Analysis of Complexity

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Example

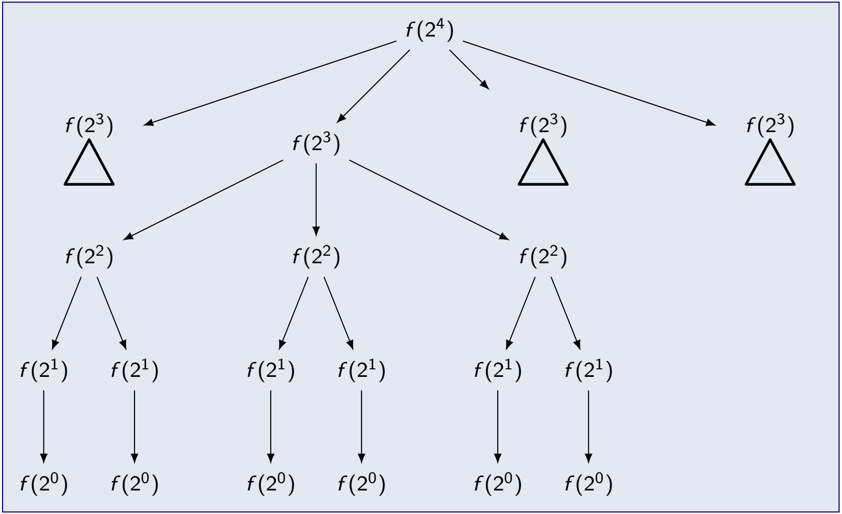
We are going to analyze the following program function:

```
static int f(int m) {
  if (m == 1) return 1;
  int s = 1;
  for (int i=0; i<log2(m); i++)
     s = s+f(m/2);
  return s;
}
```

- $f(m)$ calls $f(m/2)$ recursively $\lfloor \log_2 m \rfloor$ times .
  - $f(2^n)$ calls $f(2^{n-1})$ recursively n times.
- How often is f called in total when executing $f(m) = f(2^n)$?
  - Actually, this value is also the result of f.

The analysis of a program involving both loops and recursion.

Recursion Tree



Recursion Tree

Each node in the recursion tree denotes one function call.

- Tree of height 4:
  - Level 0: 1 node.
  - Level 1: 4 nodes.
  - Level 2: $4 \cdot 3$ nodes.
  - Level 3: $4 \cdot 3 \cdot 2$ nodes.
  - Level 4: $4 \cdot 3 \cdot 2 \cdot 1$ nodes.
- Total number of nodes (function calls):

$$1+4+4\cdot 3+4\cdot 3\cdot 2+4\cdot 3\cdot 2\cdot 1=65$$

What is the number of nodes/function calls $T(n)$ for $f(2^n)$?

A Recurrence

From the code of f, the following recurrence defines $T(n)$.

$$T(n) := \begin{cases} 1 & \text{if } n = 0 \\ 1 + n \cdot T(n-1) & \text{else} \end{cases}$$

- $m = 2^0 = 1$: 1 function call.
- $m = 2^n > 1$: $1 + n \cdot T(n-1)$ function calls.

We need an explicit solution of this recurrence.

Solving the Recurrence

Again we add the number of nodes in each level of the tree.

$$T(n) \stackrel{?}{=} 1 + n + n \cdot (n-1) + n \cdot (n-1) \cdot (n-2) + \ldots + n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

$$T(n) \stackrel{?}{=} \sum_{i=0}^{n} \frac{n!}{i!}$$

$$\begin{aligned}
n!/n! &= 1 \\
n!/(n-1)! &= n \\
n!/(n-2)! &= n \cdot (n-1) \\
n!/(n-3)! &= n \cdot (n-1) \cdot (n-2) \\
&\ldots \\
n!/0! &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1
\end{aligned}$$

We need to verify that this is indeed a valid solution of the recurrence.

Verifying the Solution

We prove $(\forall n \in \mathbb{N} : T(n) = \sum_{i=0}^{n} \frac{n!}{i!})$ by induction on n.

- Induction base:

$$T(0) = 1 = 0!/0! = \sum_{i=0}^{0} \frac{0!}{i!}$$

- Induction hypothesis: we assume

$$T(n) = \sum_{i=0}^{n} \frac{n!}{i!}$$

- Induction step: we prove

$$T(n+1) = \sum_{i=0}^{n+1} \frac{(n+1)!}{i!}$$

$$T(n+1) = 1 + (n+1) \cdot T(n) = 1 + (n+1) \cdot \sum_{i=0}^{n} \frac{n!}{i!} = 1 + \sum_{i=0}^{n} \frac{(n+1) \cdot n!}{i!}$$

$$= 1 + \sum_{i=0}^{n} \frac{(n+1)!}{i!} = \frac{(n+1)!}{(n+1)!} + \sum_{i=0}^{n} \frac{(n+1)!}{i!} = \sum_{i=0}^{n+1} \frac{(n+1)!}{i!} \qquad \square$$

Asymptotic Characterization of the Solution

The explicit solution does not give much intuition about its growth.

$$T(n) = \sum_{i=0}^{n} \frac{n!}{i!} = n! \cdot \sum_{i=0}^{n} \frac{1}{i!} < n! \cdot \sum_{i=0}^{\infty} \frac{1}{i!} = n! \cdot e = O(n!)$$

$\sum_{i=0}^{\infty} 1/i! = e$ (Euler's number).

The number of function calls is $O(n!) = O((\log_2 m)!)$.

1. Example

2. Sums

3. Recurrences

4. Divide and Conquer

5. Randomization

6. Amortized Analysis

Sums

Sort integer array $a[0\dots n-1]$ of length $n \geq 1$ in ascending order.

	Cost
procedure InsertionSort(a)	
$n \leftarrow length(a)$	1
for i **from** 1 **to** $n-1$ **do**	n
$x \leftarrow a[i]$	$n-1$
$j \leftarrow i-1$	$n-1$
while $j \geq 0 \wedge a[j] > x$ **do**	$\sum_{i=1}^{n-1} n_i$
$a[j+1] \leftarrow a[j]$	$\sum_{i=1}^{n-1}(n_i-1)$
$j \leftarrow j-1$	$\sum_{i=1}^{n-1}(n_i-1)$
end while	
$a[j+1] \leftarrow x$	$n-1$
end for	
end procedure	

Sums arise from the analysis of iterative algorithms.

Worst Case Time Complexity

$n_i = i+1$: maximum number of times **while** test is executed for value i.

- Worst case time complexity: $T(n) = 4n-2+\sum_{i=1}^{n-1}(3i+1)$

$$\begin{aligned}T(n) =&1+n+(n-1)+(n-1)+(n-1)+\\ &(\sum_{i=1}^{n-1} n_i)+(\sum_{i=1}^{n-1}(n_i-1))+(\sum_{i=1}^{n-1}(n_i-1)) = 4n-2+\sum_{i=1}^{n-1}(3n_i-2)\\ =&4n-2+\sum_{i=1}^{n-1}(3\cdot(i+1)-2) = 4n-2+\sum_{i=1}^{n-1}(3i+1)\end{aligned}$$

- Closed form: $\sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$

$$\sum_{i=1}^{n-1}(3i+1) = \sum_{i=0}^{n-1}(3i+1)-1 = n\cdot 1+3\cdot\frac{(n-1)\cdot n}{2}-1 = \frac{3n^2-n-2}{2}$$

  - Arithmetic series: $\sum_{i=0}^{n}(a+i\cdot d) = (n+1)\cdot a+d\cdot\frac{n\cdot(n+1)}{2}$
  - Geometric series: $\sum_{i=0}^{n}(a\cdot q^i) = a\cdot\frac{q^{n+1}-1}{q-1}$

High school knowledge.

