

Turing Complete Computational Models

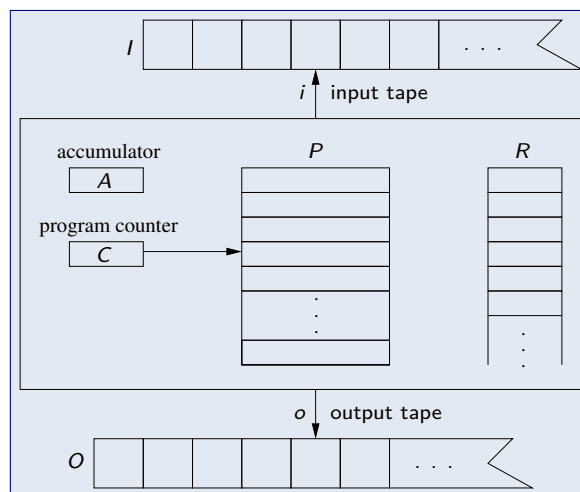
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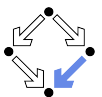
1. Random Access Machines
2. Loop and While Programs
3. Primitive Recursive and μ -recursive Functions
4. Further Turing Complete Models
5. The Chomsky Hierarchy
6. Real Computers

A Random Access Machine



A model closer to a real computer.

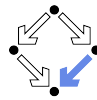
A Random Access Machine



- A **random access machine (RAM)**:
 - an infinite input tape I (whose cells can hold natural numbers of arbitrary size) with a read head position $i \in \mathbb{N}$,
 - an infinite output tape O (whose cells can hold natural numbers of arbitrary size) with a write head position $o \in \mathbb{N}$,
 - an accumulator A which can hold a natural number of arbitrary size,
 - a program counter C which can hold an arbitrary natural number,
 - a program consisting of a finite number of instructions $P[1], \dots, P[m]$,
 - a memory consisting of a countably infinite sequence of registers $R[1], R[2], \dots$, each of which can hold an arbitrary natural number.
- **Execution**:
 - Initially, $i = 0, o = 0, A = 0, C = 1, R[1] = R[2] = \dots = 0$.
 - In every step, the RAM reads $P[C]$, increments C by 1, and then performs the action indicated by the instruction.
 - Execution terminates when $C = 0$.

Program is a sequence of machine instructions.

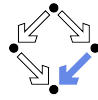
RAM Instructions



Instruction	Description	Action
IN	Read value from input tape into accumulator	$A := I[i]; i := i + 1$
OUT	Write value from accumulator to output tape	$O[o] := A; o := o + 1$
LOAD # n	Load constant n into accumulator	$A := n$
LOAD n	Load content of register n into accumulator	$A := R[n]$
LOAD (n)	Load content of register referenced by reg. n	$A := R[R[n]]$
STORE n	Store content of accumulator into register n	$R[n] := A$
STORE (n)	Store content into register referenced by reg. n	$R[R[n]] := A$
ADD # n	Increment content of accumulator by constant	$A := A + n$
SUB # n	Decrement content of accumulator by constant	$A := \max\{0, A - n\}$
JUMP n	Unconditional jump to instruction n	$C := n$
BEQ i, n	Conditional jump to instruction n	if $A = i$ then $C := n$

Immediate addressing, direct addressing, indirect addressing.

Example



```

START:  LOAD #1      A := 1
        STORE 1     R[1] := A
READ:   LOAD 1      A := R[1]
        ADD #1     A := A + 1
        STORE 1    R[1] := A
        IN        A := I[i]; i := i + 1
        BEQ 0, WRITE if A = 0 then C := WRITE
        STORE (1)  R[R[1]] := A
        JUMP READ  C := READ
WRITE:  LOAD 1      A := R[1]
        SUB #1     A := A - 1
        STORE 1    R[1] := A
        BEQ 1, HALT if A = 1 then C := HALT
        LOAD (1)  A := R[R[1]]
        OUT      O[o] := A; o := o + 1
        JUMP WRITE C := WRITE
HALT:  JUMP 0      C := 0
    
```

Reads $x_1, \dots, x_n, 0$ and writes x_n, \dots, x_1 using stack $R[2], \dots, R[n+1]$.

RAMs versus Turing Machines



Theorem: Every Turing machine can be simulated by a RAM.

- RAM uses registers $R[1], \dots, R[c-1]$ for its own purposes,
- stores in $R[c]$ the position of the tape head of the Turing machine,
- uses $R[c+1], R[c+2], \dots$ as a virtual Turing machine tape.
 - Using “indirect addressing” operations $\text{LOAD}(n)$ and $\text{STORE}(n)$.
- RAM copies the input from the input tape into its virtual tape, then it mimics the execution of the Turing machine on the virtual tape.
- When the simulated Turing machine terminates, the content of the virtual tape is copied to the output tape.

RAMs represent a Turing complete computational model.

RAMs versus Turing Machines



Theorem: Every RAM can be simulated by a Turing machine.

- The Turing machine uses 5 tapes to simulate the RAM:
 - Tape 1 represents the input tape of the RAM.
 - Tape 2 represents the output tape of the RAM.
 - Tape 3 holds a representation of that part of the memory that has been written by the simulation of the RAM.
 - Tape 4 holds a representation of the accumulator of the RAM.
 - Tape 5 serves as a working tape.
- Tape 3 holds a sequence of (address, contents) pairs that represent those registers of the RAM that have been written during the simulation (the contents of all other registers hold 0).
- Every instruction of the RAM is simulated by a sequence of steps of the Turing machine which reads respectively writes Tape 1 and 2 and updates on Tape 3 and 4 the tape representations of the contents of the memory and the accumulator.

RAMs are not more powerful than Turing machines.

Random Access Stored Program Machine



The program of a RAM is “read-only”.

- **Random Access Stored Program Machine (RASP).**
 - A RAM variant where the program is stored in memory R (there is no separate program store P).
- **Every RASP can be simulated by a RAM.**
 - RAM is interpreter for RASP instructions (like a *microprogram* in a processor interprets machine instructions).
- **Every RAM can be simulated by a RASP.**
 - Even if indirect addressing is removed from RASP.
 - RAM instructions $\text{LOAD}(n)$ and $\text{STORE}(n)$ can be interpreted by self-modifying RASP code.

