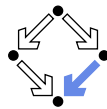


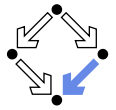
# Logic

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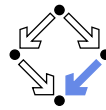
# Term Syntax



Take signature  $\Sigma = (S, \Omega)$ .

- **Variables:**
  - Family  $V = (V_s)_{s \in S}$  of infinite sets disjoint with  $\Omega$  and each other.
    - $V_s$  ... the set of variables of sort  $s$ .
  - Any family  $X \subseteq V$  is called a **set of variables** for  $\Sigma$ .
- **Terms:**
  - Family  $T_{\Sigma(X)} = (T_{\Sigma(X),s})_{s \in S}$  of terms with set of variables  $X$  for  $\Sigma$ .
    - Variables are terms:  $X_s \subseteq T_{\Sigma(X),s}$ .
    - Constants are terms: if  $n : \rightarrow s \in \Omega$ , then  $n \in T_{\Sigma(X),s}$ .
    - Applications are terms: if  $n : s_1 \times \dots \times s_k \rightarrow s \in \Omega$  and, for  $1 \leq i \leq k$ ,  $t_i \in T_{\Sigma(X),s_i}$ , then  $n(t_1, \dots, t_k) \in T_{\Sigma(X),s}$ .
- $Var(t) \subseteq X$ :
  - The set of variables occurring in term  $t \in T_{\Sigma(X)}$ .
- **Ground terms:**
  - Term  $t$  is a ground term, if  $Var(t) = \emptyset$ .
  - The set of ground terms  $T_{\Sigma} = (T_{\Sigma,s})_{s \in S}$ .

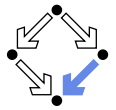
# Example



- Signature  $NATBOOL = (\{nat, bool\}, \{True : \rightarrow bool, False : \rightarrow bool, \neg : bool \rightarrow bool, \wedge : bool \times bool \rightarrow bool, 0 : \rightarrow nat, Succ : nat \rightarrow nat, + : nat \times nat \rightarrow nat, \leq : nat \times nat \rightarrow bool\})$
- Variable set  $X$  with  $X_{bool} = \{b, c\}$  and  $X_{nat} = \{m, n\}$ .
- Terms in  $T_{NATBOOL(X),bool}$ :
  - $c$
  - $\wedge(\wedge(True, b), False)$
  - $\leq(0, +(m, Succ(n)))$

All terms are strongly typed.

# Term Semantics



Take signature  $\Sigma = (S, \Omega)$ , set of variables  $X$  for  $\Sigma$ ,  $\Sigma$ -algebra  $A$ .

- **Assignment**  $\alpha : X \rightarrow A$  of  $X$  in  $A$ :
  - Family  $\alpha = (\alpha_s)_{s \in S}$  of functions  $\alpha_s : X_s \rightarrow A(s)$ .
    - Every variable is mapped to an  $A$ -value of the corresponding sort.
- **Value**  $A(\alpha)(t)$  of term  $t$  for assignment  $\alpha$ :
  - If  $t = x$  with  $x \in X_s$ , then  $\alpha_s(x)$ .
  - If  $t = n$  with  $\omega = n : \rightarrow s \in \Omega$ , then  $A(\omega)$ .
  - If  $t = n(t_1, \dots, t_k)$  with  $\omega = n : s_1 \times \dots \times s_k \rightarrow s \in \Omega$  and, for  $1 \leq i \leq k$ ,  $t_i \in T_{\Sigma(X),s_i}$ , then  $A(\omega)(A(\alpha)(t_1), \dots, A(\alpha)(t_k))$ .
- **Value**  $A(t)$  of ground term  $t$ .
  - $A(\alpha)(t)$  for any assignment  $\alpha$ .
    - Value of ground term does not depend on assignment.

Semantics maps terms to algebra values.

## Algebra Logic

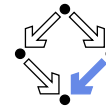


General logical framework for specifying ADTs.

- **(Algebra) Logic  $L$** : for each signature  $\Sigma$ ,
  - a set  $L(\Sigma)$  of  $\Sigma$ -formulas.
  - a relation  $\models_{\Sigma} \subseteq \text{Alg}(\Sigma) \times L(\Sigma)$  between  $\Sigma$ -algebras and  $\Sigma$ -formulas (the **satisfaction relation** for  $\Sigma$ ).
    - If  $A \models_{\Sigma} \varphi$ , we say “ $\varphi$  is valid in  $A$ ” or “ $A$  satisfies  $\varphi$ ”.
  - $L$  must satisfy the **isomorphism condition**:
    - If  $A \simeq B$ , then  $(A \models_{\Sigma} \varphi \text{ iff } B \models_{\Sigma} \varphi)$ .
    - For any signature  $\Sigma$ ,  $\Sigma$ -formula  $\varphi$ ,  $\Sigma$ -algebras  $A$  and  $B$ .
    - $L$  cannot distinguish between isomorphic algebras.
    - $L$  has no more information about  $A$  and  $B$  than visible in  $\Sigma$ .

