Problems Solved:

36	37	38	39	40

Name:

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Problem 36. Let L be a language over the alphabet $\Sigma = \{0, 1\}$ that is generated by some Turing machine N. For which L is the following problem semi-decidable? For which L is it decidable?
Input of the problem (*instance* of the problem): the code $\langle M \rangle$ of a Turing machine M.
Question of the problem: $L(M) \cap L \neq \emptyset$?

Problem 37. Is the following problem undecidable? Justify your answer.
Input of the problem (instance of the problem): The code $\langle M \rangle$ of a Turing machine with input alphabet $\{0, 1\}$.
Question of the problem: Does $L(\langle M \rangle)$ contain a word of even length?

Problem 38.

1. Consider the probability space $\Omega = \{0, 1\}^n$ of all strings over $\{0, 1\}$ of length n where each string occurs with the same probability 2^{-n}. Define a random variable $X : \Omega \to \mathbb{N}$ that gives the position of the first occurrence of the symbol 1 in a string, if 1 occurs at all. For completeness, we also define that $X(0^n) = 0$. Positions are numbered from 1 to n. Find the expected value $E(X)$ of the random variable X and justify your answer.

2. Evaluate the sum
$$\sum_{k=1}^{n} \frac{1}{2^k} k$$
in *closed form*, i.e., find a formula for the sum which does not involve a summation sign. *Hint:* Compute a closed form of the function
$$F(z) := \sum_{k=1}^{n} \frac{1}{2^k} z^k.$$
and compute its first derivative.

Problem 39. Let $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$ be a Turing machine with $Q = \{q_0, q_1\}$, $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \sqcup\}$, $F = \{q_1\}$ and the following transition function δ:

δ	0	1	$\sqcup$
q_0	$q_0 0 R$	$q_1 1 R$	–
q_1	–	–	–

1. Determine the (worst-case) time complexity $T(n)$ and the (worst-case) space complexity $S(n)$ of M.

2. Determine the average-case time complexity $\bar{T}(n)$ and the average-case space complexity $\bar{S}(n)$ of M. (Assume that all 2^n input words of length n occur with the same probability, i.e., $1/2^n$.)

Problem 40. Let $\Sigma = \{0, 1\}$ and let $L \subseteq \Sigma^*$ be the set of binary numbers divisible by 3, i.e.,

$$L = \{x_n \dots x_1 x_0 \ : \ 3 \text{ divides } \sum_{k=0}^{n} x_k 2^k\}.$$

(By convention, the empty string ε denotes the number 0 and so it is in L too.)

1. Design a Turing machine M with input alphabet Σ which recognizes L, halts on every input, and has (worst-case) time complexity $T(n) = n$. Write down your machine formally. (A picture is not needed.) *Hint:* Three states q_0, q_1, q_2 suffice. The machine is in state q_r if the bits read so far yield a binary number which leaves a remainder of r upon division by 3. The transition from one state to another represents a multiplication by 2 and the addition of 0 or 1.

2. Determine $S(n)$, $\bar{T}(n)$ and $\bar{S}(n)$ for your Turing machine.

3. Is there some faster Turing machine that achieves $\bar{T}(n) < n$? (Justify your answer.)