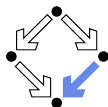
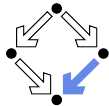


# Basics of Complexity

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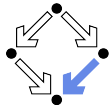
# 1. Complexity of Computations

## 2. Asymptotic Complexity

## 3. Working with Asymptotic Complexity

## 4. Complexity Classes

# Complexity of Computations

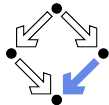


We want to determine the resource consumption of computations.

- Determine the amount of resources consumed by a computation:
  - Time
  - Space (memory)
- Determine the resource consumption for classes of inputs:
  - The **maximum** complexity for all inputs of the class.
  - The **average** complexity for these inputs.

We are going to make these notions precise.

# Resource Consumption

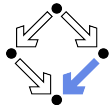


Turing machine  $M$  with input alphabet  $\Sigma$  that halts for every input.

- **Input set**  $I = \Sigma^*$ :
  - The set of input words.
- **Input size**  $|\cdot| : I \rightarrow \mathbb{N}$ 
  - $|i|$ : the length of input  $i$ .
- **Time consumption**  $t : I \rightarrow \mathbb{N}$ :
  - $t(i)$ : the number of moves that  $M$  makes for input  $i$  until it halts.
- **Space consumption**  $s : I \rightarrow \mathbb{N}$ :
  - $s(i)$ : the largest distance from the beginning of the tape that the tape head of  $M$  reaches for input  $i$  until  $M$  halts.

For any computational model,  $I$ ,  $|i|$ ,  $t(i)$  and  $s(i)$  may be defined.

# Worst-case Complexity



Computational model with  $I$ ,  $|i|$ ,  $t(i)$  and  $s(i)$  defined.

- Worst-case time complexity  $T : \mathbb{N} \rightarrow \mathbb{N}$

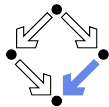
$$T(n) := \max\{t(i) \mid i \in I \wedge |i| = n\}$$

- Worst-case space complexity  $S : \mathbb{N} \rightarrow \mathbb{N}$

$$S(n) := \max\{s(i) \mid i \in I \wedge |i| = n\}$$

The maximum amount of resources consumed for any input of size  $n$  by computations in a given model.

# Average Complexity



- **Input distribution** *Input*:

- Family of (discrete) random variables  $Input_n$  that describe the distribution of inputs of each size  $n$  in  $I$
- determined by probability function  $p_I^n : I \rightarrow [0, 1]$

$p_I^n(i)$ : probability that, among all inputs of size  $n$ , input  $i$  occurs.

- **Average time/space complexity**  $\bar{T} : \mathbb{N} \rightarrow \mathbb{N}$  and  $\bar{S} : \mathbb{N} \rightarrow \mathbb{N}$

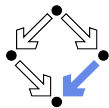
$$\bar{T}(n) := E[Time_n]$$

$$\bar{S}(n) := E[Space_n]$$

- Expected values of random variables  $Time_n$  and  $Space_n$
- determined by probability functions  $p_T^n : \mathbb{N} \rightarrow [0, 1]$  and  $p_S^n : \mathbb{N} \rightarrow [0, 1]$   
 $p_T^n(t)/p_S^n(s)$ : probabilities that time  $t$ /space  $s$  is consumed for input of size  $n$  assuming that inputs of size  $n$  are distributed according to  $Input_n$ .

The average amount of resources consumed for inputs of size  $n$  (for a given distribution of inputs) by computations in a given model.

# Example



Given non-empty integer array  $a$  of size  $n$ , find minimum index  $j$  such that  $a[j] = \max\{a[i] \mid 0 \leq i < n\}$ .

$j := 0; m := a[j]; i := 1$	1
<b>while</b> $i < \text{length}(a)$	$n$
<b>if</b> $a[i] > m$ <b>then</b>	$n - 1$
$j := i; m := a[j]$	$N \leq n - 1$
$i := i + 1$	$n - 1$

- Time: the number of lines executed.

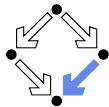
$$T(n) = 1 + n + (n - 1) + (n - 1) + (n - 1) = 4n - 2$$

- Space: the number of variables used (including elements of  $a$ ).

$$S(n) = \bar{S}(n) = n + 3$$

We are going to analyze the average time complexity  $\bar{T}(n)$ .

# Average Time Complexity



Assume  $a$  holds  $n$  distinct values  $\{1, \dots, n\}$ .

- Assume all  $n!$  permutations of  $a$  are equally probable.

$$p_i^n(i) := 1/n!$$

- Quantity  $N$  becomes random variable.

