# Analysis of Complexity

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## Example

We are going to analyze the following program function:

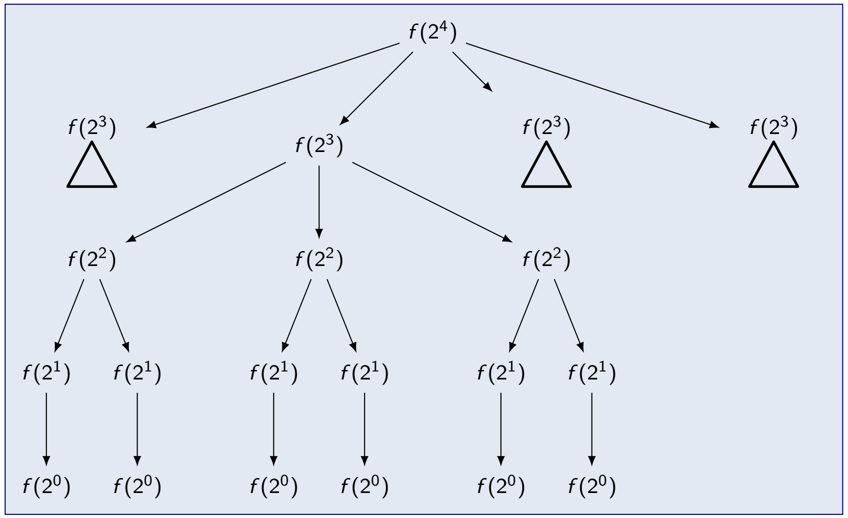
```
static int f(int m) {
  if (m == 1) return 1;
  int s = 1;
  for (int i=0; i<log2(m); i++)
     s = s+f(m/2);
  return s;
}
```

- $f(m)$ calls $f(m/2)$ recursively $\lfloor \log_2 m \rfloor$ times .
  - $f(2^n)$ calls $f(2^{n-1})$ recursively $n$ times.
- How often is $f$ called in total when executing $f(m) = f(2^n)$?
  - Actually, this value is also the result of $f$.

The analysis of a program involving both loops and recursion.



## Recursion Tree



## Recursion Tree

Each node in the recursion tree denotes one function call.

- Tree of height 4:
  - Level 0: 1 node.
  - Level 1: 4 nodes.
  - Level 2: $4 \cdot 3$ nodes.
  - Level 3: $4 \cdot 3 \cdot 2$ nodes.
  - Level 4: $4 \cdot 3 \cdot 2 \cdot 1$ nodes.
- Total number of nodes (function calls):

$$1+4+4\cdot 3+4\cdot 3\cdot 2+4\cdot 3\cdot 2\cdot 1=65$$

What is the number of nodes/function calls $T(n)$ for $f(2^n)$?



## A Recurrence

From the code of $f$, the following recurrence defines $T(n)$.

$$T(n) := \begin{cases} 1 & \text{if } n=0 \\ 1+n\cdot T(n-1) & \text{else} \end{cases}$$

- $m=2^0=1$ : 1 function call.
- $m=2^n>1$ : $1+n\cdot T(n-1)$ function calls.

We need an explicit solution of this recurrence.



## Solving the Recurrence

Again we add the number of nodes in each level of the tree.

$$T(n) \stackrel{?}{=} 1+n+n\cdot(n-1)+n\cdot(n-1)\cdot(n-2)+\ldots+n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$$

$$T(n) \stackrel{?}{=} \sum_{i=0}^{n} \frac{n!}{i!}$$

$$\begin{aligned} n!/n! &= 1 \\ n!/(n-1)! &= n \\ n!/(n-2)! &= n\cdot(n-1) \\ n!/(n-3)! &= n\cdot(n-1)\cdot(n-2) \\ &\ldots \\ n!/0! &= n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1 \end{aligned}$$

We need to verify that this is indeed a valid solution of the recurrence.



## Verifying the Solution

We prove $(\forall n\in\mathbb{N}: T(n)=\sum_{i=0}^{n}\frac{n!}{i!})$ by induction on $n$.

- Induction base:

$$T(0)=1=0!/0!=\sum_{i=0}^{0}\frac{0!}{i!}$$

- Induction hypothesis: we assume

$$T(n)=\sum_{i=0}^{n}\frac{n!}{i!}$$

- Induction step: we prove

$$T(n+1)=\sum_{i=0}^{n+1}\frac{(n+1)!}{i!}$$

$$\begin{aligned} T(n+1) &= 1+(n+1)\cdot T(n)=1+(n+1)\cdot\sum_{i=0}^{n}\frac{n!}{i!}=1+\sum_{i=0}^{n}\frac{(n+1)\cdot n!}{i!} \\ &= 1+\sum_{i=0}^{n}\frac{(n+1)!}{i!}=\frac{(n+1)!}{(n+1)!}+\sum_{i=0}^{n}\frac{(n+1)!}{i!}=\sum_{i=0}^{n+1}\frac{(n+1)!}{i!} \quad \square \end{aligned}$$

## Asymptotic Characterization of the Solution

The explicit solution does not give much intuition about its growth.

$$T(n) = \sum_{i=0}^{n} \frac{n!}{i!} = n! \cdot \sum_{i=0}^{n} \frac{1}{i!} < n! \cdot \sum_{i=0}^{\infty} \frac{1}{i!} = n! \cdot e = O(n!)$$

$\sum_{i=0}^{\infty} 1/i! = e$ (Euler's number).

The number of function calls is $O(n!) = O((\log_2 m)!)$.







## Sums

Sort integer array $a[0 \dots n-1]$ of length $n \geq 1$ in ascending order.

| | Cost |
|---|---|
| **procedure** INSERTIONSORT($a$) | |
| $n \leftarrow length(a)$ | $1$ |
| **for** $i$ **from** $1$ **to** $n-1$ **do** | $n$ |
| $x \leftarrow a[i]$ | $n-1$ |
| $j \leftarrow i-1$ | $n-1$ |
| **while** $j \geq 0 \land a[j] > x$ **do** | $\sum_{i=1}^{n-1} n_i$ |
| $a[j+1] \leftarrow a[j]$ | $\sum_{i=1}^{n-1} (n_i - 1)$ |
| $j \leftarrow j-1$ | $\sum_{i=1}^{n-1} (n_i - 1)$ |
| **end while** | |
| $a[j+1] \leftarrow x$ | $n-1$ |
| **end for** | |
| **end procedure** | |

Sums arise from the analysis of iterative algorithms.



## Worst Case Time Complexity

$n_i = i+1$: maximum number of times **while** test is executed for value $i$.

- Worst case time complexity: $T(n) = 4n - 2 + \sum_{i=1}^{n-1}(3i+1)$

$$\begin{aligned} T(n) =& 1 + n + (n-1) + (n-1) + (n-1) + \\ & (\sum_{i=1}^{n-1} n_i) + (\sum_{i=1}^{n-1}(n_i - 1)) + (\sum_{i=1}^{n-1}(n_i - 1)) = 4n - 2 + \sum_{i=1}^{n-1}(3n_i - 2) \\ =& 4n - 2 + \sum_{i=1}^{n-1}(3 \cdot (i+1) - 2) = 4n - 2 + \sum_{i=1}^{n-1}(3i+1) \end{aligned}$$

- Closed form: $\sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$

$$\sum_{i=1}^{n-1}(3i+1) = \sum_{i=0}^{n-1}(3i+1) - 1 = n \cdot 1 + 3 \cdot \frac{(n-1) \cdot n}{2} - 1 = \frac{3n^2 - n - 2}{2}$$

  - Arithmetic series: $\sum_{i=0}^{n}(a + i \cdot d) = (n+1) \cdot a + d \cdot \frac{n \cdot (n+1)}{2}$
  - Geometric series: $\sum_{i=0}^{n}(a \cdot q^i) = a \cdot \frac{q^{n+1}-1}{q-1}$

  High school knowledge.