Worst case time complexity $T(n) = 4n-2+\frac{3n^2-n-2}{2} = \frac{3n^2+7n-6}{2}$.

Average Time Complexity

Maximum value n_i is replaced by expected value $E[N_i]$.

- Expected value of random variable N_i: $E[N_i] = \frac{i+2}{2}$
  - Assume that all permutations of a have equal probability.
  - Consequently each value $1, \ldots, i+1$ of N_i has equal probability.

$$E[N_i] = \frac{1}{i+1} \cdot \sum_{j=1}^{i+1} j = \frac{(i+2)\cdot(i+1)}{2\cdot(i+1)} = \frac{i+2}{2}$$

- Average time complexity: $\overline{T}(n) = 4n-2+\frac{1}{2}\cdot\sum_{i=1}^{n-1}(3i+2)$

$$\overline{T}(n) = 4n-2+\sum_{i=1}^{n-1}\left(3\cdot\frac{i+2}{2}-2\right) = 4n-2+\frac{1}{2}\cdot\sum_{i=1}^{n-1}(3i+2)$$

- Closed form: $\overline{T}(n) = \frac{3n^2+17n-12}{4}$

$$\sum_{i=1}^{n-1}(3i+2) = \sum_{i=0}^{n-1}(3i+2)-2 = \left(2n+3\cdot\frac{(n-1)\cdot n}{2}\right)-2 = \frac{3n^2+n-4}{2}$$

$$\overline{T}(n) = 4n-2+\frac{3n^2+n-4}{4} = \frac{16n-8+3n^2+n-4}{4} = \frac{3n^2+17n-12}{4}$$

Average time complexity $\overline{T}(n) = \frac{3n^2+17n-12}{4}$.

Asymptotic Time Complexity

Worst case/average time $T(n) = \frac{3n^2+7n-6}{2}$ and $\overline{T}(n) = \frac{3n^2+17n-12}{4}$.

- $\overline{T}(n) \simeq \frac{T(n)}{2}$ (for large n)
  - In the average, the algorithm is twice as fast as in the worst case.
- $\overline{T}(n) = T(n) = \Theta(n^2)$
  - Asymptotic complexity is the same in the average as in the worst case.
- Asymptotic estimation: $\sum_{i=0}^{\Theta(n)} \Theta(i) = \Theta(n^2)$
  - a linear number of times linear complexity gives quadratic complexity.

$$\sum_{i=0}^{\Theta(n)} \Theta(i^k) = \Theta(n^{k+1})$$

Frequently, a quick estimation of asymptotic time complexity is possible.

Solving Sums by Guessing and Verifying

One may consult an (electronic/printed) table of integer sequences.

- Determine summation values for growing number of summands:

$$0, 4, 4+7, 4+7+10, 4+7+10+13, \ldots = 0, 4, 11, 21, 34, \ldots$$

- On-line Encylopedia of Integer Sequences (`http://oeis.org`)

login

This site is supported by donations to The OEIS Foundation.

Integer Sequences

0,4,11,21,34 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:0,4,11,21,34**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A115067 (3*n^2-n-2)/2. +20 18

0, 4, 11, 21, 34, 50, 69, 91, 116, 144, 175, 209, 246, 286, 329, 375, 424, 476, 531, 589, 650, 714, 781, 851, 924, 1000, 1079, 1161, 1246, 1334, 1425, 1519, 1616, 1716, 1819, 1925, 2034, 2146, 2261, 2379, 2500, 2624, 2751, 2881, 3014, 3150, 3289, 3431, 3576 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

LINKS Table of n, a(n) for n=1..49.

Alfred Hoehn, Illustration of initial terms of A000326, A005449, A045943, A115067 [temporary remark: the case n=4 appears to be incorrect in the illustration]

Index entries for sequences related to linear recurrences with constant coefficients, signature (3,-3,1).

FORMULA a(n) = (3*n+2)*(n-1)/2.

Solving Sums by Guessing and Verifying

One may consult a computer algebra system (Maple, Mathematica, ...).

```
> sum(3*i+1,i=1..n-1);
                                          2
                                     3/2 n  - 1/2 n - 1

In[1]:= Sum[3*i+1,{i,1,n-1}]

                     2
        -2 - n + 3 n
Out[1]= -------------
              2

In[2]:= FindSequenceFunction[{0,4,11,21,34},n]

        (-1 + n) (2 + 3 n)
Out[2]= ------------------
                2
```

However the solution was initially *guessed*, it must be subsequently *verified*.

Solving Sums by Guessing and Verifying

Prove $\forall n \in \mathbb{N} : n \geq 1 \Rightarrow \sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$ by induction on n.

- Base case $n = 1$:
$$\sum_{i=1}^{1-1}(3i+1) = 0 = \frac{3 \cdot 1^2 - 1 - 2}{2}$$
- We assume for fixed $n \geq 1$ $\sum_{i=1}^{n-1}(3i+1) = \frac{3n^2-n-2}{2}$ and show
$$\sum_{i=1}^{n}(3i+1) = \frac{3 \cdot (n+1)^2 - (n+1) - 2}{2}$$
This equation holds, because we have
$$\sum_{i=1}^{n}(3i+1) = \sum_{i=1}^{n-1}(3i+1) + (3n+1) = \frac{3n^2-n-2}{2} + (3n+1)$$
$$= \frac{3n^2-n-2+6n+2}{2} = \frac{3n^2+5n}{2}$$
$$\frac{3 \cdot (n+1)^2 - (n+1) - 2}{2} = \frac{3n^2+6n+3-n-1-2}{2} = \frac{3n^2+5n}{2} \qquad \square$$

1. Example

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Recurrences

Find in sorted array $a[l,r]$ the position of value x (-1, if x does not occur).

	Cost
function $\textsc{BinarySearch}(a,x,l,r)$ $\triangleright$ $n = r - l + 1$	
if $l > r$ **then**	1
return -1	1
end if	
$m \leftarrow \lfloor \frac{l+r}{2} \rfloor$	1
if $a[m] = x$ **then**	1
return m	1
else if $a[m] < x$ **then**	1
return $\textsc{BinarySearch}(a,x,m+1,r)$	$\leq 1 + T(\lfloor \frac{n}{2} \rfloor)$
else	
return $\textsc{BinarySearch}(a,x,l,m-1)$	$\leq 1 + T(\lfloor \frac{n}{2} \rfloor)$
end if	
end function	

Recurrences arise from the analysis of recursive algorithms.

