Limits of Computability

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Decision Problems

- Decision problem P.
  - A set of words $P \subseteq \Sigma^*$.

    $w \in P \ldots w$ has property P.
  - Interpretation as a property of words over Σ.

    $P(w) \ldots w$ has property P.
- Formal definition by a formula:

  $P := \{w \in \Sigma^* \mid \ldots\}$
  $P(w) :\Leftrightarrow \ldots$
- Informal definition by a decision question:

  Does word w have property ...?
- Example problem: Is the length of w a square number?

  $P := \{w \in \Sigma^* \mid \exists n \in \mathbb{N} : |w| = n^2\}$
  $P(w) :\Leftrightarrow \exists n \in \mathbb{N} : |w| = n^2$

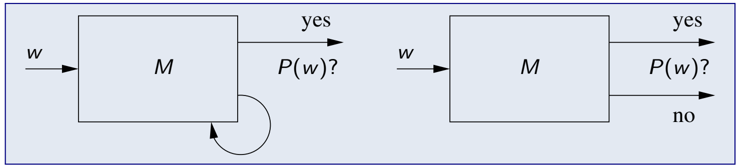
  $P = \{\varepsilon, 0, 0000, 000000000, \ldots\}$

A decision problem is the set of all words for which the answer to a decision question is "yes".

Semi-Decidability and Decidability

Problems can be the languages of Turing machines.

- A problem P is semi-decidable, if P is recursively enumerable.
  - There exists a Turing machine M that semi-decides P.
  - M must only terminate, if the answer to "$P(w)$?" is "yes".
- A problem P is decidable if P is recursive.
  - There exists a Turing machine M that decides P.
  - M must also terminate, if the answer to "$P(w)$?" is "no".



Decidability of Complement

- Theorem: If P is decidable, also its complement $\overline{P}$ is decidable.

  The answer to "$\overline{P}(w)$?" is "yes", if and only if the answer to "$P(w)$?" is "no" ($\overline{P}(w) \Leftrightarrow \neg P(w)$).

  - Proof: If P is decidable, it is recursive, thus $\overline{P}$ is recursive, thus $\overline{P}$ is decidable.
- Theorem: P is decidable, if both P and $\overline{P}$ are semi-decidable.
  - If P and $\overline{P}$ are semi-decidable, they are recursive enumerable. Thus P is recursive and therefore decidable.

Direct consequences of the previously established results about recursively enumerable and recursive languages.

Decidability and Computability

- Theorem: $P \subseteq \Sigma^*$ is semi-decidable, if and only if the partial characteristic function $1'_P : \Sigma^* \to_p \{1\}$ is Turing computable:

$$1'_P(w) := \begin{cases} 1 & \text{if } P(w) \\ \text{undefined} & \text{if } \neg P(w) \end{cases}$$

  - Proof: if P is semi-decidable, there exists M such that, for every word $w \in P = domain(1'_P)$, M accepts w. We can then construct M' which calls M on w. If M accepts w, M' writes 1 on output tape. If $1'_P$ is Turing computable, there exists M such that, for every word $w \in P = domain(1'_P)$, M accepts w and writes 1 on the tape. We can then construct M' which takes w from the tape and calls M on w. If M writes 1, M' accepts w.
- Theorem: $P \subseteq \Sigma^*$ is decidable, if and only if the characteristic function $1_P : \Sigma^* \to \{0,1\}$ is Turing computable:

$$1_P(w) := \begin{cases} 1 & \text{if } P(w) \\ 0 & \text{if } \neg P(w) \end{cases}$$

  - Proof: analogous.

Turing Machine Codes

Theorem: for every Turing machine M, there exists a bit string $\langle M \rangle$,

- the Turing machine code of M

such that

1. different Turing machines have different codes
   - if $M \neq M'$, then $\langle M \rangle \neq \langle M' \rangle$;
2. we can recognize valid Turing-machine codes
   - $w \in range(\langle . \rangle)$ is decidable
3. the encoding $\langle M \rangle$ and the decoding $\langle c \rangle^{-1}$ are Turing computable.

- Core idea: assign to all machine states, alphabet symbols, and tape directions unique natural numbers and encode every transition $\delta(q_i, a_j) = (q_k, a_l, d_r)$ by the tuple (i, j, k, l, r) in binary form.

A Turing machine code is also called a "Gödel number".

