

# Limits of Computability

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## 1. Decision Problems

## 2. The Halting Problem

## 3. Reduction Proofs

## 4. Rice's Theorem

# Decision Problems



- **Decision problem**  $P$ .

- A set of words  $P \subseteq \Sigma^*$ .

$w \in P \dots w$  has property  $P$ .

- Interpretation as a property of words over  $\Sigma$ .

$P(w) \dots w$  has property  $P$ .

- **Formal definition** by a formula:

$$P := \{w \in \Sigma^* \mid \dots\}$$

$$P(w) :\Leftrightarrow \dots$$

- **Informal definition** by a decision question:

Does word  $w$  have property  $\dots$ ?

- **Example problem:** Is the length of  $w$  a square number?

$$P := \{w \in \Sigma^* \mid \exists n \in \mathbb{N} : |w| = n^2\}$$

$$P(w) :\Leftrightarrow \exists n \in \mathbb{N} : |w| = n^2$$

$$P = \{\varepsilon, 0, 0000, 000000000, \dots\}$$

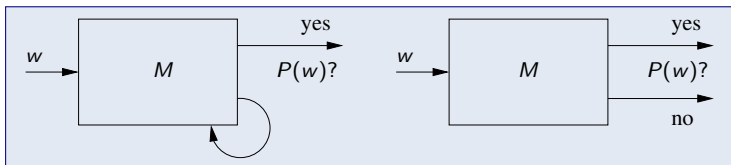
A decision problem is the set of all words for which the answer to a decision question is “yes”.



# Semi-Decidability and Decidability

Problems can be the languages of Turing machines.

- A problem  $P$  is **semi-decidable**, if  $P$  is recursively enumerable.
  - There exists a Turing machine  $M$  that semi-decides  $P$ .
  - $M$  must only terminate, if the answer to “ $P(w)$ ?” is “yes”.
- A problem  $P$  is **decidable** if  $P$  is recursive.
  - There exists a Turing machine  $M$  that decides  $P$ .
  - $M$  must also terminate, if the answer to “ $P(w)$ ?” is “no”.



# Decidability of Complement



- **Theorem:** If  $P$  is decidable, also its complement  $\overline{P}$  is decidable.

The answer to " $\overline{P}(w)$ ?" is "yes", if and only if the answer to " $P(w)$ ?" is "no" ( $\overline{P}(w) \Leftrightarrow \neg P(w)$ ).

- Proof: If  $P$  is decidable, it is recursive, thus  $\overline{P}$  is recursive, thus  $\overline{P}$  is decidable.
- **Theorem:**  $P$  is decidable, if both  $P$  and  $\overline{P}$  are semi-decidable.
  - If  $P$  and  $\overline{P}$  are semi-decidable, they are recursive enumerable. Thus  $P$  is recursive and therefore decidable.

Direct consequences of the previously established results about recursively enumerable and recursive languages.

# Decidability and Computability



- **Theorem:**  $P \subseteq \Sigma^*$  is semi-decidable, if and only if the partial characteristic function  $1'_P : \Sigma^* \rightarrow_p \{1\}$  is Turing computable:

$$1'_P(w) := \begin{cases} 1 & \text{if } P(w) \\ \text{undefined} & \text{if } \neg P(w) \end{cases}$$

- Proof: if  $P$  is semi-decidable, there exists  $M$  such that, for every word  $w \in P = \text{domain}(1'_P)$ ,  $M$  accepts  $w$ . We can then construct  $M'$  which calls  $M$  on  $w$ . If  $M$  accepts  $w$ ,  $M'$  writes 1 on output tape. If  $1'_P$  is Turing computable, there exists  $M$  such that, for every word  $w \in P = \text{domain}(1'_P)$ ,  $M$  accepts  $w$  and writes 1 on the tape. We can then construct  $M'$  which takes  $w$  from the tape and calls  $M$  on  $w$ . If  $M$  writes 1,  $M'$  accepts  $w$ .
- **Theorem:**  $P \subseteq \Sigma^*$  is decidable, if and only if the characteristic function  $1_P : \Sigma^* \rightarrow \{0, 1\}$  is Turing computable:

$$1_P(w) := \begin{cases} 1 & \text{if } P(w) \\ 0 & \text{if } \neg P(w) \end{cases}$$

- Proof: analogous.