Worst case time complexity $T(n) = 4n - 2 + \frac{3n^2-n-2}{2} = \frac{3n^2+7n-6}{2}$.

# Average Time Complexity

Maximum value $n_i$ is replaced by expected value $E[N_i]$.

- Expected value of random variable $N_i$: $E[N_i] = \frac{i+2}{2}$
  - Assume that all permutations of $a$ have equal probability.
  - Consequently each value $1, \ldots, i+1$ of $N_i$ has equal probability.

$$E[N_i] = \frac{1}{i+1} \cdot \sum_{j=1}^{i+1} j = \frac{(i+2)\cdot(i+1)}{2\cdot(i+1)} = \frac{i+2}{2}$$

- Average time complexity: $\overline{T}(n) = 4n - 2 + \frac{1}{2} \cdot \sum_{i=1}^{n-1}(3i+2)$

$$\overline{T}(n) = 4n - 2 + \sum_{i=1}^{n-1}(3 \cdot \frac{i+2}{2} - 2) = 4n - 2 + \frac{1}{2} \cdot \sum_{i=1}^{n-1}(3i+2)$$

- Closed form: $\overline{T}(n) = \frac{3n^2+17n-12}{4}$

$$\sum_{i=1}^{n-1}(3i+2) = \sum_{i=0}^{n-1}(3i+2) - 2 = (2n + 3 \cdot \frac{(n-1)\cdot n}{2}) - 2 = \frac{3n^2+n-4}{2}$$

$$\overline{T}(n) = 4n - 2 + \frac{3n^2+n-4}{4} = \frac{16n-8+3n^2+n-4}{4} = \frac{3n^2+17n-12}{4}$$

Average time complexity $\overline{T}(n) = \frac{3n^2+17n-12}{4}$.



# Asymptotic Time Complexity

Worst case/average time $T(n) = \frac{3n^2+7n-6}{2}$ and $\overline{T}(n) = \frac{3n^2+17n-12}{4}$.

- $\overline{T}(n) \simeq \frac{T(n)}{2}$ (for large $n$)
  - In the average, the algorithm is twice as fast as in the worst case.
- $\overline{T}(n) = T(n) = \Theta(n^2)$
  - Asymptotic complexity is the same in the average as in the worst case.
- Asymptotic estimation: $\sum_{i=0}^{\Theta(n)} \Theta(i) = \Theta(n^2)$
  - a linear number of times linear complexity gives quadratic complexity.

$$\sum_{i=0}^{\Theta(n)} \Theta(i^k) = \Theta(n^{k+1})$$

Frequently, a quick estimation of asymptotic time complexity is possible.



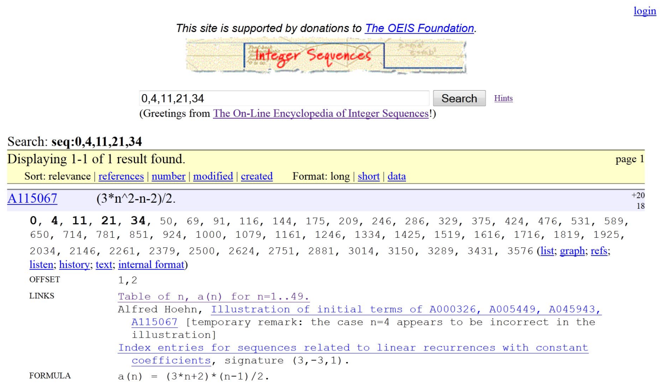
# Solving Sums by Guessing and Verifying

One may consult an (electronic/printed) table of integer sequences.

- Determine summation values for growing number of summands:

$$0, 4, 4+7, 4+7+10, 4+7+10+13, \ldots = 0, 4, 11, 21, 34, \ldots$$

- On-line Encyclopedia of Integer Sequences (`http://oeis.org`)


login

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Integer Sequences

0,4,11,21,34 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:0,4,11,21,34**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A115067 (3*n^2-n-2)/2. +20 18

**0, 4, 11, 21, 34**, 50, 69, 91, 116, 144, 175, 209, 246, 286, 329, 375, 424, 476, 531, 589, 650, 714, 781, 851, 924, 1000, 1079, 1161, 1246, 1334, 1425, 1519, 1616, 1716, 1819, 1925, 2034, 2146, 2261, 2379, 2500, 2624, 2751, 2881, 3014, 3150, 3289, 3431, 3576 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

LINKS Table of n, a(n) for n=1..49.

Alfred Hoehn, Illustration of initial terms of A000326, A005449, A045943, A115067 [temporary remark: the case n=4 appears to be incorrect in the illustration]

Index entries for sequences related to linear recurrences with constant coefficients, signature (3,-3,1).

FORMULA a(n) = (3*n+2)*(n-1)/2.



# Solving Sums by Guessing and Verifying

One may consult a computer algebra system (Maple, Mathematica, ...).

```
> sum(3*i+1,i=1..n-1);
                                 2
                          3/2 n   - 1/2 n - 1

In[1]:= Sum[3*i+1,{i,1,n-1}]

                    2
          -2 - n + 3 n
Out[1]= -------------
               2

In[2]:= FindSequenceFunction[{0,4,11,21,34},n]

          (-1 + n) (2 + 3 n)
Out[2]= ------------------
                  2
```

However the solution was initially *guessed*, it must be subsequently *verified*.

## Solving Sums by Guessing and Verifying

Prove $\forall n \in \mathbb{N} : n \geq 1 \Rightarrow \sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$ by induction on $n$.

- Base case $n=1$:
$$\sum_{i=1}^{1-1}(3i+1) = 0 = \frac{3 \cdot 1^2 - 1 - 2}{2}$$
- We assume for fixed $n \geq 1$ $\sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$ and show
$$\sum_{i=1}^{n}(3i+1) = \frac{3\cdot(n+1)^2-(n+1)-2}{2}$$
This equation holds, because we have
$$\sum_{i=1}^{n}(3i+1) = \sum_{i=1}^{n-1}(3i+1)+(3n+1) = \frac{3n^2-n-2}{2}+(3n+1)$$
$$= \frac{3n^2-n-2+6n+2}{2} = \frac{3n^2+5n}{2}$$
$$\frac{3\cdot(n+1)^2-(n+1)-2}{2} = \frac{3n^2+6n+3-n-1-2}{2} = \frac{3n^2+5n}{2} \quad \square$$







## Recurrences

Find in sorted array $a[l,r]$ the position of value $x$ ($-1$, if $x$ does not occur).

| | Cost |
|---|---|
| **function** BinarySearch$(a,x,l,r)$ $\triangleright$ $n=r-l+1$ | |
| **if** $l>r$ **then** | 1 |
| **return** $-1$ | 1 |
| **end if** | |
| $m \leftarrow \lfloor \frac{l+r}{2} \rfloor$ | 1 |
| **if** $a[m]=x$ **then** | 1 |
| **return** $m$ | 1 |
| **else if** $a[m]<x$ **then** | 1 |
| **return** BinarySearch$(a,x,m+1,r)$ | $\leq 1+T(\lfloor \frac{n}{2} \rfloor)$ |
| **else** | |
| **return** BinarySearch$(a,x,l,m-1)$ | $\leq 1+T(\lfloor \frac{n}{2} \rfloor)$ |
| **end if** | |
| **end function** | |

Recurrences arise from the analysis of recursive algorithms.



