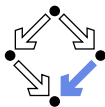


# Specifying and Verifying Programs (Part 1)

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# Specifying and Verifying Programs



We will discuss three (closely interrelated) calculi.

- **Hoare Calculus:**  $\{P\} c \{Q\}$

- If command  $c$  is executed in a pre-state with property  $P$  and terminates, it yields a post-state with property  $Q$ .

$$\{x = a \wedge y = b\} x := x + y \{x = a + y \wedge y = b\}$$

- **Predicate Transformers:**  $wp(c, Q) = P$

- If the execution of command  $c$  shall yield a post-state with property  $Q$ , it must be executed in a pre-state with property  $P$ .

$$wp(x := x + y, x = a + y \wedge y = b) = (x + y = a + y \wedge y = b)$$

- **State Relations:**  $c : [P \Rightarrow Q]^{x, \dots}$

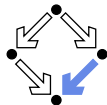
- The post-state generated by the execution of command  $c$  is related to the pre-state by  $P \Rightarrow Q$  (where only variables  $x, \dots$  have changed).

$$x = x + y : [\text{var } x = \text{old } x + \text{old } y]^x$$



- 
- 1. The Hoare Calculus**
  2. Predicate Transformers
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# The Hoare Calculus



First and best-known calculus for program reasoning (C.A.R. Hoare).

- **“Hoare triple”**:  $\{P\} c \{Q\}$ 
  - Logical propositions  $P$  and  $Q$ , program command  $c$ .
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.
- **Partial correctness** interpretation of  $\{P\} c \{Q\}$ :
  - “If  $c$  is executed in a state in which  $P$  holds, then it terminates in a state in which  $Q$  holds **unless it aborts or runs forever.**”
  - Program does not produce wrong result.
  - But program also need not produce **any** result.
    - Abortion and non-termination are not (yet) ruled out.
- **Total correctness** interpretation of  $\{P\} c \{Q\}$ :
  - “If  $c$  is executed in a state in which  $P$  holds, then it terminates in a state in which  $Q$  holds.”
  - Program produces the correct result.

**We will use the partial correctness interpretation for the moment.**



# Weakening and Strengthening

$$\frac{P \Rightarrow P' \quad \{P'\} c \{Q'\} \quad Q' \Rightarrow Q}{\{P\} c \{Q\}}$$

- Logical derivation:  $\frac{A_1 \ A_2}{B}$ 
  - Forward: If we have shown  $A_1$  and  $A_2$ , then we have also shown  $B$ .
  - Backward: To show  $B$ , it suffices to show  $A_1$  and  $A_2$ .
- Interpretation of above sentence:
  - To show that, if  $P$  holds, then  $Q$  holds after executing  $c$ , it suffices to show this for a  $P'$  weaker than  $P$  and a  $Q'$  stronger than  $Q$ .

Precondition may be weakened, postcondition may be strengthened.

# Special Commands



$\{P\}$  **skip**  $\{P\}$        $\{\text{true}\}$  **abort**  $\{\text{false}\}$

- The **skip** command does not change the state; if  $P$  holds before its execution, then  $P$  thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
  - Axiom implies  $\{P\}$  **abort**  $\{Q\}$  for arbitrary  $P, Q$ .

Useful commands for reasoning and program transformations.



# Scalar Assignments

$$\{Q[e/x]\} x := e \{Q\}$$

## ■ Syntax

- Variable  $x$ , expression  $e$ .
- $Q[e/x] \dots Q$  where every free occurrence of  $x$  is replaced by  $e$ .

## ■ Interpretation

- To make sure that  $Q$  holds for  $x$  after the assignment of  $e$  to  $x$ , it suffices to make sure that  $Q$  holds for  $e$  before the assignment.

## ■ Partial correctness

- Evaluation of  $e$  may abort.

$$\begin{array}{l} \{x + 3 < 5\} \quad x := x + 3 \quad \{x < 5\} \\ \{x < 2\} \quad x := x + 3 \quad \{x < 5\} \end{array}$$



# Array Assignments

$$\{Q[a[i \mapsto e]/a]\} \ a[i] := e \ \{Q\}$$

- An array is modelled as a **function**  $a : I \rightarrow V$ .
  - Index set  $I$ , value set  $V$ .
  - $a[i] = e$  ... array  $a$  contains at index  $i$  the value  $e$ .
- **Term**  $a[i \mapsto e]$  (“array  $a$  updated by assigning value  $e$  to index  $i$ ”)
  - A new array that contains at index  $i$  the value  $e$ .
  - All other elements of the array are the same as in  $a$ .
- **Thus array assignment becomes a special case of scalar assignment.**
  - Think of “ $a[i] := e$ ” as “ $a := a[i \mapsto e]$ ”.

$$\{\underline{a[i \mapsto x]}[1] > 0\} \ a[i] := x \ \{a[1] > 0\}$$

Arrays are here considered as basic values (no pointer semantics).