Worst Case Time Complexity

- Recurrence Relation:

$$T(0) = 2$$
$$T(n) = 5 + T\left(\lfloor \frac{n}{2} \rfloor\right), \text{ if } n \geq 1$$

- Special solution: assume $n = 2^m$.

$$\begin{aligned} T(2^m) &= 5 + T(2^{m-1}) \\ &= \underbrace{5 + \ldots + 5}_{m \text{ times}} + T(1) = \underbrace{5 + \ldots + 5}_{m+1 \text{ times}} + T(0) \\ &= 5 \cdot (m+1) + 2 = 5m + 7 \end{aligned}$$

- General solution: for all $n \geq 1$.

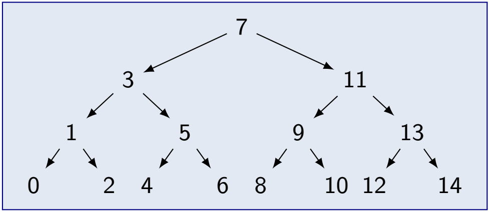
$$T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$$

  - Verified later.

Worst case time complexity $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$ ($T_{\text{found}}(n) = T(n) - 5 = 5 \cdot \lfloor \log_2 n \rfloor + 2$, if x occurs in a).

Average Time Complexity

For $n = 2^m - 1$, recursion tree of height $m - 1$ with n nodes:



- Each path describes the function calls to find a particular element:
  - $7 \to 3 \to 5$: 3 function calls to find element at position 5.
- In total $n = 2^m - 1$ paths:
  - 1 path of length 0, 2 of length 1, 4 of length 2, ..., 2^i of length i.
- Assume all paths are equally likely.
  - I.e., assume x occurs in a, at every position with equal probability.

If x is in a, average number of calls is $\frac{1}{2^m - 1} \cdot \sum_{i=0}^{m-1} i \cdot 2^i$.

Average Time Complexity

Determine the closed form of $\sum_{i=0}^{m-1} i \cdot 2^i$.

- Integer sequence: $0, 2, 10, 34, 98, \ldots$

```
A036799          2+2^(n+1)*(n-1).
        0, 2, 10, 34, 98, ...
```

- Computer algebra system:

```
> sum(i*2^i,i=0..m-1);

                                      m      m
                                   m 2  - 2 2  + 2
```

- Result: $\sum_{i=0}^{m-1} i \cdot 2^i = 2^m \cdot (m-2) + 2$
  - Verification by induction proof.
- Average number of function calls:

$$\frac{1}{2^m - 1} \cdot \sum_{i=0}^{m-1} i \cdot 2^i = \frac{1}{2^m - 1} \cdot (2^m \cdot (m-2) + 2) = \frac{2^m \cdot (m-2)}{2^m - 1} + \frac{2}{2^m - 1} \simeq m - 2$$

- Average time complexity:

$$\overline{T}_{\text{found}}(2^m - 1) \simeq 5 \cdot (m-2) + 2 = 5m - 8 = T_{\text{found}}(2^m - 1) - 5$$

One recursive call less in the average case (similar, if x is not in a).

Solving Recurrences by Guessing & Verifying

$$T(1) = 7$$
$$T(n) = 5 + T(\lfloor \frac{n}{2} \rfloor), \text{ if } n > 1$$

One may consult an (electronic/printed) table of integer sequences.

- Simplified recurrence:

$$U(1) = 0$$
$$U(n) = 1 + U(\lfloor \frac{n}{2} \rfloor), \text{ if } n > 1$$

- Integer sequence: $0, 1, 1, 2, 2, 2, 2, 3, \ldots$

```
A000523  Log_2(n) rounded down.
         0, 1, 1, 2, 2, 2, 2, 3, ...
         ...
         FORMULA ... a(n)=floor(lb(n)).
```

- $U(n) = \lfloor \log_2 n \rfloor$, $T(n) = a \cdot \lfloor \log_2 n \rfloor + b$

Closed form $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$.

Solving Recurrences by Guessing & Verifying

One may consult a computer algebra system.

```
> rsolve({T(1)=7,T(n)=5+T(n/2)},T(n));
                                        7 ln(2) + 5 ln(n)
                                        -----------------
                                              ln(2)

In[3]:= RSolve[{T[1]==7,T[n]==5+T[n/2]},T[n],n]

                       5 Log[n]
Out[3]= {{T[n] -> 7 + --------}}
                        Log[2]
```

- Real solution $T(n) = 5 \cdot \log_2(n) + 7$.
- Integer solution $T(n) = 5 \cdot \lfloor \log_2(n) \rfloor + 7$.

However the solution was initially *guessed*, it must be subsequently *verified*.

Solving Recurrences by Guessing & Verifying

$$T(1) = 7$$
$$T(n) = 5 + T(\lfloor \frac{n}{2} \rfloor), \text{ if } n > 1$$

We show for all $n \in \mathbb{N}$ with $n \geq 1$, $T(n) = 5 \cdot \lfloor \log_2 n \rfloor + 7$.

- Induction base $n = 1$:

$$5 \cdot \lfloor \log_2 1 \rfloor + 7 = 5 \cdot 0 + 7 = 7 = T(1)$$

- Ind. hypothesis: for $n > 1$ and $1 \leq m < n$, assume $T(m) = 5 \cdot \lfloor \log_2 m \rfloor + 2$.
  - Case $n = 2m$:

$$\begin{aligned} T(n) &= 5 + T(\lfloor \frac{n}{2} \rfloor) = 5 + T(m) = 5 + (5 \cdot \lfloor \log_2 m \rfloor + 2) = 5 \cdot (1 + \lfloor \log_2 m \rfloor) + 2 \\ &= 5 \cdot \lfloor 1 + \log_2 m \rfloor + 2 = 5 \cdot \lfloor \log_2 2 + \log_2 m \rfloor + 2 = 5 \cdot \lfloor \log_2 2m \rfloor + 2 \\ &= 5 \cdot \lfloor \log_2 n \rfloor + 2 \end{aligned}$$

  - Case $n = 2m + 1$:

$$\begin{aligned} T(n) &= 5 + T(\lfloor \frac{n}{2} \rfloor) = 5 + T(m) = 5 + (5 \cdot \lfloor \log_2 m \rfloor + 2) = 5 \cdot (1 + \lfloor \log_2 m \rfloor) + 2 \\ &= 5 \cdot \lfloor 1 + \log_2 m \rfloor + 2 = 5 \cdot \lfloor \log_2 2 + \log_2 m \rfloor + 2 = 5 \cdot \lfloor \log_2 2m \rfloor + 2 \\ &= 5 \cdot \lfloor \log_2 (n-1) \rfloor + 2 = 5 \cdot \lfloor \log_2 n \rfloor + 2 \end{aligned}$$

Divide and Conquer

We are going to analyze the Mergesort algorithm.

	Cost
procedure MergeSort(a, l, r) $\quad\triangleright\ n = r - l + 1$	1
if $l < r$ **then**	1
$m \leftarrow \lfloor \frac{l+r}{2} \rfloor$	
MergeSort(a, l, m)	$1 + T(\lceil \frac{n}{2} \rceil)$
MergeSort$(a, m+1, r)$	$1 + T(\lfloor \frac{n}{2} \rfloor)$
Merge(a, l, m, r)	$O(n)$
end if	
end procedure	

We will investigate the asymptotic time complexity only.