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# Turing Machine Codes

**Theorem:** for every Turing machine  $M$ , there exists a bit string  $\langle M \rangle$ ,

- the **Turing machine code** of  $M$

such that

1. different Turing machines have different codes
  - if  $M \neq M'$ , then  $\langle M \rangle \neq \langle M' \rangle$ ;
2. we can recognize valid Turing-machine codes
  - $w \in \text{range}(\langle \cdot \rangle)$  is decidable
3. the encoding  $\langle M \rangle$  and the decoding  $\langle c \rangle^{-1}$  are Turing computable.
  - Core idea: assign to all machine states, alphabet symbols, and tape directions unique natural numbers and encode every transition  $\delta(q_i, a_j) = (q_k, a_l, d_r)$  by the tuple  $(i, j, k, l, r)$  in binary form.

**A Turing machine code is also called a “Gödel number”.**





# The Halting Problem

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The most famous undecidable problem in computer science.

- The **halting problem**  $HP$  is to decide, for given Turing machine code  $\langle M \rangle$  and word  $w$ , whether  $M$  halts on input  $w$ :

$$HP := \{(\langle M \rangle, w) \mid \text{Turing machine } M \text{ halts on input word } w\}$$

- $(w_1, w_2)$ : a bit string that reversibly encodes the pair  $w_1, w_2$ .
- **Theorem:** The halting problem is undecidable.
  - There is no Turing machine that always halts and says “yes”, if its input is of form  $(\langle M \rangle, w)$  such that  $M$  halts on input  $w$ , respectively says “no”, if this is not the case.

The remainder of this section is dedicated to the proof of this theorem.

# Enumeration of Words and Turing Machines



- **Theorem:** There exists an enumeration  $w$  of all words over  $\Sigma$ .

$$w = (w_0, w_1, \dots)$$

- For every word  $w' \in \Sigma^*$ , there exists  $i \in \mathbb{N}$  such that  $w' = w_i$ .
- The enumeration  $w$  starts with the empty word, then lists the all words of length 1, then lists all the words of length 2, and so on. Thus every word eventually appears in  $w$ .
- **Theorem:** There exists an enumeration  $M$  of all Turing machines.

$$M = (M_0, M_1, \dots)$$

- For every Turing machine  $M'$  there exists  $i \in \mathbb{N}$  such that  $M' = M_i$ .
- Let  $C = (C_0, C_1, \dots)$  be the enumeration of all Turing machine codes in bit-alphabetic word order. We define  $M_i$  as the unique Turing machine denoted by  $C_i$ . Since every Turing machine has a code and  $C$  enumerates all codes,  $M$  is the enumeration of all Turing machines.

There are countably many words and countably many Turing machines.



# Undecidability of the Halting Problem

**Proof:** define  $h : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$  as

$$h(i, j) := \begin{cases} 1 & \text{if Turing machine } M_i \text{ halts on input word } w_j \\ 0 & \text{otherwise} \end{cases}$$

If the halting problem were decidable, then  $h$  were computable.

- Let  $M$  be a Turing machine that decides the halting problem.
- We construct a Turing machine  $M_h$  which computes  $h$ .
- $M_h$  takes input  $(i, j)$  and computes  $\langle M_i \rangle$  and  $w_j$ .
  - $M_h$  enumerates codes  $\langle M_0 \rangle, \dots, \langle M_i \rangle$  and words  $w_0, \dots, w_j$ .
- $M_h$  passes  $(\langle M_i \rangle, w_j)$  to  $M$  which eventually halts.
- If  $M$  accepts its input,  $M_h$  returns 1, else it returns 0.

**It thus suffices to show that  $h$  is not computable by a Turing machine.**

# Undecidability of the Halting Problem



We assume that  $h$  is computable and derive a contradiction.

- Define  $d : \mathbb{N} \rightarrow \{0,1\}$  as

$$d(i) := h(i, i)$$

- $d(i) = 1$ :  $M_i$  terminates on input word  $w_i$ .
- Diagonalization:  $d(0), d(1), d(2), \dots$  is diagonal of value table for  $h$ .