# Array Assignments

How to reason about  $a[i \mapsto e]$ ?

$$\begin{aligned} & Q[\underline{a[i \mapsto e]}[j]] \\ & \rightsquigarrow \\ & (i = j \Rightarrow Q[e]) \wedge (i \neq j \Rightarrow Q[a[j]]) \end{aligned}$$

## ■ Array Axioms

$$\begin{aligned} i = j & \Rightarrow \underline{a[i \mapsto e]}[j] = e \\ i \neq j & \Rightarrow \underline{a[i \mapsto e]}[j] = a[j] \end{aligned}$$

$$\begin{array}{lll} \{ \underline{a[i \mapsto x]}[1] > 0 \} & a[i] := x & \{ a[1] > 0 \} \\ \{ (i = 1 \Rightarrow x > 0) \wedge (i \neq 1 \Rightarrow a[1] > 0) \} & a[i] := x & \{ a[1] > 0 \} \end{array}$$

Get rid of “array update terms” when applied to indices.

# Command Sequences



$$\frac{\{P\} c_1 \{R\} \quad \{R\} c_2 \{Q\}}{\{P\} c_1; c_2 \{Q\}}$$

## ■ Interpretation

- To show that, if  $P$  holds before the execution of  $c_1; c_2$ , then  $Q$  holds afterwards, it suffices to show for some  $R$  that
  - if  $P$  holds before  $c_1$ , that  $R$  holds afterwards, and that
  - if  $R$  holds before  $c_2$ , then  $Q$  holds afterwards.

## ■ Problem: find suitable $R$ .

- Easy in many cases (see later).

$$\frac{\{x + y - 1 > 0\} y := y - 1 \{x + y > 0\} \quad \{x + y > 0\} x := x + y \{x > 0\}}{\{x + y - 1 > 0\} y := y - 1; x := x + y \{x > 0\}}$$

The calculus itself does not indicate how to find intermediate property.

# Conditionals



$$\frac{\{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{ if } b \text{ then } c_1 \text{ else } c_2 \{Q\}}$$

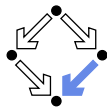
$$\frac{\{P \wedge b\} c \{Q\} \quad (P \wedge \neg b) \Rightarrow Q}{\{P\} \text{ if } b \text{ then } c \{Q\}}$$

## ■ Interpretation

- To show that, if  $P$  holds before the execution of the conditional, then  $Q$  holds afterwards,
- it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

$$\frac{\{x \neq 0 \wedge x \geq 0\} y := x \quad \{y > 0\} \quad \{x \neq 0 \wedge x < 0\} y := -x \quad \{y > 0\}}{\{x \neq 0\} \text{ if } x \geq 0 \text{ then } y := x \text{ else } y := -x \quad \{y > 0\}}$$

# Loops



$$\{ \text{true} \} \text{ loop } \{ \text{false} \} \quad \frac{\{ I \wedge b \} c \{ I \}}{\{ I \} \text{ while } b \text{ do } c \{ I \wedge \neg b \}}$$

## ■ Interpretation:

- The **loop** command does not terminate and thus trivially satisfies partial correctness.
  - Axiom implies  $\{P\} \text{ loop } \{Q\}$  for arbitrary  $P, Q$ .
- If it is the case that
  - $I$  holds before the execution of the **while**-loop and
  - $I$  also holds after every iteration of the loop body, then  $I$  holds also after the execution of the loop (together with the negation of the loop condition  $b$ ).
    - $I$  is a **loop invariant**.

## ■ Problem:

- Rule for **while**-loop does not have arbitrary pre/post-conditions  $P, Q$ .

In practice, we combine this rule with the strengthening/weakening-rule.

# Loops (Generalized)



$$\frac{P \Rightarrow I \quad \{I \wedge b\} c \{I\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\} \text{ while } b \text{ do } c \{Q\}}$$

## ■ Interpretation:

- To show that, if before the execution of a **while**-loop the property  $P$  holds, after its termination the property  $Q$  holds, it suffices to show for some property  $I$  (the **loop invariant**) that
  - $I$  holds before the loop is executed (i.e. that  $P$  implies  $I$ ),
  - if  $I$  holds when the loop body is entered (i.e. if also  $b$  holds), that after the execution of the loop body  $I$  still holds,
  - when the loop terminates (i.e. if  $b$  does not hold),  $I$  implies  $Q$ .

## ■ Problem: find appropriate loop invariant $I$ .

- Strongest relationship between all variables modified in loop body.

The calculus itself does not indicate how to find suitable loop invariant.

# Example



$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n + 1$$

$$(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I$$

$$\{I \wedge i \leq n\} s := s + i; i := i + 1 \{I\}$$

$$(I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^n j$$

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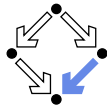
$$\{n \geq 0 \wedge i = 1 \wedge s = 0\} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \{s = \sum_{j=1}^n j\}$$

The invariant captures the “essence” of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.



