

# Turing Machines

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# 1. Turing Machines

## 2. Recognizing Languages

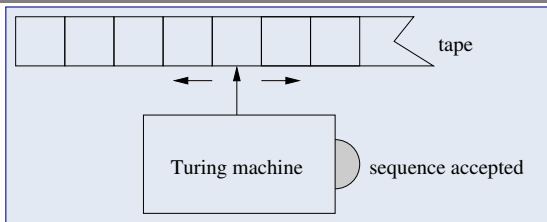
## 3. Generating Languages

## 4. Computing Functions

## 5. The Church-Turing Thesis



# Turing Machine Model



- The machine is always in one of a **finite set of states**.
  - The machine starts its execution in a fixed start state.
- An **infinite tape** holds at its beginning the input word.
  - Tape is read and written and arbitrarily moved by the machine.
- The machine proceeds in a sequence of state **transitions**.
  - Machine reads symbol, overwrites it, and moves tape head left or right.
  - The symbol read and the current state determine the symbol written, the move direction, and the next state.
- If the machine cannot make another transition, it **terminates**.
  - The machine signals whether it is in a final state.

If the machine terminates in a final state, the input word is *accepted*.



# Turing Machines

**Turing Machine**  $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$ :

- The **state set**  $Q$ , a finite set of **states**.
- A **tape alphabet**  $\Gamma$ , a finite set of **tape symbols**.
- The **blank symbol**  $\sqcup \in \Gamma$ .
- An **input alphabet**  $\Sigma \subseteq \Gamma \setminus \{\sqcup\}$ .
- The **transition function**  $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\text{'L'}, \text{'R'}\}$ ,
  - $\delta(q, x) = (q', x', \text{'L'/'R'})$  ...  $M$  reads in state  $q$  symbol  $x$ , goes to state  $q'$ , writes symbol  $x'$ , and moves the tape head left/right.
- The **start state**  $q_0 \in Q$
- A set of **final states (accepting states)**  $F \subseteq Q$ .

The crucial difference to an automaton is the infinite tape that can be arbitrarily moved and written.



# Example

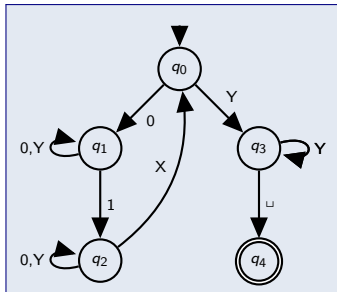
$$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$$

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

$$\Gamma = \{\sqcup, 0, 1, X, Y\}$$

$$\Sigma = \{0, 1\}$$

$$F = \{q_4\}$$



$\delta$	$\sqcup$	0	1	X	Y
$q_0$	—	$(q_1, X, R)$	—	—	$(q_3, Y, R)$
$q_1$	—	$(q_1, 0, R)$	$(q_2, Y, L)$	—	$(q_1, Y, R)$
$q_2$	—	$(q_2, 0, L)$	—	$(q_0, X, R)$	$(q_2, Y, L)$
$q_3$	$(q_4, \sqcup, R)$	—	—	—	$(q_3, Y, R)$
$q_4$	—	—	—	—	—

Machine accepts every word of form  $0^n 1^n$  (replacing it by  $X^n Y^n$ ).

# Turing Machine Simulators



For instance, <http://math.hws.edu/TCMC/java/xTuringMachine>.

CountInBinary.bt

0

1 0 0 0

Moderate

0 # # L 5 Replace

	In State	Reading	Write	Move	New State
Run	0	#	1	R	1
Step	0	0	1	R	1
Clear Tape	0	1	0	L	0
Delete Rule	1	#	#	L	0
	1	0	0	R	1
	1	1	1	R	1
Load File					
Save					