We will investigate three specific logics.

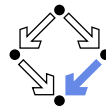
## Equational Logic $EL$



- **Formulas  $EL(\Sigma)$** :
  - $EL(\Sigma) = \{\forall X.t = u \mid X \text{ is a set of variables for } \Sigma, t, u \in T_{\Sigma(X),s} \text{ for some sort } s \text{ of } \Sigma\}$ .
  - May drop “ $\forall X$ ”, if  $X = \text{Var}(t) \cup \text{Var}(u)$ .
- **Satisfaction Relation  $\models_{\Sigma}$** :
  - $A \models_{\Sigma} \forall X.t = u$  iff for all assignments  $\alpha : X \rightarrow A$ :  
 $A(\alpha)(t) = A(\alpha)(u)$
  - For each  $\Sigma$ -algebra  $A$  and equation  $\forall X.t = u \in EL(\Sigma)$ .

The logic of universally quantified equations.

## Example

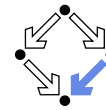


Take “classical” NATBOOL-algebra  $A$  (with  $A(\text{nat}) = \mathbb{N}$ ).

- $A \models x + 1 = 1 + x$
- $A \models (x \leq 0 \wedge \neg x \leq 0) = \text{False}$
- $A \models x = x$
- $A \not\models x = y$

Note: predicate  $\leq$  is operation of sort *bool*.

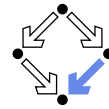
## Conditional Equational Logic $CEL$



- **Formulas  $CEL(\Sigma)$** :
  - $CEL(\Sigma) = \{\forall X.t_1 = u_1 \wedge \dots \wedge t_k = u_k \Rightarrow t_{k+1} = u_{k+1} \mid X \text{ is a set of variables for } \Sigma, t_i, u_i \in T_{\Sigma(X),s_i} \text{ for some sort } s_i\}$ .
  - Drop “ $\forall X$ ”, if  $X = \text{Var}(t_1) \cup \text{Var}(u_1) \cup \dots \cup \text{Var}(t_{k+1}) \cup \text{Var}(u_{k+1})$ .
- **Satisfaction Relation  $\models_{\Sigma}$** :
  - $A \models_{\Sigma} \forall X.t_1 = u_1 \wedge \dots \wedge t_k = u_k \Rightarrow t_{k+1} = u_{k+1}$  iff for all assignments  $\alpha : X \rightarrow A$ :  
if  $A(\alpha)(t_1) = A(\alpha)(u_1)$  and  $\dots$  and  $A(\alpha)(t_k) = A(\alpha)(u_k)$  then  
 $A(\alpha)(t_{k+1}) = A(\alpha)(u_{k+1})$ .

The logic of universally quantified conditional equations.

## Example



Take “classical” NATBOOL-algebra  $A$  (with  $A(\text{nat}) = \mathbb{N}$ ) augmented by operation  $- : \text{nat} \times \text{nat} \rightarrow \text{nat}$ .

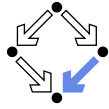
$$A \models x \leq y = \text{True} \Rightarrow (y - x) + x = y$$

$$A \models x + y = z \Rightarrow z - y = x$$

$$A \models x \leq y = \text{False} \Rightarrow y \leq x = \text{True}$$

Note: only equalities allowed as atomic predicates.

## First-Order Predicate Logic $PL$



### Formulas $PL(\Sigma)$ :

- If  $t, u \in T_{\Sigma(X),s}$  for some sort  $s$  of  $\Sigma$ , then  $t = u \in PL(\Sigma)$ .
- If  $\varphi \in PL(\Sigma)$ , then  $\neg\varphi \in PL(\Sigma)$ .
- If  $\varphi_1, \varphi_2 \in PL(\Sigma)$ , then  $\varphi_1 \wedge \varphi_2 \in PL(\Sigma)$ .
- If  $s$  is a sort of  $\Sigma$ ,  $x$  is a variable of sort  $s$ , and  $\varphi \in PL(\Sigma)$ , then  $(\forall x : s . \varphi) \in PL(\Sigma)$ .