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# Backward Reasoning



Implication of rule for command sequences and rule for assignments:

$$\frac{\{P\} c \{Q[e/x]\}}{\{P\} c; x := e \{Q\}}$$

## ■ Interpretation

- If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
- By multiple application, assignment sequences can be removed from the back to the front.

$$\begin{array}{l} \{P\} \\ x := x+1; \\ y := 2*x; \\ z := x+y \\ \{z = 15\} \end{array}$$

$$\begin{array}{l} \{P\} \\ x := x+1; \\ y := 2*x; \\ \{x + y = 15\} \end{array}$$

$$\begin{array}{l} \{P\} \\ x := x+1; \\ \{x + 2x = 15\} \\ (\Leftrightarrow 3x = 15) \\ (\Leftrightarrow x = 5) \end{array}$$

$$\begin{array}{l} \{P\} \\ \{x + 1 = 5\} \\ (\Leftrightarrow x = 4) \end{array}$$

$$P \Rightarrow x = 4$$





# Weakest Preconditions

A calculus for “backward reasoning” (E.W. Dijkstra).

- **Predicate transformer wp**
  - Function “wp” that takes a command  $c$  and a postcondition  $Q$  and returns a precondition.
  - Read  $wp(c, Q)$  as “the weakest precondition of  $c$  w.r.t.  $Q$ ”.
- $wp(c, Q)$  is a **precondition** for  $c$  that ensures  $Q$  as a postcondition.
  - Must satisfy  $\{wp(c, Q)\} c \{Q\}$ .
- $wp(c, Q)$  is the **weakest** such precondition.
  - Take any  $P$  such that  $\{P\} c \{Q\}$ .
  - Then  $P \Rightarrow wp(c, Q)$ .
- Consequence:  $\{P\} c \{Q\}$  iff  $(P \Rightarrow wp(c, Q))$ 
  - We want to prove  $\{P\} c \{Q\}$ .
  - We may prove  $P \Rightarrow wp(c, Q)$  instead.

Verification is reduced to the calculation of weakest preconditions.



# Weakest Preconditions

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The weakest precondition of each program construct.

$$\text{wp}(\text{skip}, Q) = Q$$

$$\text{wp}(\text{abort}, Q) = \text{true}$$

$$\text{wp}(x := e, Q) = Q[e/x]$$

$$\text{wp}(c_1; c_2, Q) = \text{wp}(c_1, \text{wp}(c_2, Q))$$

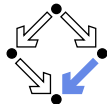
$$\text{wp}(\text{if } b \text{ then } c_1 \text{ else } c_2, Q) = (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q))$$

$$\text{wp}(\text{if } b \text{ then } c, Q) \Leftrightarrow (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)$$

$$\text{wp}(\text{while } b \text{ do } c, Q) = \dots$$

Loops represent a special problem (see later).

# Forward Reasoning



Sometimes, we want to derive a postcondition from a given precondition.

$$\{P\} x := e \{ \exists x_0 : P[x_0/x] \wedge x = e[x_0/x] \}$$

## ■ Forward Reasoning

- What is the maximum we know about the post-state of an assignment  $x := e$ , if the pre-state satisfies  $P$ ?
- We know that  $P$  holds for some value  $x_0$  (the value of  $x$  in the pre-state) and that  $x$  equals  $e[x_0/x]$ .

$$\begin{aligned} & \{x \geq 0 \wedge y = a\} \\ & \quad x := x + 1 \\ & \{ \exists x_0 : x_0 \geq 0 \wedge y = a \wedge x = x_0 + 1 \} \\ & (\Leftrightarrow (\exists x_0 : x_0 \geq 0 \wedge x = x_0 + 1) \wedge y = a) \\ & (\Leftrightarrow x > 0 \wedge y = a) \end{aligned}$$



# Strongest Postcondition

A calculus for forward reasoning.

- **Predicate transformer sp**
  - Function “sp” that takes a precondition  $P$  and a command  $c$  and returns a postcondition.
  - Read  $\text{sp}(c, P)$  as “the strongest postcondition of  $c$  w.r.t.  $P$ ”.
- $\text{sp}(c, P)$  is a **postcondition** for  $c$  that is ensured by precondition  $P$ .
  - Must satisfy  $\{P\} c \{\text{sp}(c, P)\}$ .
- $\text{sp}(c, P)$  is the **strongest** such postcondition.
  - Take any  $P, Q$  such that  $\{P\} c \{Q\}$ .
  - Then  $\text{sp}(c, P) \Rightarrow Q$ .
- **Consequence:**  $\{P\} c \{Q\}$  iff  $(\text{sp}(c, P) \Rightarrow Q)$ .
  - We want to prove  $\{P\} c \{Q\}$ .
  - We may prove  $\text{sp}(c, P) \Rightarrow Q$  instead.