Recurrence

- $T(1)$ can be solved in $O(1)$.

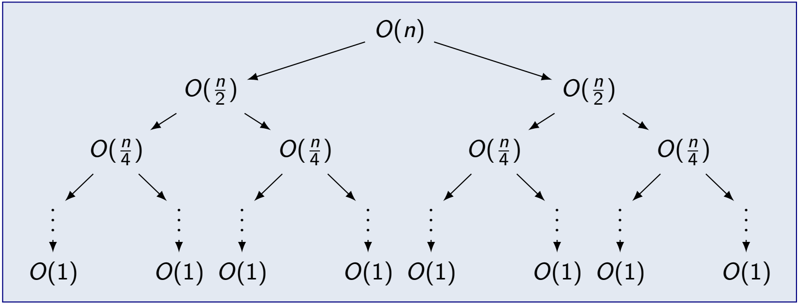
$$\begin{aligned} T(n) &= 1+1+1+T(\lceil\frac{n}{2}\rceil)+1+T(\lfloor\frac{n}{2}\rfloor)+O(n) \\ &= T(\lceil\frac{n}{2}\rceil)+T(\lfloor\frac{n}{2}\rfloor)+O(n)+4 \\ &= T(\lceil\frac{n}{2}\rceil)+T(\lfloor\frac{n}{2}\rfloor)+O(n) \end{aligned}$$

- For asymptotic analysis, it suffices to consider $\frac{n}{2} \in \{\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil\}$.

$$T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$$

We are going to guess an asymptotic solution of this recurrence.

Guessing the Asymptotic Time Complexity



- Execution is the depth-first left-to-right traversal of a binary tree.
- Total time is the sum of the times spent in all tree nodes.

$$T(n) = \sum_{i=0}^{O(\log n)} 2^i \cdot O\left(\frac{n}{2^i}\right) = \sum_{i=0}^{O(\log n)} O(n) = O(n \cdot \log n)$$

We are going to verify our guess of the asymptotic solution.

Verifying the Asymptotic Time Complexity

We prove that every solution $T(n)$ of the recurrence

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

is asymptotically bound by $T(n) = O(n \cdot \log n)$.

- From recurrence, there exist $c > 0$ and $N \geq 1$ such that, for all $n \geq N$

$$T(n) \leq 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

- From the definition of $O(n \cdot \log n) = O(n \cdot \log_2 n)$, it suffices to show

$$\exists c \in \mathbb{R}_{\geq 0}, N \in \mathbb{N} : \forall n \geq N : T(n) \leq c \cdot n \cdot \log_2 n$$

- Thus our goal is to find c' and N' for which we are able to prove

$$\forall n \geq N' : T(n) \leq c' \cdot n \cdot \log_2 n$$

Because of the characterization of T by a recurrence, this naturally leads to an induction proof with base N'.

Verifying Asymptotic Complexity (Contd)

How to choose N'?

- Clearly $N' \geq \max\{2, N\}$.
  - $n = 1 \Rightarrow c' \cdot n \cdot \log_2 n = 0$
  - $N' < N$: induction hypothesis cannot be applied.
- But $N' \geq 2$ is problematic for recurrence $T(n) = \ldots T(\frac{n}{2}) \ldots$.
  - Induction step proves goal for n based on assumption it holds for $\frac{n}{2}$.
  - $N' = 1$: every sequence of divisions by 2 which starts with $n \geq 1$ eventually leads to the base case $n = 1$.
  - $N' \geq 2$: sequence may bypass base case $n = N'$, e.g., $(N'+1)/2 < N'$.
- Solution: show induction base for all n with $2 \leq n \leq N'$.
  - Also ensure that every sequence of divisions starting with $n > N$ eventually reaches some base case, i.e., $\frac{n}{2} \geq 2$.

We choose $N' := \max\{\text{␣}, 3, N\}$.

Verifying Asymptotic Complexity (Contd)

How to choose c'?

- Have to prove $T(n) \leq c' \cdot n \cdot \log_2 n$.
  - for all $2 \leq n \leq N'$.
- It suffices to prove $T(n) \leq c'$.
  - $n \cdot \log_2 n \geq 1$, for all $2 \leq n \leq N'$.
- It suffices to prove $T(N') \leq c'$
  - $T(2) \leq T(3) \leq \ldots \leq T(N')$

We choose $c' := \max\{\textvisiblespace, T(N')\}$.

Verifying Asymptotic Complexity (Contd)

- Induction base: by choice of N' and c', we have for all $2 \leq n \leq N'$

$$T(n) \leq c' \cdot n \cdot \log_2 n$$

- Induction assumption: for $n > N'$ and $2 \leq m < n$, we assume

$$T(m) \leq c' \cdot m \cdot \log_2 m$$

- Induction step: we show

$$T(n) \leq c' \cdot n \cdot \log_2 n$$

We continue by case distinction on the interpretation of $\frac{n}{2}$.

Verifying Asymptotic Complexity (Contd)

- Case $\frac{n}{2}$ means $\lfloor\frac{n}{2}\rfloor$: since $n > N' \geq N$,

$$
\begin{aligned}
T(n) &\leq 2 \cdot T\left(\lfloor\frac{n}{2}\rfloor\right) + c \cdot n \\
&\leq 2 \cdot c' \cdot \lfloor\frac{n}{2}\rfloor \cdot \log_2\lfloor\frac{n}{2}\rfloor + c \cdot n \\
&\leq 2 \cdot c' \cdot \frac{n}{2} \cdot \log_2\left(\frac{n}{2}\right) + c \cdot n \\
&= c' \cdot n \cdot \log_2\left(\frac{n}{2}\right) + c \cdot n \\
&= c' \cdot n \cdot ((\log_2 n) - 1) + c \cdot n \\
&= c' \cdot n \cdot (\log_2 n) + (c - c') \cdot n \leq c' \cdot n \cdot \log_2 n
\end{aligned}
$$

Last inequality holds if $c' \geq c$, because then $c - c' \leq 0$.

Verifying Asymptotic Complexity (Contd)

- Case $\frac{n}{2}$ means $\lceil\frac{n}{2}\rceil$: since $n > N' \geq N$,

$$\begin{aligned}
T(n) &\leq 2 \cdot T(\lceil\frac{n}{2}\rceil) + c \cdot n \\
&\leq 2 \cdot c' \cdot \lceil\frac{n}{2}\rceil \cdot \log_2\lceil\frac{n}{2}\rceil + c \cdot n \\
&\leq c' \cdot (n+1) \cdot \log_2(\frac{n+1}{2}) + c \cdot n \\
&= c' \cdot (n+1) \cdot (\log_2(n+1) - 1) + c \cdot n \\
&\leq c' \cdot (n+1) \cdot ((\log_2 n) + \frac{1}{2} - 1) + c \cdot n
\end{aligned}$$

$\log_2(n+1) \leq (\log_2 n) + \frac{1}{2}$ holds for all $n > 2$ and thus for all $n \geq N'$.