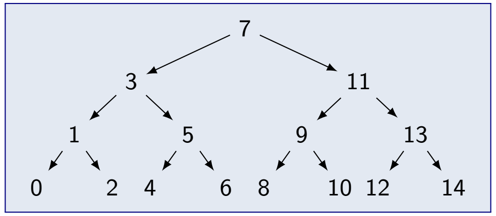
## Worst Case Time Complexity

- Recurrence Relation:
$$T(0) = 2$$
$$T(n) = 5 + T(\lfloor \frac{n}{2} \rfloor), \text{ if } n \geq 1$$
- Special solution: assume $n = 2^m$.
$$T(2^m) = 5 + T(2^{m-1})$$
$$= \underbrace{5+\ldots+5}_{m \text{ times}} + T(1) = \underbrace{5+\ldots+5}_{m+1 \text{ times}} + T(0)$$
$$= 5\cdot(m+1)+2 = 5m+7$$
- General solution: for all $n \geq 1$.
$$T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$$
  - Verified later.

Worst case time complexity $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$ ($T_{\text{found}}(n) = T(n) - 5 = 5 \cdot \lfloor \log_2 n \rfloor + 2$, if $x$ occurs in $a$).

## Average Time Complexity

For $n = 2^m - 1$, recursion tree of height $m-1$ with $n$ nodes:




- Each path describes the function calls to find a particular element:
  - $7 \to 3 \to 5$: 3 function calls to find element at position 5.
- In total $n = 2^m - 1$ paths:
  - 1 path of length 0, 2 of length 1, 4 of length 2, ..., $2^i$ of length $i$.
- Assume all paths are equally likely.
  - I.e., assume $x$ occurs in $a$, at every position with equal probability.

If $x$ is in $a$, average number of calls is $\frac{1}{2^m-1} \cdot \sum_{i=0}^{m-1} i \cdot 2^i$.



## Average Time Complexity

Determine the closed form of $\sum_{i=0}^{m-1} i \cdot 2^i$.

- Integer sequence: $0, 2, 10, 34, 98, \ldots$

```
A036799          2+2^(n+1)*(n-1).
            0, 2, 10, 34, 98, ...
```

- Computer algebra system:

```
> sum(i*2^i,i=0..m-1);

                                  m      m
                               m 2  - 2 2  + 2
```

- Result: $\sum_{i=0}^{m-1} i \cdot 2^i = 2^m \cdot (m-2) + 2$
  - Verification by induction proof.
- Average number of function calls:

$$\frac{1}{2^m-1} \cdot \sum_{i=0}^{m-1} i \cdot 2^i = \frac{1}{2^m-1} \cdot (2^m \cdot (m-2) + 2) = \frac{2^m \cdot (m-2)}{2^m-1} + \frac{2}{2^m-1} \simeq m-2$$

- Average time complexity:

$$\overline{T}_{\text{found}}(2^m-1) \simeq 5 \cdot (m-2) + 2 = 5m - 8 = T_{\text{found}}(2^m-1) - 5$$

One recursive call less in the average case (similar, if $x$ is not in $a$).



## Solving Recurrences by Guessing & Verifying

$$T(1) = 7$$
$$T(n) = 5 + T(\lfloor \tfrac{n}{2} \rfloor), \text{ if } n > 1$$

One may consult an (electronic/printed) table of integer sequences.

- Simplified recurrence:

$$U(1) = 0$$
$$U(n) = 1 + U(\lfloor \tfrac{n}{2} \rfloor), \text{ if } n > 1$$

- Integer sequence: $0, 1, 1, 2, 2, 2, 2, 3, \ldots$

```
A000523  Log_2(n) rounded down.
         0, 1, 1, 2, 2, 2, 2, 3, ...
         ...
         FORMULA ... a(n)=floor(lb(n)).
```

- $U(n) = \lfloor \log_2 n \rfloor$, $T(n) = a \cdot \lfloor \log_2 n \rfloor + b$

Closed form $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$.



## Solving Recurrences by Guessing & Verifying

One may consult a computer algebra system.

```
> rsolve({T(1)=7,T(n)=5+T(n/2)},T(n));
                         7 ln(2) + 5 ln(n)
                         -----------------
                               ln(2)

In[3]:= RSolve[{T[1]==7,T[n]==5+T[n/2]},T[n],n]

                           5 Log[n]
Out[3]= {{T[n] -> 7 + --------}}
                            Log[2]
```

- Real solution $T(n) = 5 \cdot \log_2(n) + 7$.
- Integer solution $T(n) = 5 \cdot \lfloor \log_2(n) \rfloor + 7$.

However the solution was initially *guessed*, it must be subsequently *verified*.

## Solving Recurrences by Guessing & Verifying

$$T(1) = 7$$
$$T(n) = 5 + T(\lfloor \frac{n}{2} \rfloor), \text{ if } n > 1$$

We show for all $n \in \mathbb{N}$ with $n \geq 1$, $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$.

- Induction base $n = 1$:

$$5 \cdot \lfloor \log_2 1 \rfloor + 7 = 5 \cdot 0 + 7 = 7 = T(1)$$

- Ind. hypothesis: for $n > 1$ and $1 \leq m < n$, assume $T(m) = 5 \cdot \lfloor \log_2 m \rfloor + 7$.
  - Case $n = 2m$:

$$\begin{aligned} T(n) &= 5 + T(\lfloor \tfrac{n}{2} \rfloor) = 5 + T(m) = 5 + (5 \cdot \lfloor \log_2 m \rfloor + 7) = 5 \cdot (1 + \lfloor \log_2 m \rfloor) + 7 \\ &= 5 \cdot \lfloor 1 + \log_2 m \rfloor + 7 = 5 \cdot \lfloor \log_2 2 + \log_2 m \rfloor + 7 = 5 \cdot \lfloor \log_2 2m \rfloor + 7 \\ &= 5 \cdot \lfloor \log_2 n \rfloor + 7 \end{aligned}$$

  - Case $n = 2m + 1$:

$$\begin{aligned} T(n) &= 5 + T(\lfloor \tfrac{n}{2} \rfloor) = 5 + T(m) = 5 + (5 \cdot \lfloor \log_2 m \rfloor + 7) = 5 \cdot (1 + \lfloor \log_2 m \rfloor) + 7 \\ &= 5 \cdot \lfloor 1 + \log_2 m \rfloor + 7 = 5 \cdot \lfloor \log_2 2 + \log_2 m \rfloor + 7 = 5 \cdot \lfloor \log_2 2m \rfloor + 7 \\ &= 5 \cdot \lfloor \log_2 (n-1) \rfloor + 7 = 5 \cdot \lfloor \log_2 n \rfloor + 7 \end{aligned}$$







## Divide and Conquer

We are going to analyze the Mergesort algorithm.

| | Cost |
|---|---|
| **procedure** MERGESORT$(a, l, r)$ $\triangleright$ $n = r - l + 1$ | |
|   **if** $l < r$ **then** | 1 |
|     $m \leftarrow \lfloor \frac{l+r}{2} \rfloor$ | 1 |
|     MERGESORT$(a, l, m)$ | $1 + T(\lceil \frac{n}{2} \rceil)$ |
|     MERGESORT$(a, m+1, r)$ | $1 + T(\lfloor \frac{n}{2} \rfloor)$ |
|     MERGE$(a, l, m, r)$ | $O(n)$ |
|   **end if** | |
| **end procedure** | |

We will investigate the asymptotic time complexity only.