**Verification is reduced to the calculation of strongest postconditions.**



# Strongest Postconditions

The strongest postcondition of each program construct.

$$\text{sp}(\text{skip}, P) = P$$

$$\text{sp}(\text{abort}, P) = \text{false}$$

$$\text{sp}(x := e, P) = \exists x_0 : P[x_0/x] \wedge x = e[x_0/x]$$

$$\text{sp}(c_1; c_2, P) = \text{sp}(c_2, \text{sp}(c_1, P))$$

$$\text{sp}(\text{if } b \text{ then } c_1 \text{ else } c_2, P) \Leftrightarrow \text{sp}(c_1, P \wedge b) \vee \text{sp}(c_2, P \wedge \neg b)$$

$$\text{sp}(\text{if } b \text{ then } c, P) = \text{sp}(c, P \wedge b) \vee (P \wedge \neg b)$$

$$\text{sp}(\text{while } b \text{ do } c, P) = \dots$$

Forward reasoning as a (less-known) alternative to backward-reasoning.

# Hoare Calculus and Predicate Transformers



In practice, often a combination of the calculi is applied.

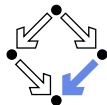
$$\{P\} c_1; \mathbf{while} \ b \ \mathbf{do} \ c; c_2 \ \{Q\}$$

- Assume  $c_1$  and  $c_2$  do not contain loop commands.
- It suffices to prove

$$\{\text{sp}(P, c_1)\} \mathbf{while} \ b \ \mathbf{do} \ c \ \{\text{wp}(c_2, Q)\}$$

Predicate transformers are applied to reduce the verification of a program to the Hoare-style verification of loops.

# Weakest Liberal Preconditions for Loops



Why not apply predicate transformers to loops?

$$\text{wp}(\mathbf{loop}, Q) = \text{true}$$

$$\text{wp}(\mathbf{while } b \mathbf{ do } c, Q) = L_0(Q) \wedge L_1(Q) \wedge L_2(Q) \wedge \dots$$

$$L_0(Q) = \text{true}$$

$$L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$$

## ■ Interpretation

- Weakest precondition that ensures that loops stops in a state satisfying  $Q$ , unless it aborts or runs forever.

## ■ Infinite sequence of predicates $L_i(Q)$ :

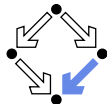
- Weakest precondition that ensures that **after less than  $i$  iterations** the state satisfies  $Q$ , unless the loop aborts or does not yet terminate.

## ■ Alternative view: $L_i(Q) = \text{wp}(\text{if}_i, Q)$

$$\text{if}_0 = \mathbf{loop}$$

$$\text{if}_{i+1} = \mathbf{if } b \mathbf{ then } (c; \text{if}_i)$$

# Example



$\text{wp}(\text{while } i < n \text{ do } i := i + 1, Q)$

$$L_0(Q) = \text{true}$$

$$L_1(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, \text{true}))$$

$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{true})$$

$$\Leftrightarrow (i \not< n \Rightarrow Q)$$

$$L_2(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, i \not< n \Rightarrow Q))$$

$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge$$

$$(i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i]))$$

$$L_3(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1,$$

$$(i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i])))$$

$$\Leftrightarrow (i \not< n \Rightarrow Q) \wedge$$

$$(i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge$$

$$(i + 1 < n \Rightarrow (i + 2 \not< n \Rightarrow Q[i + 2/i])))$$





# Weakest Liberal Preconditions for Loops

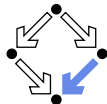
- Sequence  $L_i(Q)$  is monotonically increasing in strength:
  - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$ .
- The weakest precondition is the “lowest upper bound”:
  - $\forall i \in \mathbb{N} : \text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow L_i(Q)$ .
  - $\forall P : (\forall i \in \mathbb{N} : P \Rightarrow L_i(Q)) \Rightarrow (P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q))$ .
- We can only compute weaker **approximation**  $L_i(Q)$ .
  - $\text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow L_i(Q)$ .
- We want to prove  $\{P\} \text{ while } b \text{ do } c \{Q\}$ .
  - This is equivalent to proving  $P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q)$ .
  - Thus  $P \Rightarrow L_i(Q)$  must hold as well.
- If we can prove  $\neg(P \Rightarrow L_i(Q))$ , ...
  - $\{P\} \text{ while } b \text{ do } c \{Q\}$  does **not** hold.
  - If we fail, we may try the easier proof  $\neg(P \Rightarrow L_{i+1}(Q))$ .

Falsification is possible by use of approximation  $L_i$ , but verification is not.



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# A Constructive Definition of Arrays



```
% constructive array definition
newcontext "arrays2";

% the types
INDEX: TYPE = NAT;
ELEM:  TYPE;
ARR:   TYPE =
  [INDEX, ARRAY INDEX OF ELEM];

% error constants
any:   ARRAY INDEX OF ELEM;
anyelem: ELEM;
anyarray: ARR;

% a selector operation
content:
  ARR -> (ARRAY INDEX OF ELEM) =
    LAMBDA(a:ARR): a.1;

% the array operations
length: ARR -> INDEX =
  LAMBDA(a:ARR): a.0;
new: INDEX -> ARR =
  LAMBDA(n:INDEX): (n, any);
put: (ARR, INDEX, ELEM) -> ARR =
  LAMBDA(a:ARR, i:INDEX, e:ELEM):
    IF i < length(a)
      THEN (length(a),
            content(a) WITH [i]:=e)
      ELSE anyarray
    ENDIF;
get: (ARR, INDEX) -> ELEM =
  LAMBDA(a:ARR, i:INDEX):
    IF i < length(a)
      THEN content(a)[i]
      ELSE anyelem
    ENDIF;
```