The number of times the corresponding line of the algorithm is executed for each permutation of  $a$ .

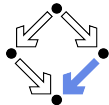
- We are interested in the expected value  $E[N]$ .
- Average time complexity  $\bar{T}(n)$ :

$$\bar{T}(n) = 1 + n + (n-1) + E[N] + (n-1) = 3n - 1 + E[N]$$

Our goal is to determine the expected value  $E[N]$ .



# Average Time Complexity (Contd)



- $p_{nk}$ : probability that  $N = k$  for array of size  $n$ .

$$p_{n0} + p_{n1} + p_{n2} + \dots + p_{n,n-1} = \sum_{k=0}^{n-1} p_{nk} = 1$$

- $p_{nk} = 0$  for  $k \geq n$ :

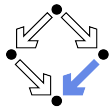
$$p_{n0} + p_{n1} + p_{n2} + \dots = \sum_k p_{nk} = 1$$

- $E[N]$  is sum of products of probability of  $N = k$  and value  $k$ :

$$E[N] = p_{n0} \cdot 0 + p_{n1} \cdot 1 + p_{n2} \cdot 2 + \dots = \sum_k p_{nk} \cdot k$$

Our goal is to determine the value of sum  $\sum_k p_{nk} \cdot k$ .

# Average Time Complexity (Contd)



We apply the technique of “generating functions”.

- $G_n(z)$ : power series with coefficients  $p_{n0}, p_{n1}, \dots$

$$G_n(z) := p_{n0} \cdot z^0 + p_{n1} \cdot z^1 + p_{n2} \cdot z^2 + \dots = \sum_k p_{nk} \cdot z^k$$

- $G'_n(z)$ : derivative of  $G_n(z)$

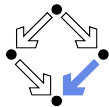
$$G'_n(z) = p_{n0} \cdot 0 \cdot z^{-1} + p_{n1} \cdot 1 \cdot z^0 + p_{n2} \cdot 2 \cdot z^1 + \dots = \sum_k p_{nk} \cdot k \cdot z^{k-1}$$

- $G'_n(1)$ :

$$G'_n(1) = p_{n0} \cdot 0 + p_{n1} \cdot 1 + p_{n2} \cdot 2 + \dots = \sum_k p_{nk} \cdot k$$

Our goal is to determine  $G'_n(1)$ .

# Average Time Complexity (Contd)



We derive a recurrence relation for  $G'_n(1)$ .

- $n = 1$ :  $p_{10} = 1$  and  $p_{1k} = 0$  for all  $k \geq 1$

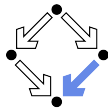
$$G'_1(1) = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 2 + \dots = 0$$

- $n > 1$ : if the loop has already found the maximum of the first  $n - 1$  array elements, the last iteration
  - will either increment  $N$  (if the last element is the largest one)
    - Probability  $1/n$ .
    - $N$  becomes  $k$  for  $n$ , if  $N$  was  $k - 1$  for  $n - 1$ .
  - or will leave  $N$  as it is (if the last element is not the largest one).
    - Probability  $(n - 1)/n$ .
    - $N$  becomes  $k$  for  $n$ , if  $N$  was  $k$  for  $n - 1$ .

$$p_{nk} = \frac{1}{n} \cdot p_{n-1, k-1} + \frac{n-1}{n} \cdot p_{n-1, k}$$

Our goal is to determine  $G'_n(1)$  for  $n > 1$ .

# Average Time Complexity (Contd)



- Determine  $G_n(z)$  from  $p_{nk}$ :

$$p_{nk} = \frac{1}{n} \cdot p_{n-1,k-1} + \frac{n-1}{n} \cdot p_{n-1,k}$$

$$G_n(z) = \frac{1}{n} \cdot z \cdot G_{n-1}(z) + \frac{n-1}{n} \cdot G_{n-1}(z) = \frac{z+n-1}{n} \cdot G_{n-1}(z)$$