## Recurrence

- $T(1)$ can be solved in $O(1)$.

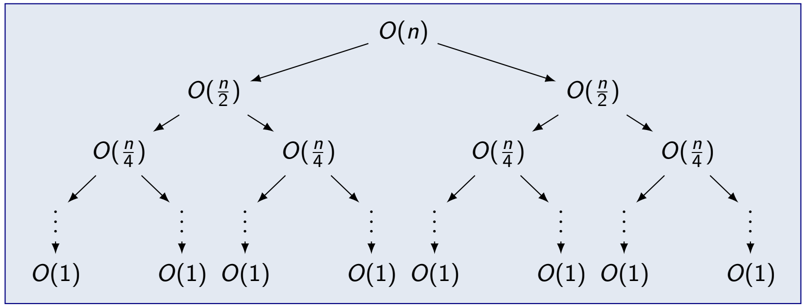
$$\begin{aligned} T(n) &= 1 + 1 + 1 + T(\lceil \tfrac{n}{2} \rceil) + 1 + T(\lfloor \tfrac{n}{2} \rfloor) + O(n) \\ &= T(\lceil \tfrac{n}{2} \rceil) + T(\lfloor \tfrac{n}{2} \rfloor) + O(n) + 4 \\ &= T(\lceil \tfrac{n}{2} \rceil) + T(\lfloor \tfrac{n}{2} \rfloor) + O(n) \end{aligned}$$

- For asymptotic analysis, it suffices to consider $\frac{n}{2} \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.

$$T(n) = 2 \cdot T(\tfrac{n}{2}) + O(n)$$

We are going to guess an asymptotic solution of this recurrence.

## Guessing the Asymptotic Time Complexity




- Execution is the depth-first left-to-right traversal of a binary tree.
- Total time is the sum of the times spent in all tree nodes.

$$T(n) = \sum_{i=0}^{O(\log n)} 2^i \cdot O(\frac{n}{2^i}) = \sum_{i=0}^{O(\log n)} O(n) = O(n \cdot \log n)$$

We are going to verify our guess of the asymptotic solution.



## Verifying the Asymptotic Time Complexity

We prove that every solution $T(n)$ of the recurrence

$$T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$$

is asymptotically bound by $T(n) = O(n \cdot \log n)$.

- From recurrence, there exist $c > 0$ and $N \geq 1$ such that, for all $n \geq N$
$$T(n) \leq 2 \cdot T(\frac{n}{2}) + c \cdot n$$
- From the definition of $O(n \cdot \log n) = O(n \cdot \log_2 n)$, it suffices to show
$$\exists c \in \mathbb{R}_{\geq 0}, N \in \mathbb{N} : \forall n \geq N : T(n) \leq c \cdot n \cdot \log_2 n$$
- Thus our goal is to find $c'$ and $N'$ for which we are able to prove
$$\forall n \geq N' : T(n) \leq c' \cdot n \cdot \log_2 n$$

Because of the characterization of $T$ by a recurrence, this naturally leads to an induction proof with base $N'$.



## Verifying Asymptotic Complexity (Contd)

How to choose $N'$?

- Clearly $N' \geq \max\{2, N\}$.
  - $n = 1 \Rightarrow c' \cdot n \cdot \log_2 n = 0$
  - $N' < N$: induction hypothesis cannot be applied.
- But $N' \geq 2$ is problematic for recurrence $T(n) = \ldots T(\frac{n}{2}) \ldots$.
  - Induction step proves goal for $n$ based on assumption it holds for $\frac{n}{2}$.
  - $N' = 1$: every sequence of divisions by 2 which starts with $n \geq 1$ eventually leads to the base case $n = 1$.
  - $N' \geq 2$: sequence may bypass base case $n = N'$, e.g., $(N'+1)/2 < N'$.
- Solution: show induction base for all $n$ with $2 \leq n \leq N'$.
  - Also ensure that every sequence of divisions starting with $n > N$ eventually reaches some base case, i.e., $\frac{n}{2} \geq 2$.

We choose $N' := \max\{\sqcup, 3, N\}$.



## Verifying Asymptotic Complexity (Contd)

How to choose $c'$?

- Have to prove $T(n) \leq c' \cdot n \cdot \log_2 n$.
  - for all $2 \leq n \leq N'$.
- It suffices to prove $T(n) \leq c'$.
  - $n \cdot \log_2 n \geq 1$, for all $2 \leq n \leq N'$.
- It suffices to prove $T(N') \leq c'$
  - $T(2) \leq T(3) \leq \ldots \leq T(N')$

We choose $c' := \max\{\sqcup, T(N')\}$.

## Verifying Asymptotic Complexity (Contd)

- Induction base: by choice of $N'$ and $c'$, we have for all $2 \leq n \leq N'$

$$T(n) \leq c' \cdot n \cdot \log_2 n$$

- Induction assumption: for $n > N'$ and $2 \leq m < n$, we assume

$$T(m) \leq c' \cdot m \cdot \log_2 m$$

- Induction step: we show

$$T(n) \leq c' \cdot n \cdot \log_2 n$$

We continue by case distinction on the interpretation of $\frac{n}{2}$.



## Verifying Asymptotic Complexity (Contd)

- Case $\frac{n}{2}$ means $\lfloor \frac{n}{2} \rfloor$: since $n > N' \geq N$,

$$\begin{aligned} T(n) &\leq 2 \cdot T(\lfloor \frac{n}{2} \rfloor) + c \cdot n \\ &\leq 2 \cdot c' \cdot \lfloor \frac{n}{2} \rfloor \cdot \log_2 \lfloor \frac{n}{2} \rfloor + c \cdot n \\ &\leq 2 \cdot c' \cdot \frac{n}{2} \cdot \log_2 (\frac{n}{2}) + c \cdot n \\ &= c' \cdot n \cdot \log_2 (\frac{n}{2}) + c \cdot n \\ &= c' \cdot n \cdot ((\log_2 n) - 1) + c \cdot n \\ &= c' \cdot n \cdot (\log_2 n) + (c - c') \cdot n \leq c' \cdot n \cdot \log_2 n \end{aligned}$$

Last inequality holds if $c' \geq c$, because then $c - c' \leq 0$.



## Verifying Asymptotic Complexity (Contd)

- Case $\frac{n}{2}$ means $\lceil \frac{n}{2} \rceil$: since $n > N' \geq N$,

$$\begin{aligned} T(n) &\leq 2 \cdot T(\lceil \frac{n}{2} \rceil) + c \cdot n \\ &\leq 2 \cdot c' \cdot \lceil \frac{n}{2} \rceil \cdot \log_2 \lceil \frac{n}{2} \rceil + c \cdot n \\ &\leq c' \cdot (n+1) \cdot \log_2 (\frac{n+1}{2}) + c \cdot n \\ &= c' \cdot (n+1) \cdot (\log_2 (n+1) - 1) + c \cdot n \\ &\leq c' \cdot (n+1) \cdot ((\log_2 n) + \frac{1}{2} - 1) + c \cdot n \end{aligned}$$

$\log_2(n+1) \leq (\log_2 n) + \frac{1}{2}$ holds for all $n > 2$ and thus for all $n \geq N'$.