# Proof of Fundamental Array Properties



% the classical array axioms as formulas to be proved

length1: FORMULA

FORALL(n:INDEX): length(new(n)) = n;

length2: FORMULA

FORALL(a:ARR, i:INDEX, e:ELEM):

i < length(a) => length(put(a, i, e)) = length(a);

get1: FORMULA

FORALL(a:ARR, i:INDEX, e:ELEM):

i < length(a) => get(put(a, i, e), i) = e;

get2: FORMULA

FORALL(a:ARR, i, j:INDEX, e:ELEM):

i < length(a) AND j < length(a) AND

i /= j =>

get(put(a, i, e), j) = get(a, j);

[adu]: expand length, get, put, content

[c3b]: scatter

[qid]: proved (CVCL)

# Proof of a Higher-Level Array Property



```
% extensionality on low-level arrays
```

```
extensionality: AXIOM
```

```
FORALL(a, b:ARRAY INDEX OF ELEM):  
  a=b <=> (FORALL(i:INDEX):a[i]=b[i]);
```

```
% unassigned parts hold identical values
```

```
unassigned: AXIOM
```

```
FORALL(a:ARR, i:INT):  
  (i >= length(a)) => content(a)[i
```

```
[adt]: expand length, get, content
```

```
[cw2]: scatter
```

```
[qey]: proved (CVCL)
```

```
[rey]: assume b_0.1 = a_0.1
```

```
[zpt]: proved (CVCL)
```

```
[1pt]: instantiate a_0.1, b_0.1 in 1fm
```

```
[y51]: scatter
```

```
[ku2]: auto
```

```
[iub]: proved (CVCL)
```

```
% extensionality on arrays to be proved  
equality: FORMULA
```

```
FORALL(a:ARR, b:ARR): a = b <=>  
  length(a) = length(b) AND  
  (FORALL(i:INDEX): i < length(a) => get(a,i) = get(b,i));
```



# A Program Verification

Verification of the following Hoare triple:

$$\{olda = a \wedge oldx = x \wedge n = |a| \wedge i = 0 \wedge r = -1\}$$

**while**  $i < n \wedge r = -1$  **do**

**if**  $a[i] = x$

**then**  $r := i$

**else**  $i := i + 1$

$$\{a = olda \wedge x = oldx \wedge$$

$$((r = -1 \wedge \forall i : 0 \leq i < |a| \Rightarrow a[i] \neq x) \vee$$

$$(0 \leq r < |a| \wedge a[r] = x \wedge \forall i : 0 \leq i < r \Rightarrow a[i] \neq x))\}$$

Find the smallest index  $r$  of an occurrence of value  $x$  in array  $a$  ( $r = -1$ , if  $x$  does not occur in  $a$ ).



# The Verification Conditions

$A : \Leftrightarrow \text{Input} \Rightarrow \text{Invariant}$

$B_1 : \Leftrightarrow \text{Invariant} \wedge i < n \wedge r = -1 \wedge a[i] = x \Rightarrow \text{Invariant}[i/r]$

$B_2 : \Leftrightarrow \text{Invariant} \wedge i < n \wedge r = -1 \wedge a[i] \neq x \Rightarrow \text{Invariant}[i + 1/i]$

$C : \Leftrightarrow \text{Invariant} \wedge \neg(i < n \wedge r = -1) \Rightarrow \text{Output}$

$\text{Input} : \Leftrightarrow \text{olda} = a \wedge \text{oldx} = x \wedge n = \text{length}(a) \wedge i = 0 \wedge r = -1$

$\text{Output} : \Leftrightarrow a = \text{olda} \wedge x = \text{oldx} \wedge$

$((r = -1 \wedge \forall i : 0 \leq i < \text{length}(a) \Rightarrow a[i] \neq x) \vee$

$(0 \leq r < \text{length}(a) \wedge a[r] = x \wedge \forall i : 0 \leq i < r \Rightarrow a[i] \neq x))$

$\text{Invariant} : \Leftrightarrow \text{olda} = a \wedge \text{oldx} = x \wedge n = \text{length}(a) \wedge$

$0 \leq i \leq n \wedge \forall j : 0 \leq j < i \Rightarrow a[j] \neq x \wedge$

$(r = -1 \vee (r = i \wedge i < n \wedge a[r] = x))$

The verification conditions  $A, B_1, B_2, C$  have to be proved.