Verifying Asymptotic Complexity (Contd)

We continue the derivation.

$$
\begin{aligned}
T(n) &\leq c' \cdot (n+1) \cdot ((\log_2 n) + \frac{1}{2} - 1) + c \cdot n \\
&= c' \cdot (n+1) \cdot ((\log_2 n) - \frac{1}{2}) + c \cdot n \\
&= c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot (\log_2 n) - n - 1) + c \cdot n \\
&\leq c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot \frac{n}{4} - n - 1) + c \cdot n
\end{aligned}
$$

Last inequality holds if $\log_2 n \leq \frac{n}{4}$, which holds for all $n \geq 16$.

Verifying Asymptotic Complexity (Contd)

We continue the derivation.

$$
\begin{aligned}
T(n) &\leq c' \cdot n \cdot (\log_2 n) + \frac{c'}{2} \cdot (2 \cdot \frac{n}{4} - n - 1) + c \cdot n \\
&\leq c' \cdot n \cdot (\log_2 n) - \frac{c' \cdot n}{4} + c \cdot n \\
&= c' \cdot n \cdot (\log_2 n) + \frac{4c - c'}{4} \cdot n \leq c' \cdot n \cdot \log_2 n
\end{aligned}
$$

Last inequality holds if $c' \geq 4c$, because then $4c - c' \leq 0$.

In both cases $N' := \max\{16, N\}, c' := \max\{4c, T(N')\}$ lets proof succeed.

The Master Theorem

Let $a \geq 1$, $b > 1$, $f : \mathbb{N} \to \mathbb{N}$, $T : \mathbb{N} \to \mathbb{N}$ satisfying the recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- $f(n) = O(n^{(\log_b a)-\varepsilon})$ for some $\varepsilon > 0$:

$$T(n) = \Theta(n^{\log_b a})$$

- $f(n) = \Theta(n^{\log_b a})$:

$$T(n) = \Theta(n^{\log_b a} \cdot \log n)$$

- $f(n) = \Omega(n^{(\log_b a)+\varepsilon})$ for some $\varepsilon > 0$ and there exist some c with $0 < c < 1$ and some $N \in \mathbb{N}$ such that $\forall n \geq N : a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$:

$$T(n) = \Theta(f(n))$$

Easy analysis of a large class of divide and conquer algorithms.

Example

Analysis of MERGESORT.

- Recurrence:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + \Theta(n)$$

- Case 2 of Master Theorem $(a = b = 2)$:

$$\log_b a = 1$$
$$\Theta(n) = \Theta(n^1) = \Theta(n^{\log_b a})$$

- Solution:

$$T(n) = \Theta(n^{\log_b a} \cdot \log n) = \Theta(n^1 \cdot \log n) = \Theta(n \cdot \log n)$$

No tedious proof required any more.

Arbitrary Precision Multiplication

Multiply two natural numbers a and b with n digits each.

	Cost
function $\textsc{Multiply}(a, b)$	
$n \leftarrow \mathit{digits}(a)$ $\quad \triangleright\ \mathit{digits}(a) = \mathit{digits}(b)$	1
$c \leftarrow 0$	1
for i **from** $n-1$ **to** 0 **do**	$n+1$
$p \leftarrow \textsc{MultiplyDigit}(a, b_i)$	$n \cdot \Theta(n)$
$c \leftarrow \textsc{Shift}(c, 1)$	$\sum_{i=0}^{n-1} \Theta(2n-i-1)$
$c \leftarrow \textsc{Add}(p, c)$	$n \cdot \Theta(n)$
end for	
return c	1
end function	

$$T(n) = 3 + (n+1) + 2n \cdot \Theta(n) + \sum_{i=0}^{n-1} \Theta(2n-i-1) = \Theta(n^2)$$

Classical (“school”) algorithm has quadratic time complexity.

Arbitrary Precision Multiplication

Split a and b into halves (a', a'') and (b', b'') of $\frac{n}{2}$ digits.

$$a = a' \cdot d^{\frac{n}{2}} + a''$$
$$b = b' \cdot d^{\frac{n}{2}} + b''$$

$$\begin{aligned} a \cdot b &= (a' \cdot d^{\frac{n}{2}} + a'') \cdot (b' \cdot d^{\frac{n}{2}} + b'') \\ &= a' \cdot b' \cdot d^n + (a' \cdot b'' + a'' \cdot b') \cdot d^{\frac{n}{2}} + a'' \cdot b'' \end{aligned}$$

Basis of a recursive multiplication algorithm.

Arbitrary Precision Multiplication (Rec.)

	Cost
function $\textsc{Multiply}(a, b)$	
$n \leftarrow digits(a)$ $\quad \triangleright\ digits(a) = digits(b) = 2^m$	1
if $n = 1$ **then**	1
$c \leftarrow \textsc{MultiplyDigit}(a_0, b_0)$	$\Theta(1)$
else	
$a' \leftarrow a_{\frac{n}{2}\ldots n-1}; a'' \leftarrow a_{0\ldots\frac{n}{2}-1}$	$\Theta(n)$
$b' \leftarrow b_{\frac{n}{2}\ldots n-1}; b'' \leftarrow b_{0\ldots\frac{n}{2}-1}$	$\Theta(n)$
$u \leftarrow \textsc{Multiply}(a', b')$	$1 + T(\frac{n}{2})$
$v \leftarrow \textsc{Multiply}(a', b'')$	$1 + T(\frac{n}{2})$
$w \leftarrow \textsc{Multiply}(a'', b')$	$1 + T(\frac{n}{2})$
$x \leftarrow \textsc{Multiply}(a'', b'')$	$1 + T(\frac{n}{2})$
$y \leftarrow \textsc{Add}(v, w)$	$\Theta(n)$
$y \leftarrow \textsc{Shift}(y, \frac{n}{2})$	$\Theta(n)$
$c \leftarrow \textsc{Shift}(u, n)$	$\Theta(n)$
$c \leftarrow \textsc{Add}(c, y)$	$\Theta(n)$
$c \leftarrow \textsc{Add}(c, x)$	$\Theta(n)$
end if	
return c	1
end function	

Four recursive calls of the algorithm with half the input size.

Arbitrary Precision Multiplication (Rec.)

Analysis of recursive algorithm by the Master Theorem.

- Recurrence:

$$\begin{aligned} T(n) &= 3 + 4 \cdot \left(1 + T\left(\frac{n}{2}\right)\right) + 7 \cdot \Theta(n) \\ &= 4 \cdot T\left(\frac{n}{2}\right) + \Theta(n) \end{aligned}$$

- Case 1 of the master theorem ($a = 4, b = 2$):

$$\begin{aligned} &(\log_b a) = (\log_2 4) = 2 \\ &f(n) = O(n) = O(n^1) = O(n^{2-1}) = O(n^{(\log_b a)-1}) \end{aligned}$$

- Solution:

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$$

Also the recursive algorithm has quadratic time complexity.