## Verifying Asymptotic Complexity (Contd)

We continue the derivation.

$$\begin{aligned} T(n) &\leq c' \cdot (n+1) \cdot ((\log_2 n) + \frac{1}{2} - 1) + c \cdot n \\ &= c' \cdot (n+1) \cdot ((\log_2 n) - \frac{1}{2}) + c \cdot n \\ &= c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot (\log_2 n) - n - 1) + c \cdot n \\ &\leq c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot \frac{n}{4} - n - 1) + c \cdot n \end{aligned}$$

Last inequality holds if $\log_2 n \leq \frac{n}{4}$, which holds for all $n \geq 16$.

## Verifying Asymptotic Complexity (Contd)

We continue the derivation.

$$
\begin{aligned}
T(n) &\leq c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot \frac{n}{4} - n - 1) + c \cdot n \\
&\leq c' \cdot n \cdot (\log_2 n) - \frac{c' \cdot n}{4} + c \cdot n \\
&= c' \cdot n \cdot (\log_2 n) + \frac{4c - c'}{4} \cdot n \leq c' \cdot n \cdot \log_2 n
\end{aligned}
$$

Last inequality holds if $c' \geq 4c$, because then $4c - c' \leq 0$.

In both cases $N' := \max\{16, N\}, c' := \max\{4c, T(N')\}$ lets proof succeed.



## The Master Theorem

Let $a \geq 1$, $b > 1$, $f : \mathbb{N} \to \mathbb{N}$, $T : \mathbb{N} \to \mathbb{N}$ satisfying the recurrence:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

- $f(n) = O(n^{(\log_b a) - \varepsilon})$ for some $\varepsilon > 0$:
$$T(n) = \Theta(n^{\log_b a})$$
- $f(n) = \Theta(n^{\log_b a})$:
$$T(n) = \Theta(n^{\log_b a} \cdot \log n)$$
- $f(n) = \Omega(n^{(\log_b a) + \varepsilon})$ for some $\varepsilon > 0$ and there exist some $c$ with $0 < c < 1$ and some $N \in \mathbb{N}$ such that $\forall n \geq N : a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$:
$$T(n) = \Theta(f(n))$$

Easy analysis of a large class of divide and conquer algorithms.



## Example

Analysis of MERGESORT.

- Recurrence:
$$T(n) = 2 \cdot T(\frac{n}{2}) + \Theta(n)$$
- Case 2 of Master Theorem ($a = b = 2$):
$$\log_b a = 1$$
$$\Theta(n) = \Theta(n^1) = \Theta(n^{\log_b a})$$
- Solution:
$$T(n) = \Theta(n^{\log_b a} \cdot \log n) = \Theta(n^1 \cdot \log n) = \Theta(n \cdot \log n)$$

No tedious proof required any more.



## Arbitrary Precision Multiplication

Multiply two natural numbers $a$ and $b$ with $n$ digits each.

| | Cost |
|---|---|
| **function** MULTIPLY$(a, b)$ | |
|   $n \leftarrow digits(a)$ $\triangleright$ $digits(a) = digits(b)$ | $1$ |
|   $c \leftarrow 0$ | $1$ |
|   **for** $i$ **from** $n-1$ **to** $0$ **do** | $n+1$ |
|     $p \leftarrow$ MULTIPLYDIGIT$(a, b_i)$ | $n \cdot \Theta(n)$ |
|     $c \leftarrow$ SHIFT$(c, 1)$ | $\sum_{i=0}^{n-1} \Theta(2n - i - 1)$ |
|     $c \leftarrow$ ADD$(p, c)$ | $n \cdot \Theta(n)$ |
|   **end for** | |
|   **return** $c$ | $1$ |
| **end function** | |

$$T(n) = 3 + (n+1) + 2n \cdot \Theta(n) + \sum_{i=0}^{n-1} \Theta(2n - i - 1) = \Theta(n^2)$$

Classical (“school”) algorithm has quadratic time complexity.

## Arbitrary Precision Multiplication

Split $a$ and $b$ into halves $(a', a'')$ and $(b', b'')$ of $\frac{n}{2}$ digits.

$$a = a' \cdot d^{\frac{n}{2}} + a''$$
$$b = b' \cdot d^{\frac{n}{2}} + b''$$

$$\begin{aligned} a \cdot b &= (a' \cdot d^{\frac{n}{2}} + a'') \cdot (b' \cdot d^{\frac{n}{2}} + b'') \\ &= a' \cdot b' \cdot d^n + (a' \cdot b'' + a'' \cdot b') \cdot d^{\frac{n}{2}} + a'' \cdot b'' \end{aligned}$$

Basis of a recursive multiplication algorithm.



## Arbitrary Precision Multiplication (Rec.)

| | Cost |
|---|---|
| **function** MULTIPLY$(a, b)$ | |
| $n \leftarrow digits(a)$ $\quad \triangleright\ digits(a) = digits(b) = 2^m$ | 1 |
| **if** $n = 1$ **then** | 1 |
| $c \leftarrow$ MULTIPLYDIGIT$(a_0, b_0)$ | $\Theta(1)$ |
| **else** | |
| $a' \leftarrow a_{\frac{n}{2}\ldots n-1}$; $a'' \leftarrow a_{0\ldots\frac{n}{2}-1}$ | $\Theta(n)$ |
| $b' \leftarrow b_{\frac{n}{2}\ldots n-1}$; $b'' \leftarrow b_{0\ldots\frac{n}{2}-1}$ | $\Theta(n)$ |
| $u \leftarrow$ MULTIPLY$(a', b')$ | $1 + T(\frac{n}{2})$ |
| $v \leftarrow$ MULTIPLY$(a', b'')$ | $1 + T(\frac{n}{2})$ |
| $w \leftarrow$ MULTIPLY$(a'', b')$ | $1 + T(\frac{n}{2})$ |
| $x \leftarrow$ MULTIPLY$(a'', b'')$ | $1 + T(\frac{n}{2})$ |
| $y \leftarrow$ ADD$(v, w)$ | $\Theta(n)$ |
| $y \leftarrow$ SHIFT$(y, \frac{n}{2})$ | $\Theta(n)$ |
| $c \leftarrow$ SHIFT$(u, n)$ | $\Theta(n)$ |
| $c \leftarrow$ ADD$(c, y)$ | $\Theta(n)$ |
| $c \leftarrow$ ADD$(c, x)$ | $\Theta(n)$ |
| **end if** | |
| **return** $c$ | 1 |
| **end function** | |

Four recursive calls of the algorithm with half the input size.



## Arbitrary Precision Multiplication (Rec.)

Analysis of recursive algorithm by the Master Theorem.

- Recurrence:

$$\begin{aligned} T(n) &= 3 + 4 \cdot \left(1 + T\left(\frac{n}{2}\right)\right) + 7 \cdot \Theta(n) \\ &= 4 \cdot T\left(\frac{n}{2}\right) + \Theta(n) \end{aligned}$$

- Case 1 of the master theorem $(a = 4, b = 2)$:

$$(\log_b a) = (\log_2 4) = 2$$
$$f(n) = O(n) = O(n^1) = O(n^{2-1}) = O(n^{(\log_b a) - 1})$$

- Solution:

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$$

Also the recursive algorithm has quadratic time complexity.