The Halting Problem

The most famous undecidable problem in computer science.

- The halting problem HP is to decide, for given Turing machine code $\langle M \rangle$ and word w, whether M halts on input w:

$$HP := \{(\langle M \rangle, w) \mid \text{Turing machine } M \text{ halts on input word } w\}$$

  - (w_1, w_2): a bit string that reversibly encodes the pair w_1, w_2.
- Theorem: The halting problem is undecidable.
  - There is no Turing machine that always halts and says "yes", if its input is of form $(\langle M \rangle, w)$ such that M halts on input w, respectively says "no", if this is not the case.

The remainder of this section is dedicated to the proof of this theorem.

Enumeration of Words and Turing Machines

- Theorem: There exists an enumeration w of all words over Σ.

  $w = (w_0, w_1, \ldots)$

  - For every word $w' \in \Sigma^*$, there exists $i \in \mathbb{N}$ such that $w' = w_i$.
  - The enumeration w starts with the empty word, then lists the all words of length 1, then lists all the words of length 2, and so on. Thus every word eventually appears in w.
- Theorem: There exists an enumeration M of all Turing machines.

  $M = (M_0, M_1, \ldots)$

  - For every Turing machine M' there exists $i \in \mathbb{N}$ such that $M' = M_i$.
  - Let $C = (C_0, C_1, \ldots)$ be the enumeration of all Turing machine codes in bit-alphabetic word order. We define M_i as the unique Turing machine denoted by C_i. Since every Turing machine has a code and C enumerates all codes, M is the enumeration of all Turing machines.

There are countably many words and countably many Turing machines.

Undecidability of the Halting Problem

Proof: define $h : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ as

$$h(i,j) := \begin{cases} 1 & \text{if Turing machine } M_i \text{ halts on input word } w_j \\ 0 & \text{otherwise} \end{cases}$$

If h were computable, then the halting problem would be decidable:

- Construct M, which for given pair (c, w) first decides whether c is a valid Turing machine code; if not, M does not accept its input.
- Otherwise, M determines $i, j \in \mathbb{N}$ such that $c = \langle M_i \rangle$ and $w = w_j$
  - M just enumerates all Turing machine codes and all words until c and w appear in the respective enumeration.
- Then M computes $h(i,j)$; if the result is 1, then M accepts its input; if the result is 0, then M does not accept it.

It thus suffices to show that h is not computable by a Turing machine.

Undecidability of the Halting Problem

We assume that h is computable and derive a contradiction.

- Define $d : \mathbb{N} \to \{0,1\}$ as

$$d(i) := h(i,i)$$

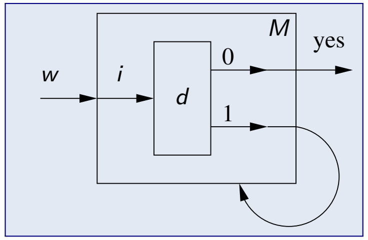
  - $d(i) = 1$: M_i terminates on input word w_i.
  - Diagonalization: $d(0), d(1), d(2), \ldots$ is diagonal of value table for h.

h	$j=0$	$j=1$	$j=2$	$\ldots$
$i=0$	$\mathbf{h(0,0)}$	$h(0,1)$	$h(0,2)$	$\ldots$
$i=1$	$h(1,0)$	$\mathbf{h(1,1)}$	$h(1,2)$	$\ldots$
$i=2$	$h(2,0)$	$h(2,1)$	$\mathbf{h(2,2)}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Since h is computable, also d is computable.

Undecidability of the Halting Problem

```
function M(w):
    let i ∈ ℕ such that w = w_i
    case d(i) of
        0: return yes
        1: loop end loop
    end case
end function
```



- Construct M which takes w and determines $i \in \mathbb{N}$ with $w = w_i$.
  - $M(w)$ halts, if and only if $d(i) = 0$.
- Let i be such that $M = M_i$ and compute $M(w_i)$.
  - $M(w_i)$ halts, if and only if $d(i) = 0$.
  - $M(w_i)$ halts, if and only if $M_i(w_i)$ does not halt.
  - $M(w_i)$ halts, if and only if $M(w_i)$ does not halt.

By letting M reason about its own behavior, we derive a contradiction.

Reduction Proofs

We can construct a partial order on decision problems.