# The Verification Conditions



```
newcontext      Input: BOOLEAN = olda = a AND oldx = x AND
  "linsearch";      n = length(a) AND i = 0 AND r = -1;

% declaration      Output: BOOLEAN = a = olda AND
% of arrays        ((r = -1 AND
...                (FORALL(j:NAT): j < length(a) =>
                    get(a,j) /= x)) OR
a: ARR;           (0 <= r AND r < length(a) AND get(a,r) = x AND
olda: ARR;        (FORALL(j:NAT):
x: ELEM;          j < r => get(a,j) /= x)));
oldx: ELEM;

i: NAT;           Invariant: (ARR, ELEM, NAT, NAT, INT) -> BOOLEAN =
n: NAT;           LAMBDA(a: ARR, x: ELEM, i: NAT, n: NAT, r: INT):
r: INT;           olda = a AND oldx = x AND
                  n = length(a) AND i <= n AND
                  (FORALL(j:NAT): j < i => get(a,j) /= x) AND
                  (r = -1 OR (r = i AND i < n AND get(a,r) = x));
...

```





# The Verification Conditions (Contd)

---

...

A: FORMULA

Input  $\Rightarrow$  Invariant(a, x, i, n, r);

B1: FORMULA

Invariant(a, x, i, n, r) AND  $i < n$  AND  $r = -1$  AND  $\text{get}(a,i) = x$   
 $\Rightarrow$  Invariant(a,x,i,n,i);

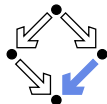
B2: FORMULA

Invariant(a, x, i, n, r) AND  $i < n$  AND  $r = -1$  AND  $\text{get}(a,i) \neq x$   
 $\Rightarrow$  Invariant(a,x,i+1,n,r);

C: FORMULA

Invariant(a, x, i, n, r) AND NOT( $i < n$  AND  $r = -1$ )  
 $\Rightarrow$  Output;

# The Proofs



A: [bca]: expand Input, Invariant  
[fuo]: scatter  
[bxg]: proved (CVCL)

(2 user actions)

B1: [p1b]: expand Invariant  
[lf6]: proved (CVCL)

(1 user action)

B2: [q1b]: expand Invariant in 6kv  
[slx]: scatter  
[a1y]: auto  
[cch]: proved (CVCL)  
[b1y]: proved (CVCL)  
[c1y]: proved (CVCL)  
[d1y]: proved (CVCL)  
[e1y]: proved (CVCL)

(3 user actions)

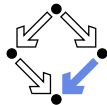
C: [dca]: expand Invariant, Output in zfg  
[tvy]: scatter  
[dca]: auto  
[t4c]: proved (CVCL)  
[ecu]: split pkg  
[kel]: proved (CVCL)  
[lel]: scatter  
[lvn]: auto  
[lap]: proved (CVCL)  
[fcu]: auto  
[blt]: proved (CVCL)  
[gcu]: proved (CVCL)

(6 user actions)



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# Termination



Hoare rules for **loop** and **while** are replaced as follows:

$$\frac{\{ \text{false} \} \text{loop} \{ \text{false} \} \quad I \Rightarrow t \geq 0 \quad \{ I \wedge b \wedge t = N \} c \{ I \wedge t < N \}}{\{ I \} \text{while } b \text{ do } c \{ I \wedge \neg b \}}$$

- New interpretation of  $\{P\} c \{Q\}$ .
  - If execution of  $c$  starts in a state where  $P$  holds, then execution **terminates** in a state where  $Q$  holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The **loop** command thus does not satisfy total correctness.
- **Termination term  $t$**  (type-checked to denote an integer).
  - Becomes smaller by every iteration of the loop.
  - But does not become negative.
  - Consequently, the loop must eventually terminate.  
The initial value of  $t$  limits the number of loop iterations.

*Any well-founded ordering may be used for the domain of  $t$ .*

# Example



$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$

$$(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I \quad I \Rightarrow n - i + 1 \geq 0$$

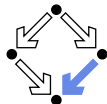
$$\{I \wedge i \leq n \wedge n - i + 1 = N\} s := s + i; i := i + 1 \{I \wedge n - i + 1 < N\}$$

$$(I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^n j$$

---

$$\{n \geq 0 \wedge i = 1 \wedge s = 0\} \text{ while } i \leq n \text{ do } (s := s + i; i := i + 1) \{s = \sum_{j=1}^n j\}$$

In practice, termination is easy to show (compared to partial correctness).



# Weakest Preconditions for Loops

$\text{wp}(\text{loop}, Q) = \text{false}$

$\text{wp}(\text{while } b \text{ do } c, Q) = L_0(Q) \vee L_1(Q) \vee L_2(Q) \vee \dots$

$L_0(Q) = \text{false}$

$L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$

## ■ New interpretation

- Weakest precondition that ensures that the loop terminates in a state in which  $Q$  holds, unless it aborts.

## ■ New interpretation of $L_i(Q)$

- Weakest precondition that ensures that the loop terminates **after less than  $i$  iterations** in a state in which  $Q$  holds, unless it aborts.

## ■ Preserves property: $\{P\} c \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$

- Now for **total correctness** interpretation of Hoare calculus.