## Arbitrary Precision Multiplication

Anatolii Karatsuba and Yuri Ofman, 1962.

$$\begin{aligned} a \cdot b &= (a' \cdot d^{\frac{n}{2}} + a'') \cdot (b' \cdot d^{\frac{n}{2}} + b'') \\ &= a' \cdot b' \cdot d^n + (a' \cdot b'' + a'' \cdot b') \cdot d^{\frac{n}{2}} + a'' \cdot b'' \\ &= a' \cdot b' \cdot d^n + ((a' + a'') \cdot (b' + b'') - a' \cdot b' - a'' \cdot b'') \cdot d^{\frac{n}{2}} + a'' \cdot b'' \end{aligned}$$

- Two multiplications $a' \cdot b'$ and $a'' \cdot b''$ of numbers with $\frac{n}{2}$ digits.
- Product $s \cdot t$ where $s = a' + a''$ and $t = b' + b''$ may have $\frac{n}{2} + 1$ digits.

$$s = s_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + s', t = t_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + t'$$

$$\begin{aligned} s \cdot t &= (s_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + s') \cdot (t_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + t') \\ &= s_{\frac{n}{2}} \cdot t_{\frac{n}{2}} \cdot d^n + (s_{\frac{n}{2}} \cdot t' + t_{\frac{n}{2}} \cdot s') \cdot d^{\frac{n}{2}} + s' \cdot t' \end{aligned}$$

- Can compute $s \cdot t$ from product $s' \cdot t'$ of numbers of length $\frac{n}{2}$.

We only need three multiplications of numbers with $\frac{n}{2}$ digits.

## Karatsuba Algorithm

| function MULTIPLY$(a,b)$ | Cost |
|---|---|
| $n \leftarrow digits(a)$ ▷ $digits(a) = digits(b) = 2^m$ | 1 |
| if $n = 1$ then | 1 |
| $c \leftarrow$ MULTIPLYDIGIT$(a_0, b_0)$ | $\Theta(1)$ |
| else | |
| $a' \leftarrow a_{\frac{n}{2}\ldots n-1}$; $a'' \leftarrow a_{0\ldots\frac{n}{2}-1}$ | $\Theta(n)$ |
| $b' \leftarrow b_{\frac{n}{2}\ldots n-1}$; $b'' \leftarrow b_{0\ldots\frac{n}{2}-1}$ | $\Theta(n)$ |
| $s \leftarrow$ ADD$(a', a'')$ | $\Theta(n)$ |
| $t \leftarrow$ ADD$(b', b'')$ | $\Theta(n)$ |
| $u \leftarrow$ MULTIPLY$(a', b')$ | $1 + T(\frac{n}{2})$ |
| $v \leftarrow$ MULTIPLY'$(s, t)$ | $\Theta(n) + T(\frac{n}{2})$ |
| $x \leftarrow$ MULTIPLY$(a'', b'')$ | $1 + T(\frac{n}{2})$ |
| $y \leftarrow$ SUBTRACT$(v, u)$ | $\Theta(n)$ |
| $y \leftarrow$ SUBTRACT$(v, x)$ | $\Theta(n)$ |
| $y \leftarrow$ SHIFT$(y, \frac{n}{2})$ | $\Theta(n)$ |
| $c \leftarrow$ SHIFT$(u, n)$ | $\Theta(n)$ |
| $c \leftarrow$ ADD$(c, y)$ | $\Theta(n)$ |
| $c \leftarrow$ ADD$(c, x)$ | $\Theta(n)$ |
| end if | |
| return $c$ | 1 |
| end function | |

Three recursive calls of the algorithm with half the input size.



## Karatsuba Algorithm

Analysis of Karatsuba algorithm by the Master Theorem.

- Recurrence:

$$T(n) = 3 + 2\cdot\Big(1 + \cdot T\big(\frac{n}{2}\big)\Big) + \Big(\Theta(n) + T\big(\frac{n}{2}\big)\Big) + 10\cdot\Theta(n)$$
$$= 3\cdot T\big(\frac{n}{2}\big) + \Theta(n)$$

- Case 1 of Master Theorem ($a = 3, b = 2$):

$$\log_b a = \log_2 3$$
$$f(n) = O(n) = O(n^1) = O(n^{(\log_b a)-\varepsilon})$$
$$1.58 < \log_2 3 < 1.59, \varepsilon = (\log_2 3) - 1 > 0.58$$

- Solution:

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3}) = o(n^2)$$

The Karatsuba algorithm has a better asymptotic time complexity than the classical algorithm and is thus implemented in computer algebra systems.







## The Quicksort Algorithm

Sort array $a$ in range $[l, r]$ of size $n = r - l + 1$ in ascending order.

| procedure QUICKSORT$(a, l, r)$ ▷ $n = r - l + 1$ | Cost |
|---|---|
| if $l < r$ then | 1 |
| choose $p \in [l, r]$ | $O(n)$ |
| $m \leftarrow$ PARTITION$(a, l, r, p)$ ▷ $i = m - l$ | $\Theta(n)$ |
| QUICKSORT$(a, l, m-1)$ | $1 + T(i)$ |
| QUICKSORT$(a, m+1, r)$ | $1 + T(n - i - 1)$ |
| end if | |
| end procedure | |

Two recursive calls with input sizes $i$ and $n - i - 1$ (for some $0 \le i \le n - 1$).

## Time Complexity of Quicksort

- Recurrence:
$$T(n) = T(i) + T(n-i-1) + \Theta(n)$$
- One interval is empty ($i = 0$ or $i = n-1$):
$$T(n) = T(0) + T(n-1) + \Theta(n) = \Theta(1) + T(n-1) + \Theta(n) = T(n-1) + \Theta(n)$$
$$= \sum_{i=0}^{n-1} \Theta(i) = \Theta(n^2)$$
  - Unbalanced binary recursion tree where every left child is a leaf and the path from root along every right child to rightmost leaf has length $n$.
- Both intervals have same size ($i = \frac{n}{2}$):
$$T(n) = T(\tfrac{n}{2}) + T(\tfrac{n}{2}) + \Theta(n) = 2 \cdot T(\tfrac{n}{2}) + \Theta(n)$$
$$= \Theta(n \cdot \log n)$$
  - Case 2 of the Master Theorem ($a = b = 2$).
  - Balanced binary recursion tree of depth $\log_2 n$.

Worst case time complexity is quadratic; best case is linear-logarithmic.



## Average Time Complexity of Quicksort

Assume that all $n$ values $0, \ldots, n-1$ of $i$ are equally likely.

$$\begin{aligned}
T(n) &= \frac{1}{n} \cdot \sum_{i=0}^{n-1} (T(i) + T(n-i-1) + \Theta(n)) \\
&= \frac{1}{n} \cdot (\sum_{i=0}^{n-1} T(i) + T(n-i-1)) + \Theta(n) \\
&= \frac{1}{n} \cdot (\sum_{i=0}^{n-1} T(i) + \sum_{i=0}^{n-1} T(n-i-1)) + \Theta(n) \\
&= \frac{1}{n} \cdot (\sum_{i=0}^{n-1} T(i) + \sum_{i=0}^{n-1} T(i)) + \Theta(n) \\
&= \frac{2}{n} \cdot \sum_{i=0}^{n-1} T(i) + \Theta(n)
\end{aligned}$$

Is the average time complexity closer to the worst or to the best case?