- Compute  $G'_n(z)$  by derivation of  $G_n(z)$ :

$$G'_n(z) = \frac{1}{n} \cdot G_{n-1}(z) + \frac{z+n-1}{n} \cdot G'_{n-1}(z)$$

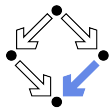
- Compute  $G'_n(1)$ :

$$G'_n(1) = \frac{1}{n} \cdot G_{n-1}(1) + \frac{z+n-1}{n} \cdot G'_{n-1}(1)$$

$$\stackrel{(*)}{=} \frac{1}{n} \cdot 1 + \frac{1+n-1}{n} \cdot G'_{n-1}(1)$$

$$= \frac{1}{n} + G'_{n-1}(1)$$

$$(*) \quad G_n(1) = p_{n0} + p_{n1} + p_{n2} + \dots = \sum_k p_{nk} = 1$$



# Average Time Complexity (Contd)

- Recurrence relation for  $G'_n(1)$ :

$$G'_1(1) = 0$$

$$G'_n(1) = \frac{1}{n} + G'_{n-1}(1), \text{ if } n > 0$$

- Solution of  $G'_n(1)$ :

$$G'_n(1) = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \sum_{k=2}^n \frac{1}{k} = H_n - 1$$

- $H(n) = \sum_{k=1}^n \frac{1}{k}$ :  $n$ -th harmonic number
- $H(n) = \ln n + \gamma + \varepsilon_n$ ,  $\gamma \approx 0.577$ ,  $0 < \varepsilon_n < 1/(2n)$ .

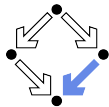
- Solution of  $E[n]$ :

$$E[N] = \ln n + \gamma + \varepsilon_n - 1$$

- Average time complexity  $\bar{T}(n)$ :

$$\bar{T}(n) = 3n - 1 + E[N] = 3n + \ln n + \varepsilon_n + \gamma - 2$$

Analysis of average complexity is more difficult than that of the worst-case.



# Complexity Approximations

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Typically, we are only interested to capture the “overall behavior” of a complexity function for large inputs.

- Exact analysis:

$$\bar{T}(n) = 3n + \ln n + \varepsilon_n + \gamma - 2$$

- Approximation:

“ $\bar{T}(n)$  is of the order  $3n + \ln n$ ”

- Coarser approximation:

“ $\bar{T}(n)$  is of the order  $3n$ ”.

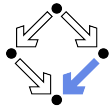
- Even coarser approximation:

“ $\bar{T}(n)$  is linear”

- Formalism:

$$\bar{T}(n) = O(n)$$

We are going to formalize such complexity approximations.



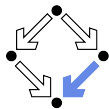
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## 1. Complexity of Computations

## 2. Asymptotic Complexity

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# The Landau Symbols

Take  $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  from the natural numbers to the non-negative reals.

- $O(g)$ : the set of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\exists c \in \mathbb{R}_{>0}, N \in \mathbb{N} : \forall n \geq N : f(n) \leq c \cdot g(n)$$

- $f(n) = O(g(n))$ :  $f \in O(g)$ .

$f$  is bounded from above by  $g$ .

- $\Omega(g)$ : the set of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\exists c \in \mathbb{R}_{>0}, N \in \mathbb{N} : \forall n \geq N : f(n) \geq c \cdot g(n)$$

- $f(n) = \Omega(g(n))$ :  $f \in \Omega(g)$ .

$f$  is bounded from below by  $g$ .

- $\Theta(g)$ : the set of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that

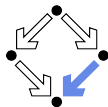
$$f \in O(g) \wedge f \in \Omega(g)$$

- $f(n) = \Theta(g(n))$ :  $f \in \Theta(g)$ .

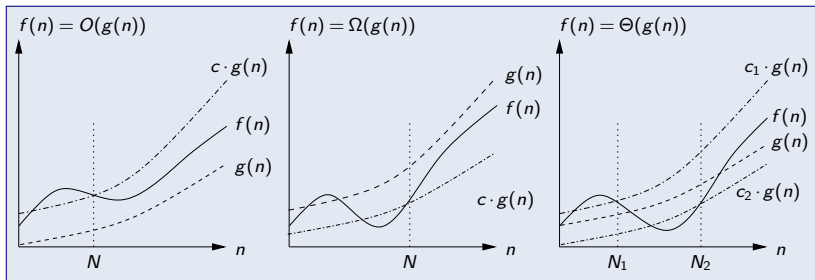
$f$  is bounded from above and below by  $g$ .



# Understanding the Landau Symbols



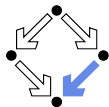
$f \in \mathcal{O}(g)$ :  $g$  represents a bound for  $f$ , from above and/or below.



- It suffices, if bound holds from a certain start value  $N$  on.
  - Finitely many exceptions are allowed.
- It suffices, if the bound holds up to arbitrarily large constant  $c$ .
  - Bounds are independent of “measurement units”.

The Landau symbols talk about the *asymptotic* behavior of functions.

# Common Practice of the Landau Symbols



We need to understand the historically developed usage of the symbols.