Self modifying programs do not add computational power to a RAM.

Loop Programs



- **Loop Program P :**

$$P ::= x_i := 0 \mid x_i := x_j + 1 \mid x_i := x_j - 1 \mid P; P \\ \mid \text{loop } x_i \text{ do } P \text{ end.}$$

- Set $\{x_0, x_1, x_2, \dots\}$ of program variables.
- Initial value of x_i determines the number of loop iterations.
- Loop must eventually terminate.

Programs with bounded iteration that necessarily terminate.

1. Random Access Machines

2. Loop and While Programs

3. Primitive Recursive and μ -recursive Functions

4. Further Turing Complete Models

5. The Chomsky Hierarchy

6. Real Computers

Semantics



- **Semantics $\llbracket P \rrbracket(m)$** maps the start memory $m : \mathbb{N} \rightarrow \mathbb{N}$ to the final memory after the termination of P :

$$\begin{aligned} \llbracket x_i := 0 \rrbracket(m) &:= m[i \leftarrow 0] \\ \llbracket x_i := x_j + 1 \rrbracket(m) &:= m[i \leftarrow m(j) + 1] \\ \llbracket x_i := x_j - 1 \rrbracket(m) &:= m[i \leftarrow \max\{0, m(j) - 1\}] \\ \llbracket P_1; P_2 \rrbracket(m) &:= \llbracket P_2 \rrbracket(\llbracket P_1 \rrbracket(m)) \\ \llbracket \text{loop } x_i \text{ do } P \text{ end} \rrbracket(m) &:= \llbracket P \rrbracket^{m(i)}(m) \end{aligned}$$

- $m[i \leftarrow n]$: memory m after updating the value x_i by value n .
- $\llbracket P \rrbracket^n(m)$: memory m after n times executing P :

$$\begin{aligned} \llbracket P \rrbracket^0(m) &:= m \\ \llbracket P \rrbracket^{n+1}(m) &:= \llbracket P \rrbracket(\llbracket P \rrbracket^n(m)) \end{aligned}$$

A loop program denotes a function over memories.

Syntactic Abbreviations



- $x_i := x_j$

```
x_i := x_j + 1; x_j := x_i - 1
```

- $x_j := n$

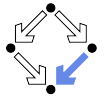
```
x_i := 0; x_i := x_i + 1; x_j := x_i + 1; ...; x_j := x_i + 1
```

- **if** $x_i = 0$ **then** P_t **else** P_e **end**

```
x_t := 1; loop x_i do x_t := 0; end;  
x_e := 1; loop x_t do x_e := 0; end;  
loop x_t do  $P_t$  end; loop x_e do  $P_e$  end;
```

The usual programming language constructs (except for unbounded iteration) can be represented.

Loop Computability



We consider the computability of functions over the natural numbers.

$f : \mathbb{N}^n \rightarrow \mathbb{N}$ is **loop computable**, if there exists a loop program P such that for all $x_1, \dots, x_n \in \mathbb{N}$ and memory $m : \mathbb{N} \rightarrow \mathbb{N}$ defined as

$$m(i) := \begin{cases} x_i & \text{if } 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

we have

$$\llbracket P \rrbracket(m)(0) = f(x_1, \dots, x_n)$$

When started in a state where x_1, \dots, x_n contain the arguments of f , the program terminates in a state where x_0 holds the result of f .

Example



- Addition is computable by the program $x_0 := x_1 + x_2$:

```
x_0 := x_1;  
loop x_2 do  
  x_0 := x_0 + 1  
end
```

- Multiplication is computable by the program $x_0 := x_1 \cdot x_2$:

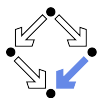
```
x_0 := 0;  
loop x_2 do  
  x_0 := x_0 + x_1  
end
```

- Exponentiation is computable by the program $x_0 := x_1^{x_2}$:

```
x_0 := 1;  
loop x_2 do  
  x_0 := x_0 · x_1  
end
```

Natural number arithmetic is loop computable.

Arithmetic



- $x_0 := x_1 \cdot x_2$:

```
x_0 := 0;  
loop x_2 do  
  x_0 := x_0 + x_1  
end
```

\rightsquigarrow

```
x_0 := 0;  
loop x_2 do  
  x_0 := x_0;  
  loop x_1 do  
    x_0 := x_0 + 1  
  end  
end
```

Higher arithmetic needs multiply nested loops.

Beyond Exponentiation



$$a \uparrow^n b := \begin{cases} a^b & \text{if } n = 1 \\ 1 & \text{if } b = 0 \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) & \text{else} \end{cases}$$

- $a \uparrow^1 b = a^b$
 $a \uparrow^1 b = a \cdot a \cdot \dots \cdot a$ (b times)
- $a \uparrow^2 b = a^{a^{\dots^a}}$ (b times)
 $a \uparrow^2 b = a \uparrow^1 a \uparrow^1 \dots \uparrow^1 a$ (b times)
- $a \uparrow^3 b$:
 $a \uparrow^3 b = a \uparrow^2 a \uparrow^2 \dots \uparrow^2 a$ (b times)

The notation allows to define arbitrary “complex” arithmetic functions.

While Programs



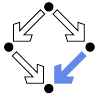
- **While Program** P :

$P ::= \dots$ (as for loop programs)
| **while** x_i **do** P **end.**

- Set $\{x_0, x_1, x_2, \dots\}$ of program variables.
- Loop is repeated as long as $x_i \neq 0$.
- If $x_i \neq 0$ forever, loop does not terminate.

Programs with unbounded iteration that may not terminate.

Limits of Loop Computability



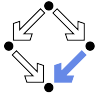
- **Theorem:** for every $n > 0$ and $f(a, b) := a \uparrow^n b$
 - f is loop computable, and
 - every loop program computing f requires at least $n+2$ nested loops.
- **Theorem:** $g : \mathbb{N}^3 \rightarrow \mathbb{N}, g(a, b, n) := a \uparrow^{n+1} b$ is not loop computable.
 - Assume g can be computed by a program P with n loops.
 - Then the computation of $g(a, b, n) = a \uparrow^{n+1} b$ requires $n+3$ loops.
 - Thus P cannot compute g .
- Also the **Ackermann Function** is not loop computable:

$$\begin{aligned} \text{ack}(0, m) &:= m + 1 \\ \text{ack}(n, 0) &:= \text{ack}(n-1, 1) \\ \text{ack}(n, m) &:= \text{ack}(n-1, \text{ack}(n, m-1)), \text{ if } n > 0 \wedge m > 0 \end{aligned}$$

- $\text{ack}(n, m) = 2 \uparrow^{n-2} (m+3) - 3$
- $\text{ack}(4, 2)$ has 20,000 digits.