## Average Time Complexity of Quicksort

Consider special form of recurrence (see lecture notes for the general case).

$$\begin{aligned}
T'(n) &= \frac{2}{n} \cdot \sum_{i=1}^{n-1} T'(i) + n \\
n \cdot T'(n) &= 2 \cdot \sum_{i=1}^{n-1} T'(i) + n^2 \\
(n-1) \cdot T'(n-1) &= 2 \cdot \sum_{i=1}^{n-2} T'(i) + (n-1)^2 \\
n \cdot T'(n) - (n-1) \cdot T'(n-1) &= 2 \cdot T'(n-1) + 2n - 1 \\
n \cdot T'(n) &= (n+1) \cdot T'(n-1) + 2n - 1 \\
\frac{T'(n)}{n+1} &= \frac{T'(n-1)}{n} + \frac{2n-1}{n \cdot (n+1)}
\end{aligned}$$

Terms involving $T'$ have now same shape on left and right side.



## Average Time Complexity of Quicksort

We solve the special recurrence.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{T'(i)}{i+1} &= \sum_{i=1}^{n} \Big( \frac{T'(i-1)}{i} + \frac{2i-1}{i \cdot (i+1)} \Big) \\
\sum_{i=1}^{n} \frac{T'(i)}{i+1} &= \sum_{i=1}^{n} \frac{T'(i-1)}{i} + \sum_{i=1}^{n} \frac{2i-1}{i \cdot (i+1)} \\
\sum_{i=1}^{n} \frac{T'(i)}{i+1} &= \sum_{i=0}^{n-1} \frac{T'(i)}{i+1} + \sum_{i=1}^{n} \frac{2i-1}{i \cdot (i+1)} \\
\frac{T'(n)}{n+1} &= \frac{T'(0)}{1} + \sum_{i=1}^{n} \frac{2i-1}{i \cdot (i+1)} \\
T'(n) &= (n+1) \cdot \Big( T'(0) + \sum_{i=1}^{n} \frac{2i-1}{i \cdot (i+1)} \Big) \\
T'(n) &= O(n \cdot \sum_{i=1}^{n} \frac{1}{i}) = O(n \cdot H_n) = O(n \cdot \log n)
\end{aligned}$$

The average time complexity is the same as that of the best case.

## Average Time Complexity of Quicksort

We could have also applied a computer algebra system.

```
> rsolve({T(0)=1,T(n)=(n+1)/n*T(n-1)+1},T(n));
                         (n + 1) (Psi(n + 2) + gamma)
```

- `Psi(n)`: $\Psi(n) = H_{n-1} - \gamma$,

```
In[1]:= RSolve[{T[0]==1,T[n]==(n+1)/n*T[n-1]+1},T[n],n]
Out[1]= {{T[n] -> EulerGamma + EulerGamma n +
                  PolyGamma[0, 2 + n] +
                  n PolyGamma[0, 2 + n]}}
```

- `PolyGamma[0,n]`: $\Psi(n) = H_{n-1} - \gamma$.

Result is in $O(n \cdot H_n)$.



## Ensuring the Average Time Complexity

We have assumed that all values of $i = m - l$ are equally likely but how realistic is this assumption?

- Assumption is satisfied if and only if the choice

  **choose** $p \in [l, r]$

  determines a pivot element $a[p]$ that is equally likely to be the element at any of the positions $l, \ldots, r$.
- We might choose a fixed index in interval $[l, r]$, e.g.,

  $p \leftarrow r$
- But then we get evenly distributed pivot elements only if all $n!$ permutations of $a$ occur with equal probability as inputs.

It is in practice typically hard to estimate how inputs are distributed; worst case situations (input array already sorted) might appear frequently.



## Randomization

We might shuffle the input array randomly before sorting it.

**procedure** RANDOMIZE($a$)
  $n \leftarrow length(a)$
  **for** $i$ **from** 0 **to** $n-1$ **do**
    $r \leftarrow$ RANDOM($i, n-1$)
    $b \leftarrow a[i]; a[i] \leftarrow a[r]; a[r] \leftarrow b$
  **end for**
**end procedure**

Indeed ensures that all permutations are equally likely but is costly.



## Randomization

We may simply choose the pivot element randomly.

$p \leftarrow$ RANDOM($l, r$)

In many algorithms, making a random choice (rather than making an arbitrary fixed choice) may yield an average case complexity that is independent of the input distribution.

## Amortized Analysis

Determine worst-case time complexity $T(n)$ of a sequence of $n$ operations.

- Assume that operations are performed on the same data structure.
  - Then it may be that the worst-case complexity $T_{\text{op}}(i)$ of operation $i$ can be only exhibited for some *few* elements of the sequence.
- $T(n) \leq \sum_{i=1}^{n} T_{\text{op}}(i)$
  - The worst case for the whole sequence may be smaller than the sum of the worst-cases of each operation.
- $\frac{T(n)}{n} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} T_{\text{op}}(i)$
  - The contribution of an individual operation to the worst-case complexity of the sequence may be smaller than the average of the individual worst case complexities.
- Amortized cost $\frac{T(n)}{n}$: some operations with high costs may be outweighed by many operations with low costs.

Amortized analysis is typically applied to operations that manipulate a certain data structure (e.g., a sequence of method calls on an object).



## Example

- Consider a stack with the following operations:
  - $\text{PUSH}(s,x)$: push an element $x$ on stack $s$.

    Time $O(1)$.
  - $\text{POP}(s)$: pop an element from the top of the non-empty stack $s$.

    Time $O(1)$.
  - $\text{MULTIPOP}(s,k)$: pop $k$ elements from the stack $s$ (if $s$ has $l < k$ elements, then only $l$ elements are popped).

    $O(\min\{l,k\})$ where $l$ is the number of elements on $s$.
- Sequence of $n$ operations can be performed in time $O(n^2)$.
  - Each $\text{MULTIPOP}$ operation has complexity $O(n)$.
  - The bound is correct but not tight.

We are interested in a much tighter bound.



## Aggregate Analysis

- Assume among the $n$ operations, there are $k$ $\text{MULTIPOP}$ operations.

$$n = k + \sum_{i=0}^{k} n_i$$

  $n_0, n_1, \ldots, n_k$: number of operations before/after a $\text{MULTIPOP}$.
- The total cost of the sequence is

$$T(n) = \sum_{i=0}^{k-1} O(p_i) + (n-k) \cdot O(1) = O(\sum_{i=0}^{k-1} p_i) + O(n)$$

$$\overset{(1)}{=} O(\sum_{i=0}^{k-1} n_i) + O(n) \overset{(2)}{=} O(n) + O(n) = O(n)$$

  $p_i$: number of elements popped in call $i$ of $\text{MULTIPOP}$.
  - (1) follows from $\sum_{i=0}^{k-1} p_i \leq \sum_{i=0}^{k-1} n_i$.
    - The total number of elements popped from the stack is bound by the total number of previously occurring ($\text{PUSH}$) operations.
  - (2) follows from $\sum_{i=0}^{k-1} n_i \leq n$.
    - Number of $\text{PUSH}$ operations is bound by number of all operations.
  - Sequence of $n$ operations can be performed in time $O(n)$.

The amortized cost of a single operation is $O(1)$, i.e., constant.