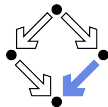
- Most wide spread:  $f(n) = O(g(n))$ .
  - Often used when actually  $f(n) = \Theta(g(n))$  is meant, i.e.,
  - when  $g(n)$  is not only a bound from above but also from below.
- Abuse of notation:  $f(n) = O(g(n))$ 
  - $=$  does not denote equality but set inclusion.
  - Notation has nevertheless been universally adopted.
- Ambiguous notation:  $f(n) = O(g(n))$ 
  - Terms  $f(n)$  and  $g(n)$  with implicit free variable  $n$ .
  - To derive  $f \in O(g)$ , we have to identify the free variable.

“Let  $c > 1$ . Then  $x^c = O(c^x)$ .”

$\rightsquigarrow$

“Let  $c > 1$ ,  $f(x) := x^c$ , and  $g(x) := c^x$ . Then  $f \in O(g)$ .”

We stick to the common practice.



# Duality of $O$ and $\Omega$

- **Theorem:** for all  $f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ , we have

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

- **Proof  $\Rightarrow$ :** We assume  $f(n) = O(g(n))$  and show  $g(n) = \Omega(f(n))$ . By the definition of  $\Omega$ , we have to find constants  $N_1, c_1$  such that

$$\forall n \geq N_1 : g(n) \geq c_1 \cdot f(n)$$

Since  $f(n) = O(g(n))$ , we have constants  $N_2, c_2$  such that

$$\forall n \geq N_2 : f(n) \leq c_2 \cdot g(n)$$

Take  $N_1 := N_2$  and  $c_1 := 1/c_2$ . Then we have, since  $N_1 = N_2$ , for all  $n \geq N_1$ ,

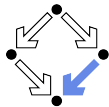
$$c_2 \cdot g(n) \geq f(n)$$

and therefore

$$g(n) \geq (1/c_2) \cdot f(n) = c_1 \cdot f(n).$$

- **Proof  $\Leftarrow$ :** analogously.

**$O$  and  $\Omega$  are dual.**



## Example

We prove  $3n^2 + 5n + 7 = O(n^2)$ .

- We have to find constants  $c$  and  $N$  such that

$$\forall n \geq N : 3n^2 + 5n + 7 \leq cn^2$$

- For  $n \geq 1$ , we have

$$3n^2 + 5n + 7 \stackrel{1 \leq n}{\leq} 3n^2 + 5n + 7n = 3n^2 + 12n$$

- For  $n \geq 12$ , we also have

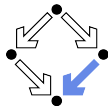
$$3n^2 + 12n \stackrel{12 \leq n}{\leq} 3n^2 + n \cdot n = 4n^2$$

- We take  $N := 12$  ( $= \max\{1, 12\}$ ) and  $c := 4$  and have for  $n \geq N$

$$3n^2 + 5n + 7 \stackrel{1 \leq n}{\leq} 3n^2 + 5n + 7n = 3n^2 + 12n \stackrel{12 \leq n}{\leq} 3n^2 + n \cdot n = 4n^2 = cn^2$$

Demonstrates general technique for asymptotics of polynomial functions.

# Asymptotic Laws



- **Theorem:** for all  $a_0, \dots, a_m \in \mathbb{R}$ , we have

$$a_m n^m + \dots + a_2 n^2 + a_1 n + a_0 = \Theta(n^m)$$

- Proof: analogous to previous example.
- **Theorem:** for all  $a, b \in \mathbb{R}_{>0}$ , we have

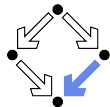
$$\log_a n = O(\log_b n)$$

- Proof: take  $c := \log_a b$  and  $N := 0$ . Then we have for all  $n \geq N$

$$\log_a n = \log_a(b^{\log_b n}) = (\log_a b) \cdot (\log_b n) = c \cdot (\log_b n)$$

Polynomials are dominated by the monomial with the highest exponent; in logarithms, bases don't matter.

# Asymptotic Laws



- **Theorem:** for all  $a, b \in \mathbb{R}$  with  $b > 1$ , we have

$$n^a = O(b^n)$$

- Proof: we know the Taylor series expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Since  $b^n = (e^{\ln b})^n = e^{n \ln b}$ , we have for all  $n \in \mathbb{N}$

$$b^n = \sum_{i=0}^{\infty} \frac{(n \ln b)^i}{i!} = 1 + (n \ln b) + \frac{(n \ln b)^2}{2!} + \frac{(n \ln b)^3}{3!} + \dots$$

Since  $b > 1$ , we have  $\ln b > 0$ ; therefore we know

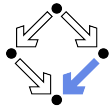
$$b^n > \frac{(n \ln b)^a}{a!} = \frac{(\ln b)^a}{a!} n^a$$

Consequently

$$n^a < \frac{a!}{(\ln b)^a} b^n$$

Thus we define  $N := 0$  and  $c := a! / (\ln b)^a$  and are done.