# Generalized Turing Machines



- Infinite tape in **both directions**.
  - Can be simulated by a machine whose tape is infinite in one direction.
- **Multiple** tapes.
  - Can be simulated by a machine with a single tape.
- **Nondeterministic** transitions.
  - We can simulate a nondeterministic  $M$  by a deterministic  $M'$ .
    - Let  $r$  be the maximum number of “choices” that  $M$  can make.
  - $M'$  operates with 3 tapes.
    - Tape 1 holds the input (tape is only read).
  - $M'$  writes to tape 2 all finite sequences of numbers  $1, \dots, r$ .
    - First all sequences of length 1, then all of length 2, etc.
  - After writing sequence  $s_1 s_2 \dots s_n$  to tape 2,  $M'$  simulates  $M$  on tape 3.
    - $M'$  copies the input to tape 3 and performs at most  $n$  transitions.
    - In transition  $i$ ,  $M$  attempts to perform choice  $s_i$ .
    - If choice  $i$  is not possible or  $M$  terminates after  $n$  transitions in a non-accepting state,  $M'$  continues with next sequence.
    - If  $M$  terminates in accepting state,  $M'$  accepts the input.

**Every generalized Turing machine can be simulated by the core form.**



- 
1. Turing Machines
  - 2. Recognizing Languages**
  3. Generating Languages
  4. Computing Functions
  5. The Church-Turing Thesis





# Turing Machine Configurations

- **Configuration**  $a_1 \dots a_k q a_{k+1} \dots a_m$ :
  - $q$ : the current state of  $M$ .
  - $a_{k+1}$ : the symbol currently under the tape head.
  - $a_1 \dots a_k$ : the portion of the tape left to the tape head.
  - $a_{k+2} \dots a_m$ : the portion right to the head (followed by  $\sqcup \dots$ ).
- **Move relation:**  $a_1 \dots a_k q a_{k+1} \dots a_m \vdash b_1 \dots b_l p b_{l+1} \dots b_m$ 

If  $M$  is a situation described by the left configuration, it can make a transition to the situation described by the right configuration.

  - $a_i = b_i$  for all  $i \neq k+1$  and one of the following:
    - $l = k+1$  and  $\delta(q, a_{k+1}) = (p, b_l, R)$ ,
    - $l = k-1$  and  $\delta(q, a_{k+1}) = (p, b_{l+2}, L)$ .
- **Extended move relation:**  $c_1 \vdash^* c_2$ 

$M$  can make in an arbitrary number of moves a transition from the situation described by configuration  $c_1$  to the one described by  $c_2$ .

$$c_1 \vdash^0 c_2 : \Leftrightarrow c_1 = c_2$$

$$c_1 \vdash^{i+1} c_2 : \Leftrightarrow \exists c : c_1 \vdash^i c \wedge c \vdash c_2$$

$$c_1 \vdash^* c_2 : \Leftrightarrow \exists i \in \mathbb{N} : c_1 \vdash^i c_2$$



# The Language of a Turing Machine

- The **language**  $L(M)$  of Turing machine  $M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$ :  
The set of all inputs that drive  $M$  from its initial configuration to a configuration with a final state such that from this configuration no further move is possible:

$$L(M) := \left\{ w \in \Sigma^* \mid \begin{array}{l} \exists a, b \in \Gamma^*, q \in Q : q_0 w \vdash^* a q b \wedge q \in F \\ \wedge \neg \exists a', b' \in \Gamma^*, q' \in Q : a q b \vdash a' q' b' \end{array} \right\}$$

- $L$  is a **recursively enumerable language**:
  - There exists a Turing machine  $M$  such that  $L = L(M)$ .
- $L$  is a **recursive language**:
  - There exists a Turing machine  $M$  such that  $L = L(M)$  and  $M$  terminates for every possible input.

Every recursive language is recursively enumerable; as we will see, the converse does not hold.

# Recursively Enumerable and Recursive Languages

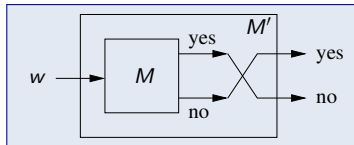


**Theorem:**  $L$  is recursive, if and only if both  $L$  and its complement  $\bar{L}$  are recursively enumerable.

Proof  $\Rightarrow$ : Let  $L$  be a recursive. Since by definition  $L$  is recursively enumerable, it remains to be shown that also  $\bar{L}$  is recursively enumerable.