- Decision problem $P \subseteq \Sigma^*$ is reducible to $P' \subseteq \Gamma^*$ ($P \leq P'$), if there is a computable function $f : \Sigma^* \to \Gamma^*$ such that
$$P(w) \Leftrightarrow P'(f(w))$$
  - w has property P if and only if $f(w)$ has property P'.
- Theorem: For all decision problems P and P' with $P \leq P'$, it holds that, if P is not decidable, then also P' is not decidable.
  - Proof: we assume that P' is decidable and show that P is decidable. Since P' is decidable, there is a Turing machine M' that decides P'. We construct M that decides P:

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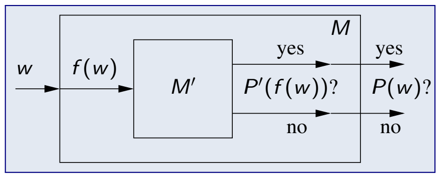
function M(w):
    w' ← f(w)
    return M'(w')
end function
```



Undecidability of the Restricted Halting Problem

To show that some problem P is not decidable, if suffices to show that, if P is decidable, also the the halting problem HP is decidable.

- Theorem: the restricted halting problem RHP is not decidable.

$$RHP := \{\langle M\rangle \mid \text{Turing machine } M \text{ halts on input word } \varepsilon\}$$

  - Decide, for given $\langle M\rangle$, whether M halts for input word ε.

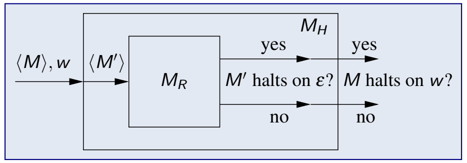
Pattern for many undecidability proofs.

Undecidability of the Restricted Halting Problem

We assume that *RHP* is decidable and show that *HP* is decidable.

- Since *RHP* is decidable, there exists M_R such that M_R accepts input c, if and only if c is the code of some M which halts on input ε.
- We can then define M_H, which accepts input (c, w), if and only if c is the code of some M that terminates on input w:
  - M_H constructs from (c, w) the code of some M' which first prints w on its tape and then behaves like M.
    - M' terminates for input ε (which is ignored and overwritten by w) if and only if M terminates on input w.
  - M_H accepts its input, if and only if M_R accepts $\langle M' \rangle$.

function $M_H(\langle M \rangle, w)$:
$\quad\langle M' \rangle :=$ compute$(\langle M \rangle, w)$
$\quad$**return** $M_R(\langle M' \rangle)$
end function



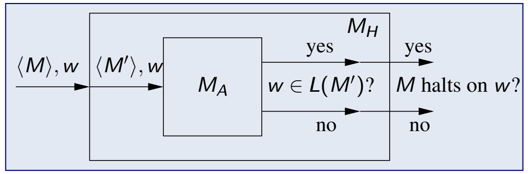
Undecidability of the Acceptance Problem

- Theorem: the acceptance problem *AP* is not decidable.

$$AP := \{(\langle M \rangle, w) \mid w \in L(M)\}$$

  - Decide, for given M and w, whether M accepts w.
- Proof: we assume *AP* is decidable and show *HP* is decidable.
  - Since *AP* is decidable, there exists M_A such that M_A accepts (c, w), if and only if c is the code of some M which accepts w.
  - We define M_H, which accepts input (c, w), if and only if c is the code of some M that halts on input w.
    - If c is not well-formed, then M_H does not accept its input.
    - Otherwise, M_H modifies $\langle M \rangle$ to $\langle M' \rangle$ where M' behaves as M, except that, if M halts and does not accept, M' halts and accepts.
      M' thus accepts input w, if and only if M halts on input w.
  - M_H accepts its input, if M_A accepts $(\langle M' \rangle, w)$.

function $M_H(\langle M \rangle, w)$:
$\quad\langle M' \rangle :=$ compute$(\langle M \rangle)$
$\quad$**return** $M_A(\langle M' \rangle, w)$
end function



Semi-Decidability of the Acceptance Problem

An undecidable problem may be semi-decidable.

- Theorem: the acceptance problem *AP* is semi-decidable.
  - There is some Turing Machine that halts and says "yes", if its input is of form $(\langle M \rangle, w)$ with $w \in L(M)$ (and does not halt or says "no", else).
- Proof: we construct a "universal Turing machine" M_u with language *AP* which acts as an "interpreter" for Turing machine codes: given input $(\langle M \rangle, w)$, M_u simulates the execution of M for input w:
  - If the real execution of M halts for input w with/without acceptance, then also the simulated execution halts with/without acceptance; thus M_u accepts its input (c, w), if in the simulation M has accepted w.