**Polynomials are dominated by all exponentials with base greater one.**



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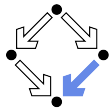
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# Asymptotic Laws



$$c \cdot f(n) = O(f(n))$$

$$n^m = O(n^{m'}), \text{ for all } m \leq m'$$

$$a_m n^m + \dots + a_2 n^2 + a_1 n + a_0 = \Theta(n^m)$$

$$\log_a n = O(\log_b n), \text{ for all } a, b > 0$$

$$n^a = O(b^n), \text{ for all } a, b \text{ with } b > 1$$

## ■ Reflexivity:

$$f = O(f), f = \Omega(f), f = \Theta(f).$$

## ■ Symmetry:

- If  $f = O(g)$ , then  $g = \Omega(f)$ .
- If  $f = \Omega(g)$ , then  $g = O(f)$ .
- If  $f = \Theta(g)$ , then  $g = \Theta(f)$ .

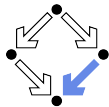
## ■ Transitivity:

- If  $f = O(g)$  and  $g = O(h)$ , then  $f = O(h)$ .
- If  $f = \Omega(g)$  and  $g = \Omega(h)$ , then  $f = \Omega(h)$ .
- If  $f = \Theta(g)$  and  $g = \Theta(h)$ , then  $f = \Theta(h)$ .

The proof of reflexivity/symmetry/transitivity is an easy exercise.



# Asymptotic Notation in Equations



A more general form of “syntactic abuse”.

■ **Equation:**

$$A[\mathcal{O}_1(f(n))] = B[\mathcal{O}_2(g(n))]$$

- (Possibly multiple) occurrences of  $\mathcal{O}_1, \mathcal{O}_2 \in \{O, \Omega, \Theta\}$ .

■ **Interpretation:**

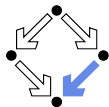
$$\forall f' \in \mathcal{O}_1(f) : \exists g' \in \mathcal{O}_2(g) :$$

$$\forall n \in \mathbb{N} : A[f'(n)] = B[g'(n)]$$

- Every occurrence of  $\mathcal{O}$  is replaced by a function in the corresponding asymptotic complexity class.
- Functions on the left side are universally quantified, functions on the right side are existentially quantified.

**A convenient shortcut to express asymptotic relationships.**

# Example



- Example:

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right)$$

- There is a function  $f \in O(1/n)$  such that, for all  $n \in \mathbb{N}$ ,  
 $H_n = \ln n + \gamma + f(n)$ .

- Example:

$$2n^2 + 3n + 1 = O(2n^2) + O(n) = O(n^2)$$

- Equation  $2n^2 + 3n + 1 = O(2n^2) + O(n)$

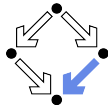
$$\exists f \in O(2n^2), g \in O(n) :$$

$$\forall n \in \mathbb{N} : 2n^2 + 3n + 1 = f(n) + g(n)$$

- Equation  $O(2n^2) + O(n) = O(n^2)$

$$\forall f \in O(2n^2), g \in O(n) : \exists h \in O(n^2) :$$

$$\forall n \in \mathbb{N} : f(n) + g(n) = h(n)$$



# Further Asymptotic Equations

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We thus can express further asymptotic relationships.

$$O(O(f(n))) = O(f(n))$$

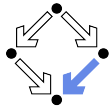
$$O(f(n)) + O(g(n)) = O(f(n) + g(n))$$

$$O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$$

$$O(f(n) \cdot g(n)) = f(n) \cdot O(g(n))$$

$$O(f(n)^m) = O(f(n))^m, \text{ for all } m \geq 0$$

The proofs are simple exercises.



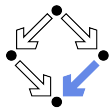
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## 1. Complexity of Computations

## 2. Asymptotic Complexity

## 3. Working with Asymptotic Complexity

## 4. Complexity Classes



## Further Landau Symbols

Take  $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  from the natural numbers to the non-negative reals.