Some arithmetic functions grow “too fast” to be loop computable.

Semantics



- **Semantics** $\llbracket P \rrbracket(m)$ maps start memory $m : \mathbb{N} \rightarrow \mathbb{N}$
 - to the final memory, if P terminates, and
 - to the special value \perp (bottom), if P does not terminate.
- Semantics generalizes that of loop programs:

$$\llbracket P \rrbracket(m) := \begin{cases} \perp & \text{if } m = \perp \\ \llbracket P \rrbracket'(m) & \text{else} \end{cases}$$

$$\llbracket \dots \rrbracket'(m) := \dots \text{ (as for loop programs)}$$

- Semantics of unbounded iteration:

$$\llbracket \text{while } x_i \text{ do } P \text{ end} \rrbracket'(m) := \begin{cases} \perp & \text{if } L_i(P, m) \\ \llbracket P \rrbracket^{T_i(P, m)}(m) & \text{else} \end{cases}$$

$$L_i(P, m) := \Leftrightarrow \forall k \in \mathbb{N} : \llbracket P \rrbracket^k(m)(i) \neq 0$$

$$T_i(P, m) := \min \{ k \in \mathbb{N} \mid \llbracket P \rrbracket^k(m)(i) = 0 \}$$

A while program denotes a function whose result is either a memory or \perp .

Syntactic Abbreviations

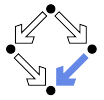


- **while** $x_i < x_j$ **do** P **end**

```
 $x_k := x_j - x_i;$   
while  $x_k$  do  $P; x_k := x_j - x_i;$  end
```

Analogous constructions possible for other termination conditions.

While Computability



$f : \mathbb{N}^n \rightarrow_p \mathbb{N}$ is **while computable**, if there exists a while program P such that for all $x_1, \dots, x_n \in \mathbb{N}$ and memory $m : \mathbb{N} \rightarrow \mathbb{N}$ defined as

$$m(i) := \begin{cases} x_i & \text{if } 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

the following holds:

- If $x_1, \dots, x_n \in \text{domain}(f)$, then $\llbracket P \rrbracket(m) : \mathbb{N} \rightarrow \mathbb{N}$ and

$$\llbracket P \rrbracket(m)(0) = f(x_1, \dots, x_n)$$

- If $x_1, \dots, x_n \notin \text{domain}(f)$, then

$$\llbracket P \rrbracket(m) = \perp$$

For a defined value of $f(x_1, \dots, x_n)$, P terminates with this value in variable x_0 . If $f(x_1, \dots, x_n)$ is undefined, the program does not terminate.

Example



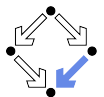
The Ackermann function is while computable with the help of a stack.

```
function  $ack(n, m)$ :  
  if  $n = 0$  then  
    return  $m + 1$   
  else if  $m = 0$  then  
    return  $ack(n - 1, 1)$   
  end if  
  return  $ack(n - 1, ack(n, m - 1))$   
end function
```

```
function  $ack(x_1, x_2)$ :  
  push( $x_1$ ); push( $x_2$ )  
  while  $size() > 1$  do  
     $x_2 \leftarrow pop()$ ;  $x_1 \leftarrow pop()$   
    if  $x_1 = 0$  then  
      push( $x_2 + 1$ )  
    else if  $x_2 = 0$  then  
      push( $x_1 - 1$ ); push(1);  
    else  
      push( $x_2 - 1$ );  
      push( $x_1$ ); push( $x_2 - 1$ )  
    end if  
  end while  
  return  $pop()$   
end function
```

While programs are computationally more powerful than loop programs.

Normal Form of a While Program



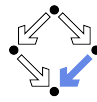
Kleene's Normal Form Theorem: every while computable function can be computed by a while program in Kleene's normal form:

```
 $x_c := 1;$   
while  $x_c \neq 0$  do  
  if  $x_c = 1$  then  $P_1$   
  else if  $x_c = 2$  then  $P_2$   
  ...  
  else if  $x_c = n$  then  $P_n$   
  end if  
end while
```

- P_1, \dots, P_n do *not* contain while loops.
- Control variable x_c determines which P_i to execute next.

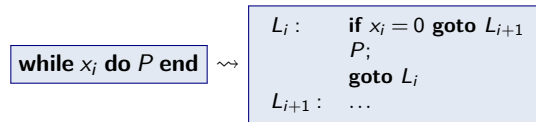
A single while loop is all that is needed.

Normal Form of a While Program

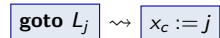


We sketch the proof of Kleene's Normal Form Theorem.

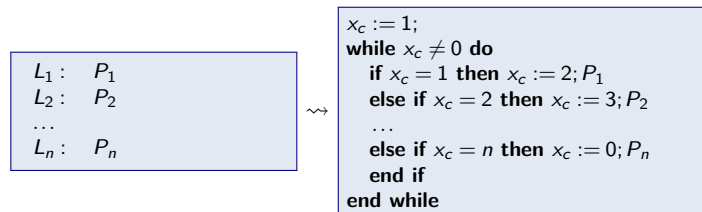
- A while program can be translated into a goto program:



- Gotos can be translated to control variable assignments:

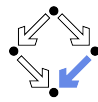


- The resulting program can be translated into normal form:



In essence, the execution loop of a processor.

Turing Machines and While Programs

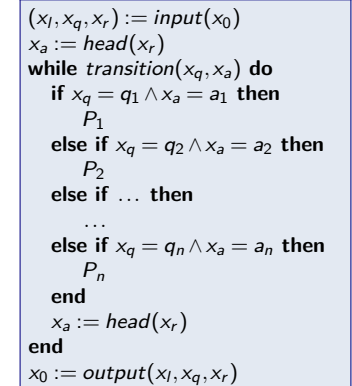


- **Theorem:** Every Turing machine can be simulated by a while program and vice versa.