  - If the real execution of M does not halt for input w, then also the simulated execution does not halt; thus M_u does not accept its input.

Turing machines can be "interpreted/simulated" by other Turing machines.

Halting versus Acceptance

We know that the halting problem is reducible to the acceptance problem.

- Theorem: the acceptance problem is reducible to the halting prob.
  - $HP \leq AP$ and $AP \leq HP$.
- Proof: assume that there exists M_H which decides the halting problem. Then we can construct M_A which decides acceptance:
  - For input (c, w), M_A decides whether bit string c is a valid code of some M, i.e., $c = \langle M \rangle$. If not, then M_A does not accept its input.
  - Otherwise, M_A forwards $(\langle M \rangle, w)$ to M_H:
    - If M_H does not accept $(\langle M \rangle, w)$, then M does not halt on input w; thus M_A halts without accepting $(\langle M \rangle, w)$.
    - If M_H does accept $(\langle M \rangle, w)$, then M halts on input w; in this case, M_A invokes M_u on $(\langle M \rangle, w)$ whose simulation of M on input w also halts with or without accepting w. Then M_A halts with the same result.

The halting problem and the acceptance problem are "equivalent".

Semi-Decidability of Other Problems

- Theorem: the halting problem HP is semi-decidable.
  - Proof: we construct Turing machine M' which takes $(\langle M \rangle, w)$ and simulates the execution of M on input w. If (the simulation of) M halts, M' accepts its input. If (the simulation of) M does not halt, M' does not halt (and thus not accept its input).
- Theorem: the non-acceptance problem NAP and the non-halting problem NHP are *not* semi-decidable.
  - Proof: if both a problem and its complement were semi-decidable, they would be complementary recursively enumerable languages; thus they would be recursive and the problem and its complement decidable.

Problem	semi-decidable	decidable
Halting	yes	no
Non-Halting	no	no
Acceptance	yes	no
Non-Acceptance	no	no

There exist problems that are not even semi-decidable.

1. Decision Problems

2. The Halting Problem

3. Reduction Proofs

4. Rice's Theorem

Properties of Recursively Enumerable Languages

- Property S of recursively enumerable languages:
  - A set of recursively enumerable languages.
- S is non-trivial:
  - there is at least one r.e. language in S, and
  - there is at least one r.e. language not in S.

  Some r.e. languages have the property and some do not.
- S is decidable: P_S is decidable.

$$P_S := \{\langle M \rangle \mid L(M) \in S\}$$

  - Given $\langle M \rangle$, it is decidable whether the language of M has property S.

Decision questions about the semantics of Turing machines.

Rice's Theorem

- Rice's Theorem: every non-trivial property of recursively enumerable languages is undecidable.
  - There is no Turing machine which for every possible Turing machine M can decide whether the language of M has a non-trivial property.
- All non-trivial questions about the behavior (semantics) of Turing machines are undecidable.
  - Also for Turing computable functions.
  - Also for other Turing complete computational models.
- Nevertheless, for some machines a decision may be possible.
  - For some machines, it is possible to decide termination.
- However, no method can perform such a decision for all machines.
  - No method can exist to decide termination for every possible machine.
- Not applicable to questions about form (syntax) of Turing machines.
  - Does Turing machine M have more than n states?
- Not applicable to trivial questions.
  - Is the language of Turing machine M recursively enumerable?

Fundamental limit to automated reasoning about Turing complete models.

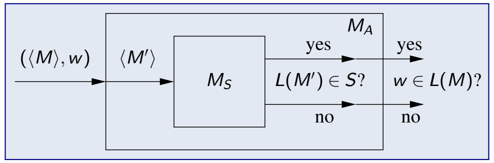
Proof of Rice's Theorem