- $o(g)$ : the set of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that
$$\forall c \in \mathbb{R}_{>0} : \exists N \in \mathbb{N} : \forall n \geq N : f(n) \leq c \cdot g(n)$$

- $f(n) = o(g(n))$ :  $f \in o(g)$ .  
 $f$  is asymptotically smaller than  $g$ .

- $\omega(g)$ : the set of all functions  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that
$$\forall c \in \mathbb{R}_{>0} : \exists N \in \mathbb{N} : \forall n \geq N : g(n) \leq c \cdot f(n)$$

- $f(n) = \omega(g(n))$ :  $f \in \omega(g)$ .  
 $f$  is asymptotically larger than  $g$ .

- **Theorem**: for all  $f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ , we have

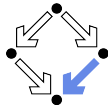
$$f \in o(g) \Leftrightarrow g \in \omega(f)$$

$$f \in o(g) \Rightarrow f \in O(g) \wedge f \notin \Theta(g)$$

$$f \in \omega(g) \Rightarrow f \in \Omega(g) \wedge f \notin \Theta(g)$$

Useful to create a hierarchy of asymptotic growth functions.

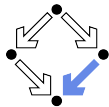
# Hierarchy of Complexity Classes



Define  $f \prec g \Leftrightarrow f = o(g)$ .

$$\begin{aligned} 1 &\prec \log \log \log n \prec \log \log n \prec \sqrt{\log n} \prec \log n \prec (\log n)^2 \prec (\log n)^3 \\ &\prec \sqrt[3]{n} \prec \sqrt{n} \prec n \prec n \log n \prec n\sqrt{n} \prec n^2 \prec n^3 \\ &\prec n^{\log n} \prec 2^{\sqrt{n}} \prec 2^n \prec 3^n \prec n! \prec n^n \prec 2^{n^2} \prec 2^{2^n} \\ &\prec 2^{2^{\dots^2}} \quad (n \text{ times}) \end{aligned}$$

Fundamental knowledge about complexity classes.

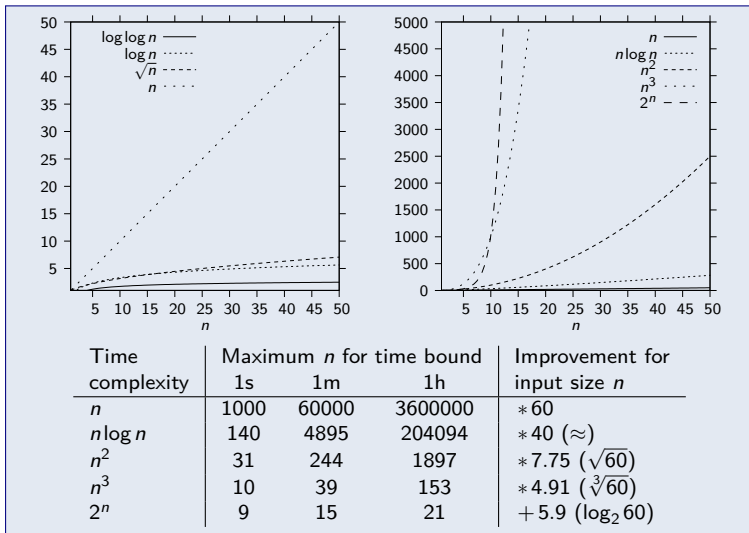
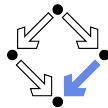


# Hierarchy of Complexity Classes

- $O(1)$  (**Constant**): upper limit on function values.
  - Space complexity of algorithms that work with fixed memory size.
- $O(\log n)$  (**Logarithmic**): values grow very slowly.
  - Time complexity of binary search.
- $O(n)$  (**Linear**): values grow proportionally with argument.
  - Time complexity of linear search.
- $O(n \log n)$  (**Linear-Logarithmic**): value growth is reasonably well behaved.
  - Time complexity of fast sorting algorithms, e.g., Mergesort.
- $O(n^c)$  (**Polynomial**): values grow rapidly but with polynomial bound.
  - Executions still “feasible” for large inputs, e.g., matrix multiplication.
- $O(c^n)$  (**Exponential**): values grow extremely rapidly.
  - Executions only reasonable for small inputs; e.g, finding exact solutions to many optimization problems (“traveling salesman problem”).
- $O(c^{d^n})$  (**Double Exponential**): Function values grow overwhelmingly rapidly.
  - Decision of statements about real numbers (“quantifier elimination”), solving multivariate polynomial equations (“Buchberger’s algorithm”).

Only computations up to polynomial complexity are considered “feasible”.

# Complexity Classes



Improvement in asymptotic complexity outperforms technological speedup.