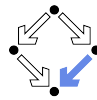
- Consequence: every Turing computable function is while computable and vice versa.

Proof \Rightarrow : construct P to simulate M .

- x_0 holds initial tape content.
 - Determines initial configuration.
- Machine configuration (x_l, x_q, x_r) :
 - x_q : the current state.
 - x_l : the tape left to the tape head.
 - x_r : the tape under/right to head.
- State x_q and symbol x_a under head determine the state transition.
 - If none is possible, final tape content is written to x_0 .

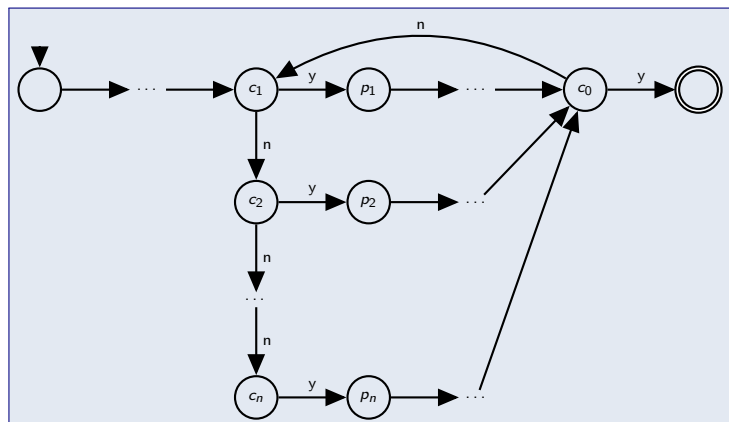


Turing Machines and While Programs



Proof \Leftarrow : construct M to simulate P (given in normal form).

- Each program fragment P_i is translated into a corresponding fragment of the transition function of M with cycle of states $c_i, p_i, \dots, c_0, c_i$.



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Primitive Recursive Functions

The following functions over the natural numbers are **primitive recursive**:

- The **constant null** function $0 \in \mathbb{N}$.
- The **successor** function $s : \mathbb{N} \rightarrow \mathbb{N}, s(x) := x + 1$.
- The **projection** functions $p_i^n : \mathbb{N}^n \rightarrow \mathbb{N}, p_i^n(x_1, \dots, x_n) := x_i$.
- Every function $h : \mathbb{N}^n \rightarrow \mathbb{N}$ defined by **composition**

$$h(x_1, \dots, x_n) := f(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n))$$

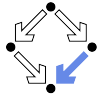
from primitive recursive $f : \mathbb{N}^k \rightarrow \mathbb{N}$ and $g_1, \dots, g_k : \mathbb{N}^n \rightarrow \mathbb{N}$.

- Every function $h : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ defined by **primitive recursion**

$$h(y, x_1 \dots x_n) := \begin{cases} f(x_1, \dots, x_n) & \text{if } y = 0 \\ g(y-1, h(y-1, x_1, \dots, x_n), x_1, \dots, x_n) & \text{else} \end{cases}$$

from primitive recursive $f : \mathbb{N}^n \rightarrow \mathbb{N}$ and $g : \mathbb{N}^{n+2} \rightarrow \mathbb{N}$.

Starting with the base functions, by composition and primitive recursion new primitive recursive functions can be defined.



Understanding Primitive Recursion

- Primitive recursion can be defined by a **pattern matching equation**:

$$\begin{aligned} h(0, x_1, \dots, x_n) &:= f(x_1, \dots, x_n) \\ h(y+1, x_1, \dots, x_n) &:= g(y, h(y, x_1, \dots, x_n), x_1, \dots, x_n) \end{aligned}$$

- Primitive recursion can be defined by a **pattern matching construct**:

$$\begin{aligned} h(y, x_1 \dots x_n) &:= \\ \text{case } y \text{ of} & \\ 0 : & f(x_1, \dots, x_n) \\ z + 1 : & g(z, h(z, x_1, \dots, x_n), x_1, \dots, x_n) \end{aligned}$$

- $h(y, x)$ denotes the $(y-1)$ -times application of g starting with $f(x)$:

$$\begin{aligned} h(0, x) &= f(x) \\ h(1, x) &= g(0, h(0, x), x) = g(0, f(x), x) \\ h(2, x) &= g(1, h(1, x), x) = g(1, g(0, f(x), x), x) \\ h(3, x) &= g(2, h(2, x), x) = g(2, g(1, g(0, f(x), x), x), x) \\ &\dots \\ h(y, x) &= g(y-1, h(y-1, x), x) = g(y-1, g(y-2, \dots, g(0, f(x), x), \dots), x) \end{aligned}$$



Example

We consider arithmetic on natural numbers.

- **Addition** $y + x$ is primitive recursive:

$$\begin{aligned} 0 + x &:= x \\ (y + 1) + x &:= (y + x) + 1 \end{aligned}$$

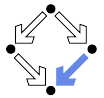
- **Multiplication** $y \cdot x$ is primitive recursive:

$$\begin{aligned} 0 \cdot x &:= 0 \\ (y + 1) \cdot x &:= y \cdot x + x \end{aligned}$$

- **Exponentiation** x^y is primitive recursive:

$$\begin{aligned} x^0 &:= 1 \\ x^{y+1} &:= x^y \cdot x \end{aligned}$$

Natural number arithmetic is primitive recursive.



Primitive Recursion and Loop Computability

Theorem: every prim. recursive function is loop computable and vice versa.

Proof \Rightarrow : we show that primitive recursive function h is loop computable.

- If h is one of the basic functions, it is clearly loop computable.
- Case $h(x_1, \dots, x_n) := f(g_1(x_1, \dots, x_n), \dots, g_k(x_1, \dots, x_n))$

```

y1 := g1(x1, ..., xn);
y2 := g2(x1, ..., xn);
...
yk := gk(x1, ..., xn);
x0 := f(y1, ..., yk)

```

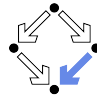
- Case $h(y, x_1 \dots x_n) := \begin{cases} f(x_1, \dots, x_n) & \text{if } y = 0 \\ g(y-1, h(y, x_1, \dots, x_n), x_1, \dots, x_n) & \text{else} \end{cases}$

```

x0 := f(x1, ..., xn); xy := 0;
loop y do
  x0 := g(xy, x0, x1, ..., xn);
  xy := xy + 1
end

```


Primitive Recursion and Loop Computability



Proof \Leftarrow : let h be computable by loop program P . Let $f_P : \mathbb{N}^{n+1} \rightarrow \mathbb{N}^{n+1}$ be the function that maps the initial values of the variables used by P to their final values. We show by induction on P that f_P is primitive recursive.