### Value $A(\alpha)(\varphi)$ of formula $\varphi$ for assignment $\alpha : \text{free}(\varphi) \rightarrow A$ : ( $\text{free}(\varphi)$ ... the set of free variables of $\varphi$ )

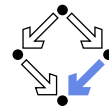
- $A(\alpha)(t = u) = \text{true}$  iff  $A(\alpha)(t) = A(\alpha)(u)$ .
- $A(\alpha)(\neg\varphi) = \text{true}$  iff  $A(\alpha)(\varphi) = \text{false}$ .
- $A(\alpha)(\varphi_1 \wedge \varphi_2) = \text{true}$  iff  $A(\alpha)(\varphi_1) = A(\alpha)(\varphi_2) = \text{true}$ .
- $A(\alpha)(\forall x : s . \varphi) = \text{true}$  iff  $A(\alpha[a/x])(\varphi) = \text{true}$  for all  $a \in A(s)$ .
  - $\alpha[a/x](x) = a$ ;  $\alpha[a/x](y) = \alpha(y)$ , if  $x \neq y$ .

### Satisfaction Relation $\models_{\Sigma}$ :

- $A \models_{\Sigma} (\varphi)$  iff  $A(\alpha)(\varphi) = \text{true}$  for all assignments  $\alpha : \text{free}(\varphi) \rightarrow A$ .

Classical predicate logic in a typed framework.

## Example



Take “classical” NATBOOL-algebra  $A$  (with  $A(\text{nat}) = \mathbb{N}$ ).

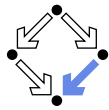
$$A \models (\forall x : \text{nat} . (0 \leq x) = \text{True})$$

$$A \models \neg(\forall x : \text{nat} . (\forall y : \text{nat} . (x \leq y) = \text{True})).$$

$$A \models (\forall x : \text{nat} . (\forall y : \text{nat} . (x \leq y) = \text{True}) \Rightarrow x = 0)$$

The connectives  $\vee, \Rightarrow, \Leftrightarrow$  and the quantifier  $\exists$  can be introduced as abbreviations of formulas that use  $\neg, \wedge, \forall$  (e.g.  $a \vee b : \Leftrightarrow \neg(\neg a \wedge \neg b)$ ).

## Models



### A model of a set of formulas $\Phi \subseteq L(\Sigma)$ :

- A  $\Sigma$ -algebra  $A$  is a model of  $\Phi$  iff  $A \models_{\Sigma} \Phi$ .
  - $A \models_{\Sigma} \Phi$  iff  $A \models_{\Sigma} \varphi$  for all  $\varphi \in \Phi$ .

### Domain (universe) for a signature $\Sigma$ (a $\Sigma$ -domain):

- A class  $\mathcal{U}$  of  $\Sigma$ -algebras closed under isomorphism.
  - Note: a domain is an abstract datatype.

### $Mod_{\mathcal{U},\Sigma}(\Phi) \subseteq \mathcal{U}$ :

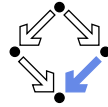
- The class of all algebras of domain  $\mathcal{U}$  that are models of  $\Phi$ .
  - If  $\Sigma$  is clear, then we write  $Mod_{\mathcal{U}}(\Phi)$ .
  - If  $\mathcal{U} = \text{Alg}(\Sigma)$ , then we write  $Mod_{\Sigma}(\Phi)$ .
  - If both holds, then we simply write  $Mod(\Phi)$ .

### Theorem: $Mod_{\mathcal{U},\Sigma}(\Phi)$ is an abstract datatype.

- Logic  $L$ , signature  $\Sigma$ , formula set  $\Phi \subseteq L(\Sigma)$ ,  $\Sigma$ -domain  $\mathcal{U}$ .

A set of formulas specifies a subset of a given  $\Sigma$ -domain as an ADT.

## Example



- $\Sigma = (\{s\}, \{0 : \rightarrow s, + : s \times s \rightarrow s\})$ .
- $\Phi = \{x + (y + z) = (x + y) + z,$   
 $x + 0 = x,$   
 $0 + x = x,$   
 $\forall x : s . \exists y : s . x + y = 0 \wedge y + x = 0\}$ .
- $Mod_{\Sigma}(\Phi) = \{A \in Alg(\Sigma) \mid A(s) \text{ and } A(+)$   
form a group with neutral element  $A(0)\}$ .

Specification of the abstract datatype “group” (polymorphic, because the group may or may not be commutative).