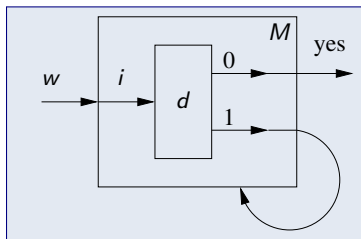
$h$	$j = 0$	$j = 1$	$j = 2$	$\dots$
$i = 0$	$\mathbf{h(0,0)}$	$h(0,1)$	$h(0,2)$	$\dots$
$i = 1$	$h(1,0)$	$\mathbf{h(1,1)}$	$h(1,2)$	$\dots$
$i = 2$	$h(2,0)$	$h(2,1)$	$\mathbf{h(2,2)}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Since  $h$  is computable, also  $d$  is computable.

# Undecidability of the Halting Problem



```
function  $M(w)$ :  
  let  $i \in \mathbb{N}$  such that  $w = w_i$   
  case  $d(i)$  of  
    0: return yes  
    1: loop end loop  
  end case  
end function
```



- Construct  $M$  which takes  $w$  and determines  $i \in \mathbb{N}$  with  $w = w_i$ .
  - $M(w)$  halts, if and only if  $d(i) = 0$ .
- Let  $i$  be such that  $M = M_i$  and compute  $M(w_i)$ .
  - $M(w_i)$  halts, if and only if  $d(i) = 0$ .
  - $M(w_i)$  halts, if and only if  $M_i(w_i)$  does not halt.
  - $M(w_i)$  halts, if and only if  $M(w_i)$  does not halt.

By letting  $M$  reason about its own behavior, we derive a contradiction.



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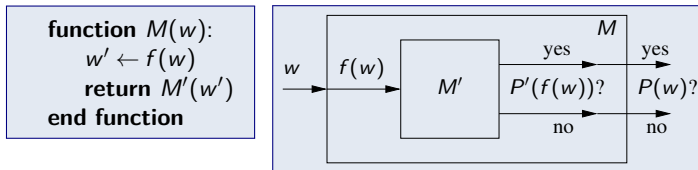
## Reduction Proofs

We can construct a partial order on decision problems.

- Decision problem  $P \subseteq \Sigma^*$  is **reducible** to  $P' \subseteq \Gamma^*$  ( $P \leq P'$ ), if there is a computable function  $f : \Sigma^* \rightarrow \Gamma^*$  such that

$$P(w) \Leftrightarrow P'(f(w))$$

- $w$  has property  $P$  if and only if  $f(w)$  has property  $P'$ .
- Theorem:** For all decision problems  $P$  and  $P'$  with  $P \leq P'$ , it holds that, if  $P$  is not decidable, then also  $P'$  is not decidable.
  - Proof: we assume that  $P'$  is decidable and show that  $P$  is decidable. Since  $P'$  is decidable, there is a Turing machine  $M'$  that decides  $P'$ . We construct  $M$  that decides  $P$ :



# Undecidability of the Restricted Halting Problem



To show that some problem  $P$  is not decidable, it suffices to show that, if  $P$  is decidable, also the halting problem  $HP$  is decidable.

- **Theorem:** the restricted halting problem  $RHP$  is not decidable.

$$RHP := \{ \langle M \rangle \mid \text{Turing machine } M \text{ halts on input word } \varepsilon \}$$

- Decide, for given  $\langle M \rangle$ , whether  $M$  halts for input word  $\varepsilon$ .

Pattern for many undecidability proofs.



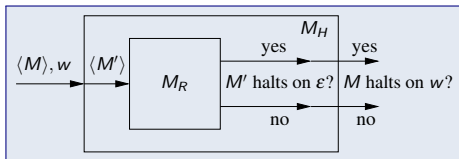
# Undecidability of the Restricted Halting Problem



We assume that  $RHP$  is decidable and show that  $HP$  is decidable.

- Since  $RHP$  is decidable, there exists  $M_R$  such that  $M_R$  accepts input  $c$ , if and only if  $c$  is the code of some  $M$  which halts on input  $\varepsilon$ .
- We can then define  $M_H$ , which accepts input  $(c, w)$ , if and only if  $c$  is the code of some  $M$  that terminates on input  $w$ :
  - $M_H$  constructs from  $(c, w)$  the code of some  $M'$  which first prints  $w$  on its tape and then behaves like  $M$ .
    - $M'$  terminates for input  $\varepsilon$  (which is ignored and overwritten by  $w$ ) if and only if  $M$  terminates on input  $w$ .
  - $M_H$  accepts its input, if and only if  $M_R$  accepts  $\langle M' \rangle$ .

```
function  $M_H(\langle M \rangle, w)$ :  
   $\langle M' \rangle := \text{compute}(\langle M \rangle, w)$   
  return  $M_R(\langle M' \rangle)$   
end function
```