- Case $x_j := k$:

$$f_P(x_0, \dots, x_n) := (x_0, \dots, x_{j-1}, k, x_{j+1}, \dots, x_n)$$

- Case $x_j := x_j \pm 1$:

$$f_P(x_0, \dots, x_n) := (x_0, \dots, x_{j-1}, x_j \pm 1, x_{j+1}, \dots, x_n)$$

- Case $P_1; P_2$:

$$f_P(x_0, \dots, x_n) := f_{P_2}(f_{P_1}(x_0, \dots, x_n))$$

- Case **loop** x_i **do** P' **end:**

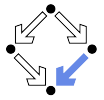
$$f_P(x_0, \dots, x_n) := g(x_i, x_0, \dots, x_n)$$

$$g(0, x_0, \dots, x_n) := (x_0, \dots, x_n)$$

$$g(m+1, x_0, \dots, x_n) := f_{P'}(g(m, x_0, \dots, x_n))$$

Thus the Ackermann function is also not primitive recursive.

μ -Recursive Functions



A partial function over the natural numbers is μ -recursive, if it

- is the constant null, successor, or a projection function,
- can be constructed from other μ -recursive functions by composition or primitive recursion, or
- is a function $h : \mathbb{N}^n \rightarrow_{\text{p}} \mathbb{N}$ defined as

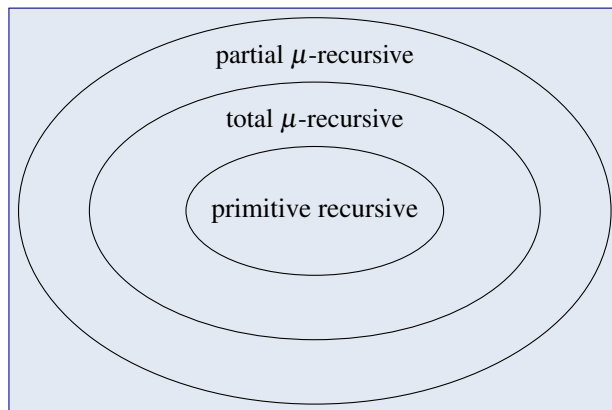
$$h(x_1, \dots, x_n) := (\mu f)(x_1, \dots, x_n)$$

with μ -recursive $f : \mathbb{N}^{n+1} \rightarrow_{\text{p}} \mathbb{N}$ and $(\mu f) : \mathbb{N}^n \rightarrow_{\text{p}} \mathbb{N}$ defined as

$$(\mu f)(x_1, \dots, x_n) := \min \left\{ y \in \mathbb{N} \mid \begin{array}{l} f(y, x_1, \dots, x_n) = 0 \wedge \\ \forall z \leq y : (z, x_1, \dots, x_n) \in \text{domain}(f) \end{array} \right\}$$

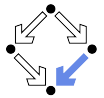
$(\mu f)(x_1, \dots, x_n)$ is the smallest y such that $f(y, x_1, \dots, x_n) = 0$ (and f is defined for all $z \leq y$); the result of h is undefined, if no such y exists.

μ -Recursive Functions



Every primitive recursive function is a total μ -recursive function; a μ -recursive function may or may not be total.

μ -Recursion and While Computability



Theorem: every μ -recursive function is while computable and vice versa.

Proof \Rightarrow : we show that μ -recursive h is while computable.

- If h is one of the basic functions or defined by composition or primitive recursion, it is clearly while computable.
- Case $h(x_1, \dots, x_n) := (\mu f)(x_1, \dots, x_n)$

```

x0 := 0;
y := f(x0, x1, ..., xn);
while y ≠ 0 do
  x0 := x0 + 1;
  y := f(x0, x1, ..., xn)
end
    
```

μ -recursion denotes unbounded iterative search.

μ -Recursion and While Computability



Proof \Leftarrow : let $h: \mathbb{N}^k \rightarrow_{\mathcal{P}} \mathbb{N}$ be computable by while program P that uses variables x_0, \dots, x_n . Then $h(x_1, \dots, x_k) := \text{var}_0(f_P(0, x_1, \dots, x_k, 0, \dots, 0))$ where $\text{var}_i(x_0, \dots, x_n) := x_i$. We show that $f_P: \mathbb{N}^n \rightarrow_{\mathcal{P}} \mathbb{N}^n$ is μ -recursive by induction on P .

- If P is an assignment, a sequence, of a bounded loop, then f_P is clearly μ -recursive.
- Case **while x_i do P' end**:

$$f_P(x_0, \dots, x_n) := g((\mu g_i)(x_0, \dots, x_n), x_0, \dots, x_n)$$

$$g_i: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$$

$$g_i(m, x_0, \dots, x_n) := \text{var}_i(g(m, x_0, \dots, x_n))$$

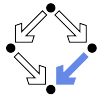
$$g(0, x_0, \dots, x_n) := (x_0, \dots, x_n)$$

$$g(m+1, x_0, \dots, x_n) := f_{P'}(g(m, x_0, \dots, x_n))$$

- $g_i(m, x_0, \dots, x_n)$: the value of program variable i after m iterations
- $g(m, x_0, \dots, x_n)$: the values of all variables after m iterations.

Thus the Ackermann function is also μ -recursive.

Normal Form of a μ -Recursive Function



Kleene's Normal Form Theorem: every μ -recursive function h can be defined in Kleene's normal form:

$$h(x_1, \dots, x_k) := f_2(x_1, \dots, x_k, (\mu g)(f_1(x_1, \dots, x_k)))$$

- f_1, f_2, g are **primitive** recursive functions.

A single application of μ is all that is needed.

Normal Form of a μ -Recursive Function



We sketch the proof of Kleene's Normal Form Theorem.

Since h is μ -recursive, it is computable by a while program in normal form

$x_c := 1$; while x_c do ... end

with memory function

$$f_P(x_0, \dots, x_n) := g((\mu g_c)(\text{init}(x_0, \dots, x_n)), \text{init}(x_0, \dots, x_n))$$

with primitive recursive g and g_c and $\text{init}(x_0, \dots, x_c, \dots, x_n) := (x_0, \dots, 1, \dots, x_n)$.