## The Potential Method

- Assign to operation $i$ its actual cost $c_i$ and amortized cost $\hat{c}_i$:
  - If $\hat{c}_i > c_i$, operation $i$ saves $\hat{c}_i - c_i$ "credit".
  - If $\hat{c}_i < c_i$, operation $i$ uses uses up $c_i - \hat{c}_i$ credit.
- The potential function $\Phi(s)$ maps data structure $s$ to a real number.
  - $\Phi(s)$: the credit accumulated so far (the "potential" of $s$).

$$\hat{c}_i - c_i = C \cdot (\Phi(s_i) - \Phi(s_{i-1}))$$

    - $\hat{c}_i - c_i$: the credit saved/used by operation $i$.
    - $s_0$: the initial value of $s$; $s_i$: its value after operation $i$.
    - Constant factor $C \geq 0$.
- Sum of amortized costs:

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \Big(c_i + C \cdot (\Phi(s_i) - \Phi(s_{i-1}))\Big) = \sum_{i=1}^{n} c_i + C \cdot (\Phi(s_n) - \Phi(s_0))$$

  - We can ensure $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ by ensuring $\Phi(s_n) \geq \Phi(s_0)$.

We have $T(n) \leq \sum_{i=1}^{n} \hat{c}_i$, i.e., can use amortized costs in the analysis.



## Example

Stack $s$ with $n$ PUSH, POP, MULTIPOP operations.

- Potential $\Phi(s)$: the number of elements in stack $s$.

$$\Phi(s_n) \geq 0 = \Phi(s_0)$$

- Amortized cost $\hat{c}_i = c_i + C \cdot \big(\Phi(s_i) - \Phi(s_{i-1})\big)$

  $C \geq 0$ upper bound for execution time of PUSH and POP; $C \cdot k'$ upper bound for the execution time of MULTIPOP$(s, k)$.

  $k' = \min\{k, m\}$, $m = |s|$.

  - PUSH$(s,x)$: $\hat{c}_i \leq C + C \cdot \big((m+1) - m\big) = C + C = 2 \cdot C$.
  - POP$(s)$: $\hat{c}_i \leq C + C \cdot \big((m-1) - (m)\big) = C - C = 0$.
  - MULTIPOP$(s,k)$: $\hat{c}_i \leq C \cdot k' + C \cdot \big((m-k') - m\big) = C \cdot k' - C \cdot k' = 0$.

Each operation has amortized cost $O(1)$, a sequence of $n$ operations has worst case time complexity $O(n)$.



## Dynamic Tables

Table $t$ for which a certain amount of space is allocated.

- Operation INSERT$(t,x)$ inserts value $x$ into $t$.
  - Sequence of $n$ such operations starting with an empty table.
- If space gets exhausted, $t$ is expanded:
  - More space is allocated and elements are copied from the old space to the new one.
- If $n$ elements are to be copied, time complexity of INSERT is $O(n)$.
  - However, most of the time, there is space available and INSERT can be performed in time $O(1)$.
- Time complexity $O(n^2)$ of a sequence of $n$ INSERT operations.
  - However, this bound is not tight.

We are interested in a much tighter bound.



## Aggregate Analysis

How much to expand table of size $m$?

- $m+1$: every call of INSERT triggers an expansion.

$$T(n) = 1+2+3+\ldots(n-1) = \sum_{i=1}^{n-1} i = O(n^2)$$

- $m+c$: every $c$-th call triggers an expansion.

$$T(n) = \sum_{i=1}^{\lceil \frac{n}{c} \rceil} O((i-1) \cdot c) + (n - \lceil \frac{n}{c} \rceil) \cdot O(1)$$
$$= O(\sum_{i=1}^{\lceil \frac{n}{c} \rceil} i) + O(n) = O(n^2) + O(n) = O(n^2)$$

- $m \cdot c$: every call $c^i + 1$ triggers an expansion (typically $c := 2$).

$$T(n) = \sum_{i=0}^{\lceil \log_2 n \rceil + 1} O(2^i) + (n - \lceil \log_2 n \rceil - 1) \cdot O(1)$$
$$= O(\sum_{i=0}^{\lceil \log_2 n \rceil + 1} 2^i) + O(n) = O(\sum_{i=0}^{\lfloor \log_2 n \rfloor} 2^i) + O(n) = O(2n) + O(n) = O(n)$$

With the last strategy, amortized cost of INSERT is $O(1)$.

## Analysis with the Potential Method

- Potential: $\Phi(t) := 2 \cdot \mathit{num}(t) - \mathit{size}(t)$
  - $\mathit{num}(t)$: number of elements in $t$.
  - $\mathit{size}(t)$ : the number of slots in $t$.
  - Since $\mathit{num}(t) \geq \frac{\mathit{size}(t)}{2}$, we have $\Phi(t) \geq 0$.
- Change of Potential:
  - After expansion of $t_{i-1}$: $\Phi(t_i) = 0$.
  - After insertion into $t_{i-1}$: $\Phi(t_i) = 2 + \Phi(t_{i-1})$.
  - Before expansion of $t_{i-1}$: $\Phi(t_{i-1}) = \mathit{size}(t_{i-1})$

Cost of expansion can be covered by accumulated credit.



## Analysis with the Potential Method

- Amortized cost: $\hat{c}_i = c_i + C \cdot \big(\Phi(t_i) - \Phi(t_{i-1})\big)$.

  $C$: upper bound for the cost of insertion, $C \cdot k$: upper bound for the cost of copying $k$ elements.
- INSERT does not trigger an expansion:

  $\mathit{num}(t_i) = \mathit{num}(t_{i-1}) + 1$, $\mathit{size}(t_i) = \mathit{size}(t_{i-1})$.

$$\begin{aligned}\hat{c}_i &\leq C \cdot 1 + C \cdot (2 \cdot \mathit{num}(t_i) - \mathit{size}(t_i) - (2 \cdot \mathit{num}(t_{i-1}) - \mathit{size}(t_{i-1})))\\ &= C \cdot 1 + C \cdot (2 \cdot (\mathit{num}(t_{i-1}) + 1) - \mathit{size}(t_{i-1}) - (2 \cdot \mathit{num}(t_{i-1}) - \mathit{size}(t_{i-1})))\\ &= C \cdot 1 + C \cdot 2 = 3C\end{aligned}$$

- INSERT triggers an expansion:

  $\mathit{num}(t_i) = \mathit{num}(t_{i-1}) + 1$, $\mathit{size}(t_{i-1}) = \mathit{num}(t_{i-1})$,
  $\mathit{size}(t_i) = 2 \cdot \mathit{size}(t_{i-1}) = 2 \cdot \mathit{num}(t_{i-1})$

$$\begin{aligned}\hat{c}_i &\leq C \cdot \mathit{num}(t_{i-1}) + C \cdot (2 \cdot \mathit{num}(t_i) - \mathit{size}(t_i) - (2 \cdot \mathit{num}(t_{i-1}) - \mathit{size}(t_{i-1})))\\ &= C \cdot \mathit{num}(t_{i-1}) + C \cdot (2 \cdot (\mathit{num}(t_{i-1}) + 1) - 2 \cdot \mathit{num}(t_{i-1}) - (2 \cdot \mathit{num}(t_{i-1}) - \mathit{num}(t_{i-1})))\\ &= C \cdot \mathit{num}(t_{i-1}) + C \cdot (2 - \mathit{num}(t_{i-1})) = 2C\end{aligned}$$

In average, we can perform INSERT in time $O(1)$.