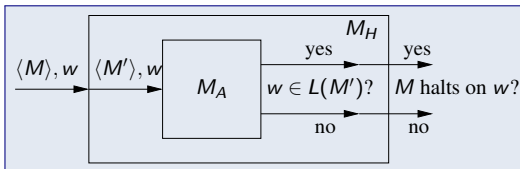
# Undecidability of the Acceptance Problem

- **Theorem:** the acceptance problem  $AP$  is not decidable.

$$AP := \{(\langle M \rangle, w) \mid w \in L(M)\}$$

- Decide, for given  $M$  and  $w$ , whether  $M$  accepts  $w$ .
- **Proof:** we assume  $AP$  is decidable and show  $HP$  is decidable.
  - Since  $AP$  is decidable, there exists  $M_A$  such that  $M_A$  accepts  $(c, w)$ , if and only if  $c$  is the code of some  $M$  which accepts  $w$ .
  - We define  $M_H$ , which accepts input  $(c, w)$ , if and only if  $c$  is the code of some  $M$  that halts on input  $w$ .
    - If  $c$  is not well-formed, then  $M_H$  does not accept its input.
    - Otherwise,  $M_H$  modifies  $\langle M \rangle$  to  $\langle M' \rangle$  where  $M'$  behaves as  $M$ , except that, if  $M$  halts and does not accept,  $M'$  halts and accepts.  
 $M'$  thus accepts input  $w$ , if and only if  $M$  halts on input  $w$ .
  - $M_H$  accepts its input, if  $M_A$  accepts  $(\langle M' \rangle, w)$ .

```
function  $M_H(\langle M \rangle, w)$ :  
   $\langle M' \rangle := \text{compute}(\langle M \rangle)$   
  return  $M_A(\langle M' \rangle, w)$   
end function
```





## Semi-Decidability of the Acceptance Problem

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An undecidable problem may be semi-decidable.

- **Theorem:** the acceptance problem  $AP$  is semi-decidable.
  - There is some Turing Machine that halts and says “yes”, if its input is of form  $(\langle M \rangle, w)$  with  $w \in L(M)$  (and does not halt or says “no”, else).
- **Proof:** we construct a “universal Turing machine”  $M_u$  with language  $AP$  which acts as an “interpreter” for Turing machine codes: given input  $(\langle M \rangle, w)$ ,  $M_u$  simulates the execution of  $M$  for input  $w$ :
  - If the real execution of  $M$  halts for input  $w$  with/without acceptance, then also the simulated execution halts with/without acceptance; thus  $M_u$  accepts its input  $(c, w)$ , if in the simulation  $M$  has accepted  $w$ .
  - If the real execution of  $M$  does not halt for input  $w$ , then also the simulated execution does not halt; thus  $M_u$  does not accept its input.

Turing machines can be “interpreted/simulated” by other Turing machines.

# Halting versus Acceptance



We know that the halting problem is reducible to the acceptance problem.

- **Theorem:** the acceptance problem is reducible to the halting prob.
  - $HP \leq AP$  and  $AP \leq HP$ .
- **Proof:** assume that there exists  $M_H$  which decides the halting problem. Then we can construct  $M_A$  which decides acceptance:
  - For input  $(c, w)$ ,  $M_A$  decides whether bit string  $c$  is a valid code of some  $M$ , i.e.,  $c = \langle M \rangle$ . If not, then  $M_A$  does not accept its input.
  - Otherwise,  $M_A$  forwards  $(\langle M \rangle, w)$  to  $M_H$ :
    - If  $M_H$  does not accept  $(\langle M \rangle, w)$ , then  $M$  does not halt on input  $w$ ; thus  $M_A$  halts without accepting  $(\langle M \rangle, w)$ .
    - If  $M_H$  does accept  $(\langle M \rangle, w)$ , then  $M$  halts on input  $w$ ; in this case,  $M_A$  invokes  $M_U$  on  $(\langle M \rangle, w)$  whose simulation of  $M$  on input  $w$  also halts with or without accepting  $w$ . Then  $M_A$  halts with the same result.

