoals).
The conjunction of the subgoals implies the parent goal.

Constants: $x_0 \in S_0, \dots$

$[L_1] \quad A_1$

\dots

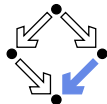
$[L_n] \quad A_n$

$[L_{n+1}] \quad B_1$

\dots

$[L_{n+m}] \quad B_m$

An Open Proof Tree



Proof Tree

▼ [tca]: induction n in byu

[dbj]: proved (CVCL)

[ebj]

Formula [S] proof state [dbj]

Constants (with types): sum.

[xe] $\forall n \in \mathbb{N} : n > 0 \Rightarrow \text{sum}(n) = n + \text{sum}(n-1)$

[d3i] $\text{sum}(0) = 0$

[nfq] $\text{sum}(0) = \frac{(0+1) \cdot 0}{2}$

Parent: [tca]

Closed goals are indicated in blue; goals that are open (or have open subgoals) are indicated in red. The red bar denotes the “current” goal.

A Completed Proof Tree

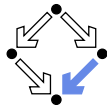


Proof Tree





- ▾ [tca]: induction n in byu
 - [dbj]: proved (CVCL)
- ▾ [ebj]: instantiate n_0+1 in lxe
 - [k5f]: proved (CVCL)

The visual representation of the complete proof structure; by clicking on a node, the corresponding proof state is displayed.

Navigation Commands

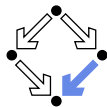


Various buttons support navigation in a proof tree.






- : prev
 - Go to previous open state in proof tree.
- : next
 - Go to next open state in proof tree.
- : undo
 - Undo the proof command that was issued in the parent of the current state; this discards the whole proof tree rooted in the parent.
- : redo
 - Redo the proof command that was previously issued in the current state but later undone; this restores the discarded proof tree.

Single click on a node in the proof tree displays the corresponding state;
double click makes this state the current one.

Proving Commands

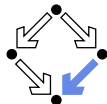


The most important proving commands can be also triggered by buttons.

-  (scatter)
 - Recursively applies decomposition rules to the current proof state and to all generated child states; attempts to close the generated states by the application of a validity checker.
-  (decompose)
 - Like scatter but generates a single child state only (no branching).
-  (split)
 - Splits current state into multiple children states by applying rule to current goal formula (or a selected formula).
-  (auto)
 - Attempts to close current state by instantiation of quantified formulas.
-  (autostar)
 - Attempts to close current state and its siblings by instantiation.

Automatic decomposition of proofs and closing of proof states.

Proving Commands



More commands can be selected from the menus.





- **assume**
 - Introduce a new assumption in the current state; generates a sibling state where this assumption has to be proved.
- **case:**
 - Split current state by a formula which is assumed as true in one child state and as false in the other.
- **expand:**
 - Expand the definitions of denoted constants, functions, or predicates.
- **lemma:**
 - Introduce another (previously proved) formula as new knowledge.
- **instantiate:**
 - Instantiate a universal assumption or an existential goal.
- **induction:**
 - Start an induction proof on a goal formula that is universally quantified over the natural numbers.

Here the creativity of the user is required!

Auxiliary Commands

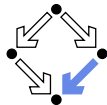


Some buttons have no command counterparts.

- : counterexample
 - Generate a “counterexample” for the current proof state, i.e. an interpretation of the constants that refutes the current goal.
- 
 - Abort current prover activity (proof state simplification or counterexample generation).
- 
 - Show menu that lists all commands and their (optional) arguments.
- 
 - Simplify current state (if automatic simplification is switched off).

More facilities for proof control.

Proving Strategies



- Initially: semi-automatic proof decomposition.
 - `expand` expands constant, function, and predicate definitions.
 - `scatter` aggressively decomposes a proof into subproofs.
 - `decompose` simplifies a proof state without branching.
 - `induction` for proofs over the natural numbers.
- Later: critical hints given by user.
 - `assume` and `case cut` proof states by conditions.
 - `instantiate` provide specific formula instantiations.
- Finally: simple proof states are yielded that can be automatically closed by the validity checker.
 - `auto` and `autostar` may help to close formulas by the heuristic instantiation of quantified formulas.

Appropriate combination of semi-automatic proof decomposition, critical hints given by the user, and the application of a validity checker is crucial.