

Hagenberg Research

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Abstract

This is just an exercise.

1 Computer-Assisted Discovery and Proving

First we comment on *computer-assisted guessing* in the context of mathematical discovery. Then we turn to the activity of proving, more precisely, to *proving methods* where *computer algebra algorithms* are used. Here we restrict to this special type of computer-assisted proving; for *general mathematical proving machines* like the THEOREMA system developed at RISC.

1.1 I.Q. Tests, Rabbits, and the Golden Section

Let us consider the following problem taken from an I.Q. test [2, Aufgabe 13, Denksport I fuer Superintelligente] from the sixties of the last century:

Continue the sequence 1, 1, 2, 3, 5, 8, 13, 21.

In the 21st century we let the computer do the problem. To this end we load the RISC package **GeneratingFunctions** written by C. Mallinger [4] in the computer algebra system Mathematica:

```
In[1]:= <<GeneratingFunctions.m
```

In the next step we input a little program that can be used to solve such I.Q. tests automatically:

*This is an exercise for the RISC course “Computer based working environments”.

```
In[2]:= GuessNext2Values[Li_] := Module[{rec},
      rec = GuessRE[Li,c[k],{1,2},{0,3}];
      RE2L[rec[[1]],c[k],Length[Li]+1]]
Finally the problem is solved automatically with
In[3]:= GuessNext2Values[{1,1,2,3,5,8,13,21}]
Out[3]= {1,1,2,3,5,8,13,21,34,55}
```

To produce additional values is no problem:

```
In[4]:= GuessNext2Values[{1,1,2,3,5,8,13,21,34,55}]
Out[4]= {1,1,2,3,5,8,13,21,34,55,89,144}
```

What is the mathematical basis for such automatic guessing? The answer originates in a simple observation: Many of the sequences $(x_n)_{n \geq 0}$ arising in practical applications (and in I.Q. tests!) are produced from a very simple pattern; namely, linear recurrences of the form

$$p_d(n)x_{n+d} + p_{d-1}(n)x_{n+d-1} + \cdots + p_0(n)x_n = 0, \quad n \geq 0,$$

with coefficients $p_i(n)$ being polynomials in n . So packages like Mallinger's **GeneratingFunctions** try to compute-via an ansatz using undetermined coefficients-a recurrence of exactly this type. For the I.Q. example above a recurrence is obtained by

```
In[5]:= GuessRE[{1,1,2,3,5,8,13,21},f[n]]
Out[5]= {{-f[n] - f[1+n] + f[2+n] == 0, f[0] == 1, f[1] == 1}, ogf}
```

Since only finitely many values are given as input, the output recurrence $f_{n+2} = f_{n+1} + f_n$ ($n \geq 0$) can be only a *guess* about a possible building principle of an *infinite* sequence. However, such kind of automated guessing is becoming more and more relevant to concrete applications. For instance, an application from mathematical chemistry can be found in [1] where a prediction for the total number of benzenoid hydrocarbons was made. Three years later this predication was confirmed [5]. Recently, quite sophisticated applications arose in connection with the enumeration of lattice paths and also with quantum field theory.

In 1202 Leonard Fibonacci introduced the numbers f_n . The fact that $f_0 = f_1 = 1$, and

$$f_{n+2} = f_{n+1} + f_n, \quad n \geq 0, \tag{1}$$

in Fibonacci's book was given the following interpretation: If baby rabbits become adults after one month, and if each pair of adult rabbits produces one pair of baby rabbits every month, how many pairs of rabbits, starting with one pair, are present after n months?

A non-recursive representation of (1) is the celebrated Euler-Binet formula

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right), \quad n \geq 0.$$

The number $(1 + \sqrt{5})/2 \approx 1.611803$, the *golden ratio*, is important in many parts of mathematics as well as in the art world. For instance, Phidias is said to have used it consciously in his sculpture.

Mathematicians gradually began to discover more and more interesting things about Fibonacci numbers f_n ; see e.g. [3]. For example, a typical sunflower has a large head that contains spirals of tightly packed florets, usually with $f_8 = 34$ winding in one direction and $f_9 = 55$ in another.

Another observation is this: Define g_n as a sum over binomial coefficients of the form

$$g_n := \sum_{k=0}^n \binom{n-k}{k}.$$

From the values $g_0 = 1$, $g_1 = 1$, $g_2 = 2$, $g_3 = 3$, $g_4 = 5$, and $g_5 = 8$ it is straight-forward to conjecture that the sequence $(g_n)_{n \geq 0}$ is nothing but the Fibonacci sequence $(f_n)_{n \geq 0}$. In the next subsection we shall see that nowadays such statements can be proved automatically with the computer.

References

- [1] F. Chyzak, I. Gutman, and P. Paule. Predicting the Number of Hexagonal Systems with 24 and 25 Hexagons. *MATCH*, 40:139–151, 1999.
- [2] Hans J. Eysenck. *Check Your Own I.Q.* Rowohlt, 1966.
- [3] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 2nd edition edition, 1994.

- [4] Christian Mallinger. Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences. Master's thesis, RISC-Linz, August 1996.
- [5] Markus Voegelé, Anthony J. Guttmann, and Iwan Jensen. On the Number of Benzenoid Hydrocarbons. *Journal of Chemical Information and Computer Sciences*, 42(3):456–466, 2002.