Arbitrary Precision Multiplication

Anatolii Karatsuba and Yuri Ofman, 1962.

$$
\begin{aligned}
a \cdot b &= (a' \cdot d^{\frac{n}{2}} + a'') \cdot (b' \cdot d^{\frac{n}{2}} + b'') \\
&= a' \cdot b' \cdot d^n + (a' \cdot b'' + a'' \cdot b') \cdot d^{\frac{n}{2}} + a'' \cdot b'' \\
&= a' \cdot b' \cdot d^n + ((a' + a'') \cdot (b' + b'') - a' \cdot b' - a'' \cdot b'') \cdot d^{\frac{n}{2}} + a'' \cdot b''
\end{aligned}
$$

- Two multiplications $a' \cdot b'$ and $a'' \cdot b''$ of numbers with $\frac{n}{2}$ digits.
- Product $s \cdot t$ where $s = a' + a''$ and $t = b' + b''$ may have $\frac{n}{2} + 1$ digits.

$$s = s_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + s', t = t_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + t'$$

$$
\begin{aligned}
s \cdot t &= (s_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + s') \cdot (t_{\frac{n}{2}} \cdot d^{\frac{n}{2}} + t') \\
&= s_{\frac{n}{2}} \cdot t_{\frac{n}{2}} \cdot d^n + (s_{\frac{n}{2}} \cdot t' + t_{\frac{n}{2}} \cdot s') \cdot d^{\frac{n}{2}} + s' \cdot t'
\end{aligned}
$$

- Can compute $s \cdot t$ from product $s' \cdot t'$ of numbers of length $\frac{n}{2}$.

We only need three multiplications of numbers with $\frac{n}{2}$ digits.

Karatsuba Algorithm

	Cost
function $\text{MULTIPLY}(a,b)$	
$n \leftarrow digits(a)$ $\quad \triangleright\ digits(a) = digits(b) = 2^m$	1
if $n = 1$ **then**	1
$c \leftarrow \text{MULTIPLYDIGIT}(a_0, b_0)$	$\Theta(1)$
else	
$a' \leftarrow a_{\frac{n}{2}\ldots n-1}$; $a'' \leftarrow a_{0\ldots\frac{n}{2}-1}$	$\Theta(n)$
$b' \leftarrow b_{\frac{n}{2}\ldots n-1}$; $b'' \leftarrow b_{0\ldots\frac{n}{2}-1}$	$\Theta(n)$
$s \leftarrow \text{ADD}(a', a'')$	$\Theta(n)$
$t \leftarrow \text{ADD}(b', b'')$	$\Theta(n)$
$u \leftarrow \text{MULTIPLY}(a', b')$	$1 + T(\frac{n}{2})$
$v \leftarrow \text{MULTIPLY'}(s, t)$	$\Theta(n) + T(\frac{n}{2})$
$x \leftarrow \text{MULTIPLY}(a'', b'')$	$1 + T(\frac{n}{2})$
$y \leftarrow \text{SUBTRACT}(v, u)$	$\Theta(n)$
$y \leftarrow \text{SUBTRACT}(v, x)$	$\Theta(n)$
$y \leftarrow \text{SHIFT}(y, \frac{n}{2})$	$\Theta(n)$
$c \leftarrow \text{SHIFT}(u, n)$	$\Theta(n)$
$c \leftarrow \text{ADD}(c, y)$	$\Theta(n)$
$c \leftarrow \text{ADD}(c, x)$	$\Theta(n)$
end if	
return c	1
end function	

Three recursive calls of the algorithm with half the input size.

Karatsuba Algorithm

Analysis of Karatsuba algorithm by the Master Theorem.

- Recurrence:

$$\begin{aligned} T(n) &= 3 + 2 \cdot \left(1 + \cdot T(\tfrac{n}{2})\right) + \left(\Theta(n) + T(\tfrac{n}{2})\right) + 10 \cdot \Theta(n) \\ &= 3 \cdot T(\tfrac{n}{2}) + \Theta(n) \end{aligned}$$

- Case 1 of Master Theorem $(a = 3, b = 2)$:

$$\begin{aligned} &\log_b a = \log_2 3 \\ &f(n) = O(n) = O(n^1) = O(n^{(\log_b a) - \varepsilon}) \\ &1.58 < \log_2 3 < 1.59, \varepsilon = (\log_2 3) - 1 > 0.58 \end{aligned}$$

- Solution:

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3}) = o(n^2)$$

The Karatsuba algorithm has a better asymptotic time complexity than the classical algorithm and is thus implemented in computer algebra systems.

The Quicksort Algorithm

Sort array a in range $[l, r]$ of size $n = r - l + 1$ in ascending order.

Algorithm	Cost
procedure $\textsc{QuickSort}(a, l, r)$ $\quad \triangleright\ n = r - l + 1$	
if $l < r$ **then**	1
choose $p \in [l, r]$	$O(n)$
$m \leftarrow \textsc{Partition}(a, l, r, p)$ $\quad \triangleright\ i = m - l$	$\Theta(n)$
$\textsc{QuickSort}(a, l, m - 1)$	$1 + T(i)$
$\textsc{QuickSort}(a, m + 1, r)$	$1 + T(n - i - 1)$
end if	
end procedure	

Two recursive calls with input sizes i and $n - i - 1$ (for some $0 \leq i \leq n - 1$).

Time Complexity of Quicksort

- Recurrence:

$$T(n) = T(i) + T(n-i-1) + \Theta(n)$$

- One interval is empty ($i = 0$ or $i = n-1$):

$$T(n) = T(0) + T(n-1) + \Theta(n) = \Theta(1) + T(n-1) + \Theta(n) = T(n-1) + \Theta(n)$$
$$= \sum_{i=0}^{n-1} \Theta(i) = \Theta(n^2)$$

  - Unbalanced binary recursion tree where every left child is a leaf and the path from root along every right child to rightmost leaf has length n.

- Both intervals have same size ($i = \frac{n}{2}$):

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + \Theta(n) = 2 \cdot T\left(\frac{n}{2}\right) + \Theta(n)$$
$$= \Theta(n \cdot \log n)$$

  - Case 2 of the Master Theorem ($a = b = 2$).
  - Balanced binary recursion tree of depth $\log_2 n$.

Worst case time complexity is quadratic; best case is linear-logarithmic.

Average Time Complexity of Quicksort

Assume that all n values $0, \ldots, n-1$ of i are equally likely.

$$
\begin{aligned}
T(n) &= \frac{1}{n} \cdot \sum_{i=0}^{n-1} (T(i) + T(n-i-1) + \Theta(n)) \\
&= \frac{1}{n} \cdot \big(\sum_{i=0}^{n-1} T(i) + T(n-i-1) \big) + \Theta(n) \\
&= \frac{1}{n} \cdot \big(\sum_{i=0}^{n-1} T(i) + \sum_{i=0}^{n-1} T(n-i-1) \big) + \Theta(n) \\
&= \frac{1}{n} \cdot \big(\sum_{i=0}^{n-1} T(i) + \sum_{i=0}^{n-1} T(i) \big) + \Theta(n) \\
&= \frac{2}{n} \cdot \sum_{i=0}^{n-1} T(i) + \Theta(n)
\end{aligned}
$$

Is the average time complexity closer to the worst or to the best case?