Since  $L$  is recursive, there exists a Turing machine  $M$  such that  $M$  halts for every input  $w$ : if  $w \in L$ , then  $M$  accepts  $w$ ; if  $w \notin L$ , then  $M$  does not accept  $w$ . With the help of  $M$ , we can construct the following  $M'$  with  $L(M') = \bar{L}$ :

```
function  $M'(w)$ :  
  case  $M(w)$  of  
    yes: return no  
    no:  return yes  
  end case  
end function
```

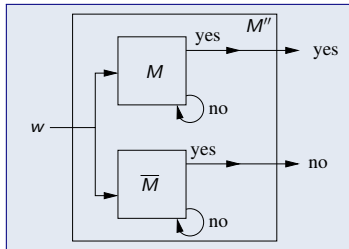


# Recursively Enumerable and Recursive Languages



Proof  $\Leftarrow$ : Let  $L$  be such that both  $L$  and  $\bar{L}$  are recursively enumerable. We show that  $L$  is recursive. Since  $L$  is r.e., there exists  $M$  such that  $L = L(M)$  and  $M$  halts for  $w \in L$  with  $M(w) = \text{yes}$ . Since  $\bar{L}$  is r.e., there exists  $\bar{M}$  with  $\bar{L} = L(\bar{M})$  and  $\bar{M}$  halts for  $w \in \bar{L}$  with  $\bar{M}(w) = \text{yes}$ . We can thus construct  $M''$  with  $L(M'') = L$  that always halts:

```
function  $M''(w)$ :  
  parallel  
    begin  
      if  $M(w) = \text{yes}$  then  
        return yes  
      end if  
      loop forever  
    end  
    begin  
      if  $\bar{M}(w) = \text{yes}$  then  
        return no  
      end if  
      loop forever  
    end  
  end parallel  
end function
```





# Closure of Recursive Languages

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Let  $L, L_1, L_2$  be recursive languages. Then also

- the complement  $\bar{L}$ ,
- the union  $L_1 \cup L_2$ ,
- the intersection  $L_1 \cap L_2$

are recursive languages.

Proof by construction of the corresponding Turing machines.



- 
1. Turing Machines
  2. Recognizing Languages
  - 3. Generating Languages**
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  5. The Church-Turing Thesis

# Enumerators



Turing machine  $M = (Q, \Gamma, \sqcup, \emptyset, \delta, q_0, F)$  with special symbol  $\# \in \Gamma$ .

- $M$  is an **enumerator**, if  $M$  has an additional **output tape** on which
  - $M$  moves its tape head only to the right, and
  - $M$  writes only symbols different from  $\sqcup$ .
- The **generated language**  $Gen(M)$  of enumerator  $M$  is the set of all words that  $M$  eventually writes on its output tape.
  - The end of each word is marked by a trailing  $\#$ .

$M$  may run forever and thus  $Gen(M)$  may be infinite.

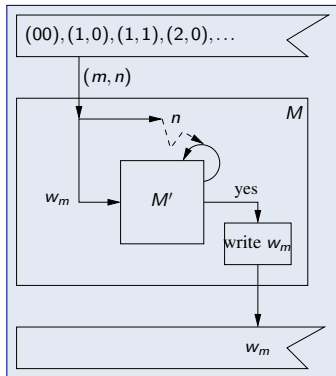
# Recognizing versus Generating Languages



**Theorem:**  $L$  is recursively enumerable, if and only if there exists some enumerator  $M$  such that  $L = \text{Gen}(M)$ .