## ■ Preserves alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

$\text{if}_0 = \text{loop}$

$\text{if}_{i+1} = \text{if } b \text{ then } (c; \text{if}_i)$

# Example



$wp(\text{while } i < n \text{ do } i := i + 1, Q)$

$$L_0(Q) = \text{false}$$

$$\begin{aligned} L_1(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow wp(i := i + 1, L_0(Q))) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{false}) \\ &\Leftrightarrow i \not< n \wedge Q \end{aligned}$$

$$\begin{aligned} L_2(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow wp(i := i + 1, L_1(Q))) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\ &\quad i < n \Rightarrow (i + 1 \not< n \wedge Q[i + 1/i]) \end{aligned}$$

$$\begin{aligned} L_3(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow wp(i := i + 1, L_2(Q))) \\ &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\ &\quad (i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge \\ &\quad (i + 1 < n \Rightarrow (i + 2 \not< n \wedge Q[i + 2/i]))) \end{aligned}$$

...



# Weakest Preconditions for Loops

- Sequence  $L_i(Q)$  is now monotonically **decreasing** in strength:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q)$ .
- The weakest precondition is the “greatest lower bound”:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q)$ .
  - $\forall P : (\forall i \in \mathbb{N} : L_i(Q) \Rightarrow P) \Rightarrow (\text{wp}(\text{while } b \text{ do } c, Q) \Rightarrow P)$ .
- We can only compute a stronger approximation  $L_i(Q)$ .
  - $L_i(Q) \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q)$ .
- We want to prove  $\{P\} c \{Q\}$ .
  - It suffices to prove  $P \Rightarrow \text{wp}(\text{while } b \text{ do } c, Q)$ .
  - It thus also suffices to prove  $P \Rightarrow L_i(Q)$ .
  - If proof fails, we may try the easier proof  $P \Rightarrow L_{i+1}(Q)$

However, verifications are typically not successful with any finite approximation of the weakest precondition.





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# Abortion



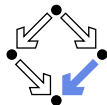
New rules to prevent abortion.

$$\begin{aligned} & \{\text{false}\} \mathbf{abort} \{\text{true}\} \\ & \{Q[e/x] \wedge D(e)\} x := e \{Q\} \\ & \{Q[a[i \mapsto e]/a] \wedge D(e) \wedge D(i) \wedge 0 \leq i < \text{length}(a)\} a[i] := e \{Q\} \end{aligned}$$

- New interpretation of  $\{P\} c \{Q\}$ .
  - If execution of  $c$  starts in a state, in which property  $P$  holds, then it does not abort and eventually terminates in a state in which  $Q$  holds.
- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

$D(e)$  makes sure that every subexpression of  $e$  is well defined.

# Definedness of Expressions



$$D(0) = \text{true.}$$

$$D(1) = \text{true.}$$

$$D(x) = \text{true.}$$

$$D(a[i]) = D(i) \wedge 0 \leq i < \text{length}(a).$$

$$D(e_1 + e_2) = D(e_1) \wedge D(e_2).$$

$$D(e_1 * e_2) = D(e_1) \wedge D(e_2).$$

$$D(e_1 / e_2) = D(e_1) \wedge D(e_2) \wedge e_2 \neq 0.$$

$$D(\text{true}) = \text{true.}$$

$$D(\text{false}) = \text{true.}$$

$$D(\neg b) = D(b).$$

$$D(b_1 \wedge b_2) = D(b_1) \wedge D(b_2).$$

$$D(b_1 \vee b_2) = D(b_1) \wedge D(b_2).$$

$$D(e_1 < e_2) = D(e_1) \wedge D(e_2).$$

$$D(e_1 \leq e_2) = D(e_1) \wedge D(e_2).$$

$$D(e_1 > e_2) = D(e_1) \wedge D(e_2).$$

$$D(e_1 \geq e_2) = D(e_1) \wedge D(e_2).$$

Assumes that expressions have already been type-checked.

# Abortion



Slight modification of existing rules.

$$\frac{P \Rightarrow D(b) \quad \{P \wedge b\} c_1 \{Q\} \quad \{P \wedge \neg b\} c_2 \{Q\}}{\{P\} \text{ if } b \text{ then } c_1 \text{ else } c_2 \{Q\}}$$

$$\frac{P \Rightarrow D(b) \quad \{P \wedge b\} c \{Q\} \quad (P \wedge \neg b) \Rightarrow Q}{\{P\} \text{ if } b \text{ then } c \{Q\}}$$

$$\frac{I \Rightarrow (t \geq 0 \wedge D(b)) \quad \{I \wedge b \wedge t = N\} c \{I \wedge t < N\}}{\{I\} \text{ while } b \text{ do } c \{I \wedge \neg b\}}$$

Expressions must be defined in any context.