Average Time Complexity of Quicksort

Consider special form of recurrence (see lecture notes for the general case).

$$\begin{aligned}
T'(n) &= \frac{2}{n} \cdot \sum_{i=1}^{n-1} T'(i) + n \\
n \cdot T'(n) &= 2 \cdot \sum_{i=1}^{n-1} T'(i) + n^2 \\
(n-1) \cdot T'(n-1) &= 2 \cdot \sum_{i=1}^{n-2} T'(i) + (n-1)^2 \\
n \cdot T'(n) - (n-1) \cdot T'(n-1) &= 2 \cdot T'(n-1) + 2n - 1 \\
n \cdot T'(n) &= (n+1) \cdot T'(n-1) + 2n - 1 \\
\frac{T'(n)}{n+1} &= \frac{T'(n-1)}{n} + \frac{2n-1}{n \cdot (n+1)}
\end{aligned}$$

Terms involving T' have now same shape on left and right side.

Average Time Complexity of Quicksort

We solve the special recurrence.

$$\sum_{i=1}^{n} \frac{T'(i)}{i+1} = \sum_{i=1}^{n} \Big(\frac{T'(i-1)}{i} + \frac{2i-1}{i\cdot(i+1)} \Big)$$

$$\sum_{i=1}^{n} \frac{T'(i)}{i+1} = \sum_{i=1}^{n} \frac{T'(i-1)}{i} + \sum_{i=1}^{n} \frac{2i-1}{i\cdot(i+1)}$$

$$\sum_{i=1}^{n} \frac{T'(i)}{i+1} = \sum_{i=0}^{n-1} \frac{T'(i)}{i+1} + \sum_{i=1}^{n} \frac{2i-1}{i\cdot(i+1)}$$

$$\frac{T'(n)}{n+1} = \frac{T'(0)}{1} + \sum_{i=1}^{n} \frac{2i-1}{i\cdot(i+1)}$$

$$T'(n) = (n+1)\cdot \Big(T'(0) + \sum_{i=1}^{n} \frac{2i-1}{i\cdot(i+1)} \Big)$$

$$T'(n) = O(n\cdot \sum_{i=1}^{n} \frac{1}{i}) = O(n\cdot H_n) = O(n\cdot \log n)$$

The average time complexity is the same as that of the best case.

Average Time Complexity of Quicksort

We could have also applied a computer algebra system.

```
> rsolve({T(0)=1,T(n)=(n+1)/n*T(n-1)+1},T(n));
                              (n + 1) (Psi(n + 2) + gamma)
```

- `Psi(n)`: $\Psi(n) = H_{n-1} - \gamma$,

```
In[1]:= RSolve[{T[0]==1,T[n]==(n+1)/n*T[n-1]+1},T[n],n]
Out[1]= {{T[n] -> EulerGamma + EulerGamma n +
                  PolyGamma[0, 2 + n] +
                  n PolyGamma[0, 2 + n]}}
```

- `PolyGamma[0,n]`: $\Psi(n) = H_{n-1} - \gamma$.

Result is in $O(n \cdot H_n)$.

Ensuring the Average Time Complexity

We have assumed that all values of $i = m - l$ are equally likely but how realistic is this assumption?

- Assumption is satisfied if and only if the choice

  $\boxed{\textbf{choose } p \in [l, r]}$

  determines a pivot element $a[p]$ that is equally likely to be the element at any of the positions $l, \ldots, r$.
- We might choose a fixed index in interval $[l, r]$, e.g.,

  $\boxed{p \leftarrow r}$
- But then we get evenly distributed pivot elements only if all $n!$ permutations of a occur with equal probability as inputs.

It is in practice typically hard to estimate how inputs are distributed; worst case situations (input array already sorted) might appear frequently.

Randomization

We might shuffle the input array randomly before sorting it.

procedure $\textsc{Randomize}(a)$
  $n \leftarrow length(a)$
  for i **from** 0 **to** $n-1$ **do**
    $r \leftarrow \textsc{Random}(i, n-1)$
    $b \leftarrow a[i]; a[i] \leftarrow a[r]; a[r] \leftarrow b$
  end for
end procedure

Indeed ensures that all permutations are equally likely but is costly.

Randomization

We may simply choose the pivot element randomly.

$$\boxed{p \leftarrow \textsc{Random}(l, r)}$$

In many algorithms, making a random choice (rather than making an arbitrary fixed choice) may yield an average case complexity that is independent of the input distribution.

1. Example

2. Sums

3. Recurrences

4. Divide and Conquer

5. Randomization

6. Amortized Analysis

Amortized Analysis

Determine worst-case time complexity $T(n)$ of a sequence of n operations.

- Assume that operations are performed on the same data structure.
  - Then it may be that the worst-case complexity $T_{\text{op}}(i)$ of operation i can be only exhibited for some *few* elements of the sequence.
- $T(n) \leq \sum_{i=1}^{n} T_{\text{op}}(i)$
  - The worst case for the whole sequence may be smaller than the sum of the worst-cases of each operation.
- $\frac{T(n)}{n} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} T_{\text{op}}(i)$
  - The contribution of an individual operation to the worst-case complexity of the sequence may be smaller than the average of the individual worst case complexities.
- Amortized cost $\frac{T(n)}{n}$: some operations with high costs may be outweighed by many operations with low costs.

Amortized analysis is typically applied to operations that manipulate a certain data structure (e.g., a sequence of method calls on an object).

Example

- Consider a stack with the following operations:
  - $\textsc{Push}(s, x)$: push an element x on stack s.

    Time $O(1)$.
  - $\textsc{Pop}(s)$: pop an element from the top of the non-empty stack s.

    Time $O(1)$.
  - $\textsc{MultiPop}(s, k)$: pop k elements from the stack s (if s has $l < k$ elements, then only l elements are popped).

    $O(\min\{l, k\})$ where l is the number of elements on s.
- Sequence of n operations can be performed in time $O(n^2)$.
  - Each $\textsc{MultiPop}$ operation has complexity $O(n)$.
  - The bound is correct but not tight.

We are interested in a much tighter bound.

Aggregate Analysis

- Assume among the n operations, there are k MULTIPOP operations.

$$n = k + \sum_{i=0}^{k} n_i$$

$n_0, n_1, \ldots, n_k$: number of operations before/after a MULTIPOP.

- The total cost of the sequence is

$$T(n) = \sum_{i=0}^{k-1} O(p_i) + (n-k) \cdot O(1) = O(\sum_{i=0}^{k-1} p_i) + O(n)$$

$$\stackrel{(1)}{=} O(\sum_{i=0}^{k-1} n_i) + O(n) \stackrel{(2)}{=} O(n) + O(n) = O(n)$$

p_i: number of elements popped in call i of MULTIPOP.

  - (1) follows from $\sum_{i=0}^{k-1} p_i \leq \sum_{i=0}^{k-1} n_i$.
    - The total number of elements popped from the stack is bound by the total number of previously occurring (PUSH) operations.
  - (2) follows from $\sum_{i=0}^{k-1} n_i \leq n$.
    - Number of PUSH operations is bound by number of all operations.
  - Sequence of n operations can be performed in time $O(n)$.