Thus we can define

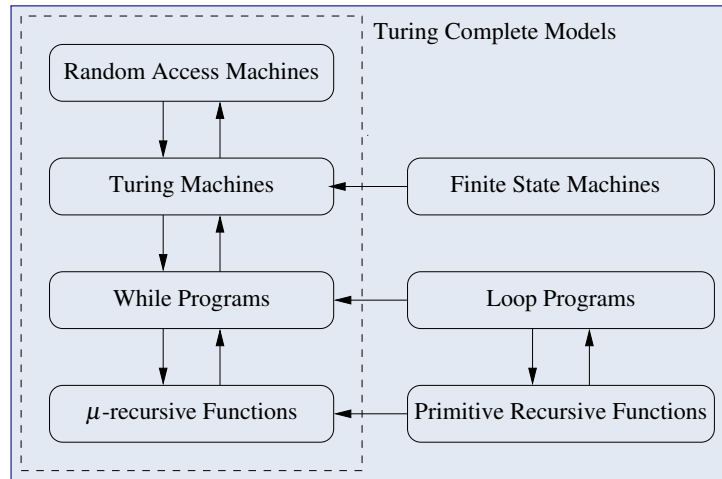
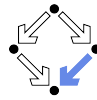
$$\begin{aligned} h(x_1, \dots, x_k) &:= \text{var}_0(f_P(0, x_1, \dots, x_k, 0, \dots, 0)) \\ &= \text{var}_0(g((\mu g_c)(\text{init}(0, x_1, \dots, x_k, 0, \dots, 0)), \text{init}(0, x_1, \dots, x_k, 0, \dots, 0))) \\ &= f_2(x_1, \dots, x_k, (\mu g_c)(f_1(x_1, \dots, x_k))) \end{aligned}$$

with primitive recursive

$$\begin{aligned} f_1(x_1, \dots, x_k) &:= \text{init}(0, x_1, \dots, x_k, 0, \dots, 0) \\ f_2(x_1, \dots, x_k, r) &:= \text{var}_0(g(r, \text{init}(0, x_1, \dots, x_k, 0, \dots, 0))) \end{aligned}$$

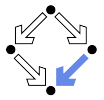
1. Random Access Machines
2. Loop and While Programs
3. Primitive Recursive and μ -recursive Functions
4. Further Turing Complete Models
5. The Chomsky Hierarchy
6. Real Computers

The Big Picture So Far



We are going to sketch some more Turing complete models.

Goto Programs



- A **goto program** has form

$$L_1 : P_1; L_2 : P_2; \dots; P_n : A_n$$

where L_k denotes a label and P_k an action:

$$P ::= x_i := 0 \mid x_i := x_j + 1 \mid x_i := x_j - 1 \mid \text{if } x_i \text{ goto } L_j$$

- Semantics** $\llbracket P \rrbracket(k, m)$:

- A partial function which maps the initial state (k, m) of P , consisting of program counter $k \in \mathbb{N}$ and memory $m : \mathbb{N} \rightarrow \mathbb{N}$, to its final state (unless the program does not terminate).

$$\begin{aligned} \llbracket P \rrbracket(0, m) &:= m \\ \llbracket P = \dots; P_k : x_i := 0; \dots \rrbracket(k, m) &:= \llbracket P \rrbracket(k+1, m[i \leftarrow 0]) \\ \llbracket P = \dots; P_k : x_i := x_j + 1; \dots \rrbracket(k, m) &:= \llbracket P \rrbracket(k+1, m[i \leftarrow m[j] + 1]) \\ \llbracket P = \dots; P_k : x_i := x_j - 1; \dots \rrbracket(k, m) &:= \llbracket P \rrbracket(k+1, m[i \leftarrow \max\{0, m[j] - 1\}]) \\ \llbracket P = \dots; P_k : \text{if } x_i \text{ goto } L_j; \dots \rrbracket(k, m) &:= \begin{cases} \llbracket P \rrbracket(k+1, m), & \text{if } m(i) = 0 \\ \llbracket P \rrbracket(j, m), & \text{if } m(i) \neq 0 \end{cases} \end{aligned}$$

We have already seen how goto programs can be translated to while programs and vice versa; goto programs are therefore Turing complete.

λ-Calculus



- A **λ-term** T :

$$T ::= x_i \mid (T T) \mid (\lambda x_i. T)$$

- x_i : a variable.
- $(T T)$: an **application**.
- $(\lambda x_i. T)$: an **abstraction**.

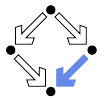
- Reduction relation** \rightarrow :

$$((\lambda x_i. T_1) T_2) \rightarrow (T_1[x_i \leftarrow T_2])$$

- The result of the application of a function to an argument.
- Reduction sequence** $T_1 \rightarrow^* T_2$
 $T_1 \rightarrow \dots \rightarrow T_2$
 - T_2 is in **normal form**, if no further reduction is possible.
- Church-Rosser Theorem**: If $T_1 \rightarrow^* T_2$ and $T_1 \rightarrow^* T'_2$ such that both T_2 and T'_2 are in normal form, then $T_2 = T'_2$.

Every computable function can be represented by a λ-term.

λ-Calculus



How can we represent unbounded iteration (recursion)?

- Can define **fixpoint operator** Y :

$$(YF) \rightarrow^* (F(YF))$$

- $Y := (\lambda f. (\lambda x. (f(xx)))) (\lambda x. (f(xx)))$
- Can translate **recursive function definition to λ-term**:

$$f(x) := \dots f(g(x)) \dots \rightsquigarrow f := YF$$

$$F := \lambda h. \lambda x. \dots h(g(x)) \dots$$

- λ-term **behaves like recursive function**.

$$fa = (YF)a \rightarrow^* F(YF)a \rightarrow^* \dots (YF)(g(a)) \dots = \dots f(g(a)) \dots$$

Formal basis of functional programming languages.