The halting problem and the acceptance problem are “equivalent”.



## Semi-Decidability of Other Problems

- **Theorem:** the halting problem  $HP$  is semi-decidable.
  - **Proof:** we construct Turing machine  $M'$  which takes  $(\langle M \rangle, w)$  and simulates the execution of  $M$  on input  $w$ . If (the simulation of)  $M$  halts,  $M'$  accepts its input. If (the simulation of)  $M$  does not halt,  $M'$  does not halt (and thus not accept its input).
- **Theorem:** the non-acceptance problem  $NAP$  and the non-halting problem  $NHP$  are *not* semi-decidable.
  - **Proof:** if both a problem and its complement were semi-decidable, they would be complementary recursively enumerable languages; thus they would be recursive and the problem and its complement decidable.

Problem	semi-decidable	decidable
Halting	yes	no
Non-Halting	no	no
Acceptance	yes	no
Non-Acceptance	no	no

There exist problems that are not even semi-decidable.



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# Properties of Recursively Enumerable Languages



- **Property  $S$**  of recursively enumerable languages:

- A set of recursively enumerable languages.

- **$S$  is non-trivial:**

- there is at least one r.e. language in  $S$ , and

- there is at least one r.e. language not in  $S$ .

Some r.e. languages have the property and some do not.

- **$S$  is decidable:**  $P_S$  is decidable.

$$P_S := \{ \langle M \rangle \mid L(M) \in S \}$$

- Given  $\langle M \rangle$ , it is decidable whether the language of  $M$  has property  $S$ .

Decision questions about the semantics of Turing machines.

# Rice's Theorem



- **Rice's Theorem:** every non-trivial property of recursively enumerable languages is undecidable (proof: see lecture notes).
  - There is no Turing machine which for every possible Turing machine  $M$  can decide whether the language of  $M$  has a non-trivial property.
- **Relevance:** all non-trivial questions about the **input/output behavior** of Turing machines are undecidable.
  - Also for Turing computable functions.
  - Also for other Turing complete computational models.
- Nevertheless, for **some** machines a decision may be possible.
  - For some machines, it is possible to decide termination.
- However, no method can perform such a decision for **all** machines.
  - No method can exist to decide termination for every possible machine.
- Not applicable to **arbitrary questions** about Turing machines.
  - Form/syntax: does Turing machine  $M$  have more than  $n$  states?
  - Non-functional property: does  $M$  stop in less than  $n$  steps?
- Not applicable to **trivial questions**.
  - Is the language of Turing machine  $M$  recursively enumerable?

**Fundamental limit to automated reasoning about Turing complete models.**



# Undecidable Turing Machine Problems



Many interesting problems about Turing machines are undecidable:

- The **halting problem** (also in its restricted form).
- The **acceptance problem**  $w \in L(M)$  (also restricted to  $\varepsilon \in L(M)$ ).
- The **emptiness problem**: is  $L(M)$  empty?
- The problem of **language finiteness**: is  $L(M)$  finite?
- The problem of **language equivalence**:  $L(M_1) = L(M_2)$ ?
- The problem of **language inclusion**:  $L(M_1) \subseteq L(M_2)$ ?
- The problem whether  $L(M)$  is regular, context-free, context-sensitive.

Also the complements of these problems are not decidable; however, some of these problems (respectively their complements) may be semi-decidable.



## Undecidable Problems from Other Domains

- The **Entscheidungsproblem**: given a formula and a finite set of axioms, all in first order predicate logic, decide whether the formula is valid in every structure that satisfies the axioms.
- **Post's correspondence problem**: given pairs  $(x_1, y_1), \dots, (x_n, y_n)$  of non-empty words  $x_i$  and  $y_i$ , find a sequence  $i_1, \dots, i_k$  such that

$$x_{i_1} \dots x_{i_k} = y_{i_1} \dots y_{i_k}?$$

- The **word problem** for groups: given a group with finitely many generators  $g_1, \dots, g_n$  find two sequences  $i_1, \dots, i_k, j_1, \dots, j_l$  such that

$$g_{i_1} \circ \dots \circ g_{i_k} = g_{j_1} \circ \dots \circ g_{j_l}$$

- The **ambiguity problem** for context-free grammars: are there two different derivations for the same sentence?

Theory of decidability/undecidability has profound impact on many areas in computer science, mathematics, and logic.