- We assume that non-trivial property S of recursively enumerable languages is decidable and derive a contradiction.
  - We may assume that $\emptyset \notin S$.
  - Otherwise continue with $\overline{S}$.
    - If $\overline{S}$ is not decidable, S is not decidable.
  - Since S is decidable, there exists M_S which decides S.
- Assume we can compute, for every $(\langle M \rangle, w)$, code $\langle M' \rangle$ such that

$$L(M') \in S \Leftrightarrow w \in L(M)$$

  - M' has property S if and only if M accepts w.
- Then we can construct M_A which decides the acceptance problem:

function $M_A(\langle M \rangle, w)$:
  $\langle M' \rangle :=$ compute$(\langle M \rangle)$
  return $M_S(\langle M' \rangle)$
end function



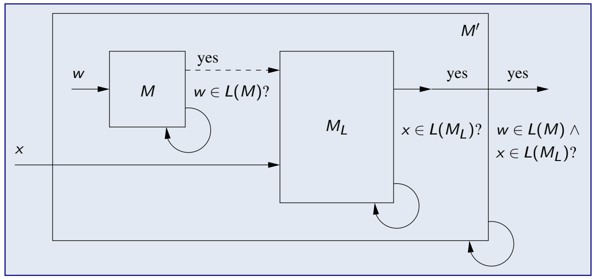
It remains to be shown how to compute appropriate $\langle M' \rangle$.

Proof of Rice's Theorem (Contd)

We construct $\langle M' \rangle$ with $L(M') \in S \Leftrightarrow w \in L(M)$.

- M_A enumerates all possible Turing machine codes and applies M_S to decide whether this code has property S.
  - Since S is not-trivial and thus not empty, eventually code $\langle M_L \rangle \in S$ with $L(M_L) \in S$ is found.
- Then M_A constructs the code $\langle M' \rangle$ of the following M':

function $M'(x)$:
  if $w \in L(M)$ **then**
    if $x \in L(M_L)$ **then**
      return yes
    else
      loop end loop
    end if
  else
    loop end loop
  end if
end function



It remains to be shown $L(M') \in S \Leftrightarrow w \in L(M)$.

Proof of Rice's Theorem (Contd)

We show $L(M') \in S \Leftrightarrow w \in L(M)$.

- M' accepts x only, if M accepts w and if M_L accepts x.
  - If M does not accept w, M' does not accept any input.
  - If M accepts w, then M' accepts the same words as M_L;

$$L(M') = \begin{cases} \emptyset & \text{if } w \notin L(M) \\ L(M_L) & \text{if } w \in L(M) \end{cases}$$

$$w \notin L(M) \Rightarrow L(M') = \emptyset$$
$$w \in L(M) \Rightarrow L(M') = L(M_L)$$

- We know $\emptyset \notin S$ and $L(M_L) \in S$.

$$w \notin L(M) \Rightarrow L(M') \notin S$$
$$w \in L(M) \Rightarrow L(M') \in S$$

- Thus $L(M') \in S \Leftrightarrow w \in L(M)$.

Core idea is to encode the answer to the acceptance problem as a property of the language of M'.

Undecidable Turing Machine Problems

Many interesting problems about Turing machines are undecidable:

- The halting problem (also in its restricted form).
- The acceptance problem $w \in L(M)$ (also restricted to $\varepsilon \in L(M)$).
- The emptiness problem: is $L(M)$ empty?
- The problem of language finiteness: is $L(M)$ finite?
- The problem of language equivalence: $L(M_1) = L(M_2)$?
- The problem of language inclusion: $L(M_1) \subseteq L(M_2)$?
- The problem whether $L(M)$ is regular, context-free, context-sensitive.

Also the complements of these problems are not decidable; however, some of these problems (respectively their complements) may be semi-decidable.

Undecidable Problems from Other Domains

- The Entscheidungsproblem: given a formula and a finite set of axioms, all in first order predicate logic, decide whether the formula is valid in every structure that satisfies the axioms.
- Post's correspondence problem: given pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of non-empty words x_i and y_i, find a sequence $i_1, \ldots, i_k$ such that

$$x_{i_1} \ldots x_{i_k} = y_{i_1} \ldots y_{i_k}?$$

- The word problem for groups: given a group with finitely many generators $g_1, \ldots, g_n$ find two sequences $i_1, \ldots, i_k$, $j_1, \ldots, j_l$ such that

$$g_{i_1} \circ \ldots \circ g_{i_k} = g_{j_1} \circ \ldots \circ g_{j_l}$$

- The ambiguity problem for context-free grammars: are there two different derivations for the same sentence?

Theory of decidability/undecidability has profound impact on many areas in computer science, mathematics, and logic.