Rewriting Systems



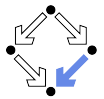
- A **term rewriting system** is a set of rules of form

$$L \rightarrow R$$

- L, R : terms such that L is not a variable and every variable that appears in R must also appear in L .
- **Rewriting Step** $T \rightarrow T'$:
 - There is some rule $L \rightarrow R$ and a substitution σ (a mapping of variables to terms) such that
 - some subterm U of T matches the left hand side L of the rule under the substitution σ , i.e., $U = L\sigma$,
 - T' is derived from T by replacing U with $R\sigma$, i.e with the right hand side of the rule after applying the variable replacement.
- **Rewriting Sequence** $T_1 \rightarrow^* T_2$
 $T_1 \rightarrow \dots \rightarrow T_2$
 - T_2 is in **normal form**, if no further reduction is possible.

Every computable function can be represented by a term rewriting system.

Rewriting Systems



- Term rewriting system:

$$f(x, f(y, z)) \rightarrow f(f(x, y), z)$$

$$f(x, e) \rightarrow x$$

$$f(x, i(x)) \rightarrow e$$

- Rewriting sequence:

$$f(a, f(i(a), e)) \rightarrow f(f(a, i(a)), e) \rightarrow f(e, e) = e$$

$$f(a, f(i(a), e)) \rightarrow f(a, i(a)) \rightarrow e$$

Rewriting systems can be also defined over strings and graphs; the later form the basis of tools for model driven architectures.

1. Random Access Machines

2. Loop and While Programs

3. Primitive Recursive and μ -recursive Functions

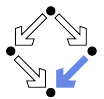
4. Further Turing Complete Models

5. The Chomsky Hierarchy

6. Real Computers



Languages and Machines



- **Regular languages:**

- Representable by regular expressions.
- Recognizable by finite state machines.

- **Recursively enumerable languages:**

- Representable by ... ?
- Recognizable by Turing machines.

- **Relationship:**

- Every regular language is recursively enumerable.
- Every finite state machine can be simulated by a Turing machine.
But not vice versa.

Are there any other interesting classes of languages and associated machine models and how do they relate to those above?

Grammars



Grammar $G = (N, \Sigma, P, S)$:

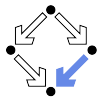
- N : a finite set of **nonterminal symbols**,
- Σ : a finite set of **terminal symbols** disjoint from N .
 $N \cap \Sigma = \emptyset$
- P : a finite set of **production rules** of form $l \rightarrow r$ such that
 $l \in (N \cup \Sigma)^* \circ N \circ (N \cup \Sigma)^*$
 $r \in (N \cup \Sigma)^*$
 - l and r consist of nonterminal and/or terminal symbols.
 - l must contain at least one nonterminal symbol.
 - Multiple rules $l \rightarrow r_1, l \rightarrow r_2, \dots, l \rightarrow r_n$ can be abbreviated:

$$l \rightarrow r_1 \mid r_2 \mid \dots \mid r_n$$

- S : the **start symbol**.
 $S \in N$

Grammar G describes a language over alphabet Σ .

The Language of a Grammar



Grammar $G = (N, \Sigma, P, S)$, words $w, w_1, w_2 \in (N \cup \Sigma)^*$.

- **Direct derivation** $w_1 \Rightarrow w_2$ in G :
 $w_1 = ulv$ and $w_2 = urv$
for $u, v \in (N \cup \Sigma)^*$ and $(l \rightarrow r) \in P$
- **Derivation** $w_1 \Rightarrow^* w_2$ in G :
 $w_1 \Rightarrow \dots \Rightarrow w_2$ in G .
- w is a **sentential form** in G :
 $S \Rightarrow^* w$
- w is a **sentence** in G :
 - w is a sentential form in G and $w \in \Sigma^*$.
- **Language** $L(G)$ of G :
 $L(G) := \{w \text{ is a sentence in } G\}$

The language of a grammar is the set of all words that consist only of terminal symbols and that are derivable from the start symbol.

Example



- Grammar $G = (N, \Sigma, P, S)$:

$$N = \{S, A, B\}$$

$$\Sigma = \{a, b, c\}$$

$$P = \{S \rightarrow Ac, A \rightarrow aB, A \rightarrow BBb, B \rightarrow b, B \rightarrow ab\}$$

- Derivations:

$$S \Rightarrow Ac \Rightarrow aBc \Rightarrow abc$$

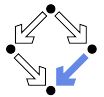
$$S \Rightarrow Ac \Rightarrow BBbc \Rightarrow abBbc \Rightarrow ababbc$$

- Language:

$$L(G) = \{abc, aabc, bbbc, babbc, abbbc, ababbc\}$$

This grammar defines a finite language.

Example



- Grammar $G = (N, \Sigma, P, S)$:

$$N = \{S\}$$

$$\Sigma = \{(' , ') , '[,]'\}$$

$$P = \{S \rightarrow \varepsilon \mid SS \mid [S] \mid (S)\}$$

- Derivations:

$$S \Rightarrow [S] \Rightarrow [SS] \Rightarrow [(S)S] \Rightarrow [(S)] \Rightarrow [(S)] \Rightarrow [(S)] \Rightarrow [(S)] \Rightarrow [(S)]$$

- Language: the "Dyck-Language"

$L(G)$ is the language of all expressions with matching pairs of parentheses "(" and brackets "["

This grammar defines an infinite language.

Right-Linear Grammars and Regular Languages



- Grammar $G = (N, \Sigma, P, S)$ is **right linear** if each rule in P has form
 - $A \rightarrow \varepsilon$, $A \rightarrow a$, $A \rightarrow aB$
 with nonterminal symbols $A, B \in N$ and terminal symbol $a \in \Sigma$.
- Theorem:** The languages of right linear grammars are exactly the regular languages.
 - For every right linear grammar G , there exists a FSM M with $L(M) = L(G)$ and vice versa.
 - Proof \Rightarrow : we construct from right linear grammar G a NFSM M . The states are the nonterminal symbols extended by a final state q_f ; the start state is the start symbol.
 - For every rule $A \rightarrow \varepsilon$, the state A becomes final.
 - For every rule $A \rightarrow a$, we add a transition $\delta(A, a) = q_f$.
 - For every rule $A \rightarrow aB$, we add a transition $\delta(A, a) = B$.
 - Proof \Leftarrow : we construct from DFSM M right linear grammar G . The nonterminal symbols are the states; the start symbol is the start state.
 - For every transition $\delta(q, a) = q'$ we add a production rule $q \rightarrow aq'$.
 - For every final state q , we add a production rule $q \rightarrow \varepsilon$.