Proof  $\Rightarrow$ : Let  $L$  be recursively enumerable, i.e.,  $L = L(M')$  for some  $M'$ . We construct enumerator  $M$  such that  $L = \text{Gen}(M)$ .

```
procedure  $M'$ :  
  loop  
    produce next  $(m, n)$  on working tape  
    if  $M(w_m) = \text{yes}$  in at most  $n$  steps then  
      write  $w_m$  to output tape  
    end if  
  end loop  
end procedure
```



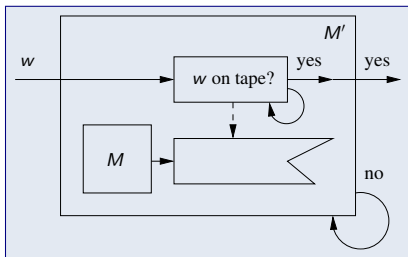


# Recognizing versus Generating Languages



Proof  $\Leftarrow$ : Let  $L$  be such that  $L = \text{Gen}(M)$  for some enumerator  $M$ . We show that there exists some Turing machine  $M'$  such that  $L = L(M')$ .

```
function  $M'(w)$ :  
  while  $M$  is not terminated do  
     $M$  writes next word  $w'$   
    if  $w = w'$  then  
      return yes  
    end if  
  end while  
  return no  
end function
```



Recognizing is possible, if and only if generating is possible.



- 
1. Turing Machines
  2. Recognizing Languages
  3. Generating Languages
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Take binary relation  $f \subseteq A \times B$ .

- $f : A \rightarrow B$ :  $f$  is a **total function** from  $A$  to  $B$ .
  - For every  $a \in A$ , there is **exactly one**  $b \in B$  such that  $(a, b) \in f$ .
- $f : A \rightarrow_p B$ :  $f$  is a **partial function** from  $A$  to  $B$ .
  - For every  $a \in A$ , there is **at most one**  $b \in B$  such that  $(a, b) \in f$ .
- Auxiliary notions:

$$\text{domain}(f) := \{a \mid \exists b : (a, b) \in f\}$$

$$\text{range}(f) := \{b \mid \exists a : (a, b) \in f\}$$

$$f(a) := \text{such } b : (a, b) \in f$$

Every total function  $f : A \rightarrow B$  is a partial function  $f : A \rightarrow_p B$ ; every partial function  $f : A \rightarrow_p B$  is a total function  $f : \text{domain}(f) \rightarrow B$ .



- Let  $f : \Sigma^* \rightarrow_p \Gamma^*$  where  $\sqcup \notin \Sigma \cup \Gamma$ .
  - $f$  is a function over words in some alphabets.
- $f$  is **Turing computable**, if there exists a Turing machine  $M$  such that
  - for input  $w$  (i.e. initial tape content  $w\sqcup\dots$ ),  $M$  terminates in a final state, if and only if  $w \in \text{domain}(f)$ ;
  - for input  $w$ ,  $M$  terminates in a final state with output  $w'$  (i.e. final tape content  $w'\sqcup\dots$ ), if and only if  $w' = f(w)$ .
- **Not every function  $f : \Sigma^* \rightarrow_p \Gamma^*$  is Turing computable:**
  - The set of all Turing machines is countably infinite: all machines can be ordered in a single list (in the alphabetic order of their definitions).
  - The set of all functions  $\Sigma^* \rightarrow_p$  is more than countably infinite (Cantor's diagonalization argument).
  - Consequently, there are more functions than Turing machines.

$M$  computes  $f$ , if  $M$  terminates for arguments in the domain of  $f$  with output  $f(a)$  and does not terminate for arguments outside the domain.



## Example

We show that natural number subtraction is Turing computable.

- Subtraction  $\ominus$  on  $\mathbb{N}$ :

$$m \ominus n := \begin{cases} m - n & \text{if } m \geq n \\ 0 & \text{else} \end{cases}$$

- Unary representation of  $n \in \mathbb{N}$ :

$$\underbrace{000\dots 0}_{n \text{ times}} \in L(0^*)$$

- Input  $00 \sqcup 0$  shall lead to output  $0$ .
  - $2 \ominus 1 = 1$ .

Idea: replace every pair of  $0$  in  $m$  and  $n$  by  $\sqcup$ .