# Abortion

Similar modifications of weakest preconditions.

$$\text{wp}(\mathbf{abort}, Q) = \text{false}$$

$$\text{wp}(x := e, Q) = Q[e/x] \wedge D(e)$$

$$\text{wp}(\mathbf{if } b \mathbf{ then } c_1 \mathbf{ else } c_2, Q) =$$

$$D(b) \wedge (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q))$$

$$\text{wp}(\mathbf{if } b \mathbf{ then } c, Q) = D(b) \wedge (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)$$

$$\text{wp}(\mathbf{while } b \mathbf{ do } c, Q) = (L_0(Q) \vee L_1(Q) \vee L_2(Q) \vee \dots)$$

$$L_0(Q) = \text{false}$$

$$L_{i+1}(Q) = D(b) \wedge (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$$

$\text{wp}(c, Q)$  now makes sure that the execution of  $c$  does not abort but eventually terminates in a state in which  $Q$  holds.



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# Procedure Specifications



global  $g$ ;  
requires  $Pre$ ;  
ensures  $Post$ ;  
 $o := p(i) \{ c \}$

- Specification of a procedure  $p$  implemented by a command  $c$ .
  - Input parameter  $i$ , output parameter  $o$ , global variable  $g$ .
    - Command  $c$  may read/write  $i$ ,  $o$ , and  $g$ .
  - Precondition  $Pre$  (may refer to  $i, g$ ).
  - Postcondition  $Post$  (may refer to  $i, o, g, g_0$ ).
    - $g_0$  denotes the value of  $g$  before the execution of  $p$ .

## ■ Proof obligation

$$\{ Pre \wedge i_0 = i \wedge g_0 = g \} c \{ Post[i_0/i] \}$$

Proof of the correctness of the implementation of a procedure with respect to its specification.

# Example



## ■ Procedure specification:

global  $g$

requires  $g \geq 0 \wedge i > 0$

ensures  $g_0 = g \cdot i + o \wedge 0 \leq o < i$

$o := p(i) \{ o := g \% i; g := g / i \}$

## ■ Proof obligation:

$\{g \geq 0 \wedge i > 0 \wedge i_0 = i \wedge g_0 = g\}$

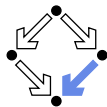
$o := g \% i; g := g / i$

$\{g_0 = g \cdot i_0 + o \wedge 0 \leq o < i_0\}$

A procedure that divides  $g$  by  $i$  and returns the remainder.



# Procedure Calls



A call of  $p$  provides actual input argument  $e$  and output variable  $x$ .

$$x := p(e)$$

Similar to assignment statement; we thus first give an alternative (equivalent) version of the assignment rule.

- Original:

$$\begin{array}{c} \{D(e) \wedge Q[e/x]\} \\ x := e \\ \{Q\} \end{array}$$

- Alternative:

$$\begin{array}{c} \{D(e) \wedge \forall x' : x' = e \Rightarrow Q[x'/x]\} \\ x := e \\ \{Q\} \end{array}$$

The new value of  $x$  is given name  $x'$  in the precondition.

# Procedure Calls



From this, we can derive a rule for the correctness of procedure calls.

$$\frac{\{D(e) \wedge Pre[e/i] \wedge \forall x', g' : Post[e/i, x'/o, g'/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]\}}{x := p(e) \{Q\}}$$

- $Pre[e/i]$  refers to the values of the actual argument  $e$  (rather than to the formal parameter  $i$ ).
- $x'$  and  $g'$  denote the values of the vars  $x$  and  $g$  after the call.
- $Post[...]$  refers to the argument values before and after the call.
- $Q[x'/x, g'/g]$  refers to the argument values after the call.

**Modular reasoning:** rule only relies on the *specification* of  $p$ , not on its implementation.

# Corresponding Predicate Transformers



$$\begin{aligned} \text{wp}(x = p(e), Q) = & \\ & D(e) \wedge \text{Pre}[e/i] \wedge \\ & \forall x', g' : \\ & \quad \text{Post}[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g] \end{aligned}$$

$$\begin{aligned} \text{sp}(P, x = p(e)) = & \\ & \exists x_0, g_0 : \\ & \quad P[x_0/y, g_0/g] \wedge \\ & \quad (\text{Pre}[e[x_0/x, g_0/g]/i, g_0/g] \Rightarrow \text{Post}[e[x_0/x, g_0/g]/i, x/o]) \end{aligned}$$

Explicit naming of old/new values required.



# Example

## ■ Procedure specification:

global  $g$

requires  $g \geq 0 \wedge i > 0$

ensures  $g_0 = g \cdot i + o \wedge 0 \leq o < i$

$o = p(i) \{ o := g \% i; g := g / i \}$

## ■ Procedure call:

$\{g \geq 0 \wedge g = N \wedge b \geq 0\}$

$x = p(b + 1)$

$\{g \cdot (b + 1) \leq N < (g + 1) \cdot (b + 1)\}$

## ■ To be proved:

$g \geq 0 \wedge g = N \wedge b \geq 0 \Rightarrow$

$D(b + 1) \wedge g \geq 0 \wedge b + 1 > 0 \wedge$

$\forall x', g' :$

$g = g' \cdot (b + 1) + x' \wedge 0 \leq x' < b + 1 \Rightarrow$

$g' \cdot (b + 1) \leq N < (g' + 1) \cdot (b + 1)$