The amortized cost of a single operation is $O(1)$, i.e., constant.

The Potential Method

- Assign to operation i its actual cost c_i and amortized cost $\hat{c}_i$:
  - If $\hat{c}_i > c_i$, operation i saves $\hat{c}_i - c_i$ "credit".
  - If $\hat{c}_i < c_i$, operation i uses uses up $c_i - \hat{c}_i$ credit.
- The potential function $\Phi(s)$ maps data structure s to a real number.
  - $\Phi(s)$: the credit accumulated so far (the "potential" of s).

$$\hat{c}_i - c_i = C \cdot \big(\Phi(s_i) - \Phi(s_{i-1})\big)$$

    - $\hat{c}_i - c_i$: the credit saved/used by operation i.
    - s_0: the initial value of s; s_i: its value after operation i.
    - Constant factor $C \geq 0$.
- Sum of amortized costs:

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \Big(c_i + C \cdot \big(\Phi(s_i) - \Phi(s_{i-1})\big)\Big) = \sum_{i=1}^{n} c_i + C \cdot \big(\Phi(s_n) - \Phi(s_0)\big)$$

  - We can ensure $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ by ensuring $\Phi(s_n) \geq \Phi(s_0)$.

We have $T(n) \leq \sum_{i=1}^{n} \hat{c}_i$, i.e., can use amortized costs in the analysis.

Example

Stack s with n PUSH, POP, MULTIPOP operations.

- Potential $\Phi(s)$: the number of elements in stack s.

$$\Phi(s_n) \geq 0 = \Phi(s_0)$$

- Amortized cost $\hat{c}_i = c_i + C \cdot \big(\Phi(s_i) - \Phi(s_{i-1})\big)$

  $C \geq 0$ upper bound for execution time of PUSH and POP; $C \cdot k'$ upper bound for the execution time of MULTIPOP(s, k).

  $k' = \min\{k, m\}, \ m = |s|$.

  - PUSH(s,x): $\hat{c}_i \leq C + C \cdot \big((m+1) - m\big) = C + C = 2 \cdot C$.
  - POP(s): $\hat{c}_i \leq C + C \cdot \big((m-1) - (m)\big) = C - C = 0$.
  - MULTIPOP(s,k): $\hat{c}_i \leq C \cdot k' + C \cdot \big((m-k') - m\big) = C \cdot k' - C \cdot k' = 0$.

Each operation has amortized cost $O(1)$, a sequence of n operations has worst case time complexity $O(n)$.

Dynamic Tables

Table t for which a certain amount of space is allocated.

- Operation $\textsc{Insert}(t,x)$ inserts value x into t.
  - Sequence of n such operations starting with an empty table.
- If space gets exhausted, t is expanded:
  - More space is allocated and elements are copied from the old space to the new one.
- If n elements are to be copied, time complexity of $\textsc{Insert}$ is $O(n)$.
  - However, most of the time, there is space available and $\textsc{Insert}$ can be performed in time $O(1)$.
- Time complexity $O(n^2)$ of a sequence of n $\textsc{Insert}$ operations.
  - However, this bound is not tight.

We are interested in a much tighter bound.

Aggregate Analysis

How much to expand table of size m?

- $m+1$: every call of INSERT triggers an expansion.

$$T(n) = 1+2+3+\ldots(n-1) = \sum_{i=1}^{n-1} i = O(n^2)$$

- $m+c$: every c-th call triggers an expansion.

$$T(n) = \sum_{i=1}^{\lceil \frac{n}{c} \rceil} O((i-1)\cdot c) + (n - \lceil \frac{n}{c} \rceil) \cdot O(1)$$

$$= O(\sum_{i=1}^{\lceil \frac{n}{c} \rceil} i) + O(n) = O(n^2) + O(n) = O(n^2)$$

- $m \cdot c$: every call c^i+1 triggers an expansion (typically $c := 2$).

$$T(n) = \sum_{i=0}^{\lceil \log_2 n \rceil + 1} O(2^i) + (n - \lceil \log_2 n \rceil - 1) \cdot O(1)$$

$$= O(\sum_{i=0}^{\lceil \log_2 n \rceil + 1} 2^i) + O(n) = O(\sum_{i=0}^{\lfloor \log_2 n \rfloor} 2^i) + O(n) = O(2n) + O(n) = O(n)$$

With the last strategy, amortized cost of INSERT is $O(1)$.

Analysis with the Potential Method

- Potential: $\Phi(t) := 2 \cdot num(t) - size(t)$
  - $num(t)$: number of elements in t.
  - $size(t)$: the number of slots in t.
  - Since $num(t) \geq \frac{size(t)}{2}$, we have $\Phi(t) \geq 0$.
- Change of Potential:
  - After expansion of t_{i-1}: $\Phi(t_i) = 0$.
  - After insertion into t_{i-1}: $\Phi(t_i) = 2 + \Phi(t_{i-1})$.
  - Before expansion of t_{i-1}: $\Phi(t_{i-1}) = size(t_{i-1})$

Cost of expansion can be covered by accumulated credit.

Analysis with the Potential Method

- Amortized cost: $\hat{c}_i = c_i + C \cdot \big(\Phi(t_i) - \Phi(t_{i-1})\big)$.

  C: upper bound for the cost of insertion, $C \cdot k$: upper bound for the cost of copying k elements.

- INSERT does not trigger an expansion:

  $num(t_i) = num(t_{i-1}) + 1$, $size(t_i) = size(t_{i-1})$.

$$\begin{aligned}
\hat{c}_i &\leq C \cdot 1 + C \cdot \big(2 \cdot num(t_i) - size(t_i) - (2 \cdot num(t_{i-1}) - size(t_{i-1}))\big) \\
&= C \cdot 1 + C \cdot \big(2 \cdot (num(t_{i-1}) + 1) - size(t_{i-1}) - (2 \cdot num(t_{i-1}) - size(t_{i-1}))\big) \\
&= C \cdot 1 + C \cdot 2 = 3C
\end{aligned}$$

- INSERT triggers an expansion:

  $num(t_i) = num(t_{i-1}) + 1$, $size(t_{i-1}) = num(t_{i-1})$,
  $size(t_i) = 2 \cdot size(t_{i-1}) = 2 \cdot num(t_{i-1})$

$$\begin{aligned}
\hat{c}_i &\leq C \cdot num(t_{i-1}) + C \cdot \big(2 \cdot num(t_i) - size(t_i) - (2 \cdot num(t_{i-1}) - size(t_{i-1}))\big) \\
&= C \cdot num(t_{i-1}) + C \cdot \big(2 \cdot (num(t_{i-1}) + 1) - 2 \cdot num(t_{i-1}) - (2 \cdot num(t_{i-1}) - num(t_{i-1}))\big) \\
&= C \cdot num(t_{i-1}) + C \cdot \big(2 - num(t_i - 1)\big) = 2C
\end{aligned}$$

In average, we can perform INSERT in time $O(1)$.