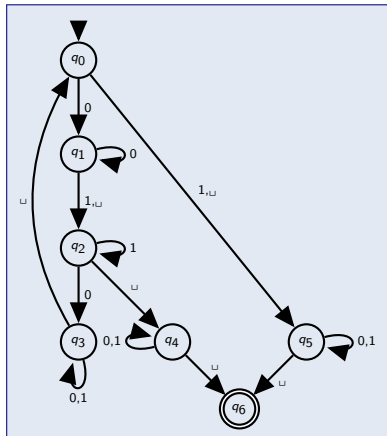
## Example (Contd)

$$M = (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F)$$

$$Q = \{q_0, \dots, q_6\}$$

$$\Sigma = \{0\}, \Gamma = \{0, 1, \sqcup\}, F = \{q_6\}$$

$\delta$	0	1	$\sqcup$
$q_0$	$(q_1, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$
$q_1$	$(q_1, 0, R)$	$(q_2, 1, R)$	$(q_2, 1, R)$
$q_2$	$(q_3, 1, L)$	$(q_2, 1, R)$	$(q_4, \sqcup, L)$
$q_3$	$(q_3, 0, L)$	$(q_3, 1, L)$	$(q_0, \sqcup, R)$
$q_4$	$(q_4, 0, L)$	$(q_4, \sqcup, L)$	$(q_6, 0, R)$
$q_5$	$(q_5, \sqcup, R)$	$(q_5, \sqcup, R)$	$(q_6, \sqcup, R)$
$q_6$	—	—	—



- In  $q_0$ , the leading 0 is replaced by  $\sqcup$ .
- In  $q_1$ ,  $M$  searches for the next  $\sqcup$  and replaces it by a 1.
- In  $q_2$ ,  $M$  searches for the next 0 and replaces it by 1, then moves left.
- In  $q_3$ ,  $M$  searches for previous  $\sqcup$ , moves right and starts from begin.
- In  $q_4$ ,  $M$  has found a  $\sqcup$  instead of 0 and replaces all previous 1 by  $\sqcup$ .
- In  $q_5$ ,  $n$  is (has become) 0; the rest of the tape is erased.
- In  $q_6$ , the computation successfully terminates.



## Example (Contd)

### ■ $2 \ominus 1 = 1$ :

$$\begin{aligned} & q_0 0 0 \sqcup 0 \vdash \sqcup q_1 0 \sqcup 0 \vdash \sqcup 0 q_1 \sqcup 0 \vdash \sqcup 0 1 q_2 0 \\ & \vdash \sqcup 0 q_3 1 1 \vdash \sqcup q_3 0 1 1 \vdash q_3 \sqcup 0 1 1 \vdash \sqcup q_0 0 1 1 \\ & \vdash \sqcup \sqcup q_1 1 1 \vdash \sqcup \sqcup 1 q_2 1 \vdash \sqcup \sqcup 1 1 q_2 \vdash \sqcup \sqcup 1 q_4 1 \\ & \vdash \sqcup \sqcup q_4 1 \vdash \sqcup q_4 \vdash \sqcup 0 q_6 \end{aligned}$$

### ■ $1 - 2 \ominus 0$ :

$$\begin{aligned} & q_0 0 \sqcup 0 0 \vdash \sqcup q_1 \sqcup 0 0 \vdash \sqcup 1 q_2 0 0 \vdash \sqcup q_3 1 1 0 \\ & \vdash q_3 \sqcup 1 1 0 \vdash \sqcup q_0 1 1 0 \vdash \sqcup \sqcup q_5 1 0 \vdash \sqcup \sqcup \sqcup q_5 0 \\ & \vdash \sqcup \sqcup \sqcup \sqcup q_5 \vdash \sqcup \sqcup \sqcup \sqcup \sqcup q_6. \end{aligned}$$

For  $m > n$ , leading blanks still have to be removed.