The Chomsky Hierarchy



Noam Chomsky, 1959.

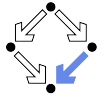
Type i	Grammar $G(i)$	Language $L(i)$	Machine $M(i)$
0	unrestricted	recursively enumerable	Turing machine
1	context sensitive	context sensitive	linear bounded automaton
2	context free	context free	push down automaton
3	right linear	regular	finite state machine

$L(i)$ is the set of languages of grammars $G(i)$ and machines $M(i)$.

- For $i > 0$, the set of languages of type $L(i)$ is a proper subset of the set of languages $L(i-1)$, i.e. $L(i) \subset L(i-1)$.
- For $i > 0$, every machine in $M(i)$ can be simulated by a machine in $M(i-1)$ (but not vice versa).

Grammars correspond to machine models.

Grammars and Recursively Enumerable Languages

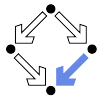


Theorem: The languages of (unrestricted) grammars are exactly the recursively enumerable languages.

- Proof \Rightarrow : construct 2-tape nondeterministic M with $L(M) = L(G)$.
 M uses the second tape to construct some sentence of $L(G)$: it starts by writing S on the tape and then nondeterministically chooses some rule $l \rightarrow r$ and applies it to some occurrence of l on the tape, replacing it by r . Then M checks whether the result equals the word on the first tape. If yes, M accepts the word, otherwise, it continues with another production rule.
- Proof \Leftarrow : construct grammar G with $L(G) = L(M)$.
 Sentential forms encode pairs (w, c) of input w and configuration c of M ; every form contains a non-terminal symbol such that by a rule application the current configuration is replaced by the successor configuration. The rules ensure that
 - from the start symbol, every pair matching (w, c) of M can be derived;
 - for every transition that moves c to c' , a rule is constructed that allows a derivation $(w, c) \Rightarrow (w, c')$;
 - if configuration c describes a final state from which no further transition is possible, the derivation $(w, c) \Rightarrow w$ is possible.

Unrestricted grammars represent another Turing complete model.

Context Free Languages (Type 2)



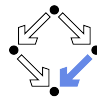
- Context free grammar G :** every rule has form $A \rightarrow r$ with $A \in N$.
 - Independent of the context, any occurrence of A can be replaced.
- Example:** $L := \{a^i b^i \mid i \in \mathbb{N}\}$

$$S \rightarrow \varepsilon \mid aSb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$
- Pushdown automaton M :** nondeterministic FSM with unbounded stack of symbols as “working memory”:
 - in every transition $\delta(q, a, b) = (q', w)$,
 - M reads the next input symbol a (a may be ε , i.e., M may not read a symbol) and the symbol b on the top of the stack, and
 - replaces b by a (possibly empty) sequence w of symbols.

Most languages in computer science are context free.

Generation of Syntax Analyzers



“Compiler generators” for the generation of syntax analyzers (parsers).

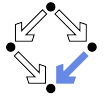
- Input: a (deterministic) context free grammar.

```
statement: assignment | conditional | whileloop | ... ;
whileloop: 'while' '(' valexp ')' statement ;
```

- Output: a (deterministic) push down automaton (as a program)

```
public final LoopStatement whileloop() throws ... {
    ...
    pushFollow(FOLLOW_valexp_in_whileloop1457);
    valexp();
    state._fsp--;
    if (state.failed) return value;
    ...
    pushFollow(FOLLOW_statement_in_whileloop1484);
    statement();
    state._fsp--;
    if (state.failed) return value;
    ...
}
```

Context Sensitive Languages (Type 1)



- **Context sensitive grammar G :**

- in every rule $l \rightarrow r$, we have $|l| \leq |r|$, i.e., the length of left side l is less than or equal the length of right side r ,
- the rule $S \rightarrow \varepsilon$ is only allowed, if the start symbol S does not appear on the right hand side of any rule.

- **Example:** $L := \{a^i b^i c^i \mid i \in \mathbb{N}\}$

$S \rightarrow ABC \mid SABC$

$BA \rightarrow AB, CB \rightarrow BC, CA \rightarrow AC$

$AB \rightarrow ab, BC \rightarrow bc, Aa \rightarrow aa, bB \rightarrow bb, cC \rightarrow cc$

$S \Rightarrow \underline{S}ABC \Rightarrow \underline{A}BC\underline{A}BC \Rightarrow \underline{A}B\underline{A}C\underline{B}C \Rightarrow \underline{A}A\underline{B}C\underline{B}C \Rightarrow \underline{A}A\underline{B}B\underline{C}C$

$\Rightarrow \underline{A}a\underline{b}B\underline{C}C \Rightarrow a\underline{a}b\underline{B}C\underline{C} \Rightarrow a\underline{a}b\underline{b}c\underline{C} \Rightarrow a\underline{a}b\underline{b}c\underline{c}$

- **Linear bounded automaton M :** nondeterministic Turing machine with k tapes (for some k).

- For input of length n , only the first n cells of each tape are used.
- The “space” used is a fixed multiple of the length of the input word.

Less practical importance.

Summary



We have seen examples of each type of language.

- **Type 3:** $\{(ab)^n \mid n \in \mathbb{N}\}$
 - Language is regular.
- **Type 2:** $\{a^n b^n \mid n \in \mathbb{N}\}$
 - Language is context free.
- **Type 1:** $\{a^n b^n c^n \mid n \in \mathbb{N}\}$
 - Language is context sensitive.
- **Type 0:** $\{a^{2^n} \mid n \in \mathbb{N}\}$
 - Language is recursively enumerable.

None of these languages of type i is also of type $i + 1$.

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Are real computers Turing complete?

■ **Hardware view:**

- Finite number of digital elements and thus a finite number of states.
- Cannot simulate the infinite Turing machine tape.
- Cannot perform unbounded arithmetic.
- A computer is thus a **finite state machine** (i.e., not Turing complete).

View taken by model checkers.

■ **Algorithm theory view:**

- On demand, arbitrary much (e.g., virtual) memory may be added.
- Can thus simulate arbitrary large portion of the Turing machine tape.
- Can thus perform unbounded arithmetic.
- A computer is **Turing complete**.

View taken by algorithm design.

A matter of the point of view respectively the goal of the modeling.