# Turing Computability



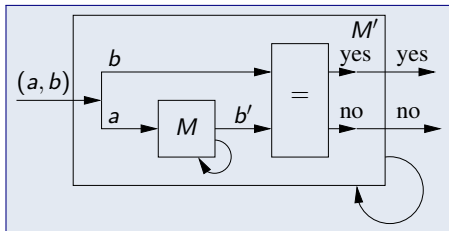
**Theorem:**  $f : \Sigma^* \rightarrow_p \Gamma^*$  is Turing computable, if and only if

$$L_f := \{(a, b) \in \Sigma^* \times \Gamma^* \mid a \in \text{domain}(f) \wedge b = f(a)\}$$

is recursively enumerable.

Proof  $\Rightarrow$ : Since  $f : \Sigma^* \rightarrow_p \Gamma^*$  is Turing computable, there exists a Turing machine  $M$  which computes  $f$ . To show that  $L_f$  is r.e., we construct  $M'$  with  $L(M') = L_f$ :

```
function  $M'(a, b)$ :  
   $b' \leftarrow M(a)$   
  if  $b' = b$  then  
    return yes  
  else  
    return no  
  end if  
end function
```



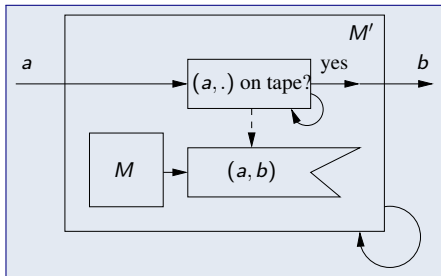




# Turing Computability

Proof  $\Leftarrow$ : Since  $L_f$  is recursively enumerable, there exists an enumerator  $M$  with  $\text{Gen}(M) = L_f$ . We construct the following Turing machine  $M'$  which computes  $f$ :

```
function  $M'(a)$ :  
  while  $M$  is not terminated do  
     $M$  writes  $(a', b')$  to tape  
    if  $a = a'$  then  
      return  $b'$   
    end if  
  end while  
loop forever  
end function
```



Computing is possible, if and only if recognizing is possible.



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1. Turing Machines
  2. Recognizing Languages
  3. Generating Languages
  4. Computing Functions
  5. **The Church-Turing Thesis**

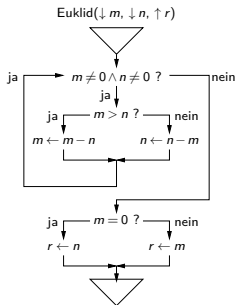


# Algorithms

Computer science is based on algorithms.

Compute as follows the greatest common divisor of two natural numbers  $m, n$  that are not both 0:

1. If  $m = 0$ , the result is  $n$ .
2. If  $n = 0$ , the result is  $m$ .
3. If  $m > n$ , subtract  $n$  from  $m$  and continue with step 1.
4. Otherwise subtract  $m$  from  $n$  and continue with step 1.



Euklid( $\downarrow m, \downarrow n, \uparrow r$ ):  
**while**  $m \neq 0 \wedge n \neq 0$  **do**  
    **if**  $m > n$   
        **then**  $m \leftarrow m - n$   
    **else**  $n \leftarrow n - m$   
**if**  $m = 0$   
    **then**  $r \leftarrow n$   
    **else**  $r \leftarrow m$   
**end** Euklid.

What is an “algorithm” and what is computable by an algorithm?



# The Church-Turing Thesis

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**Church-Turing Thesis:** Every problem that is solvable by an algorithm (in an intuitive sense) is solvable by a Turing machine. Thus the set of intuitively computable functions is identical with the set of Turing computable functions.

- Replaces fuzzy notion “algorithm” by precise notion “Turing machine”.
- Unprovable thesis, exactly because the notion “algorithm” is fuzzy.
- Substantially validated, because many different computational models have no more computational power than Turing machines.
  - Random access machines, loop programs, recursive functions, goto programs,  $\lambda$ -calculus, rewriting systems, grammars, ...

Turing machines represent the most powerful computational model known, but there are many other equally powerful (“Turing complete”) models.