Monads in Category Theory

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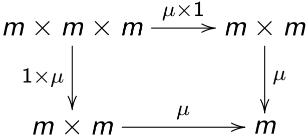
What a monoid is

$$(m, m \times m \xrightarrow{\mu} m, \star \xrightarrow{e} m)$$

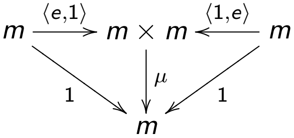
m	a set
$m \times m \xrightarrow{\mu} m$	operation
$\star \xrightarrow{e} m$	element

obeying conditions

Associativity



Identity element



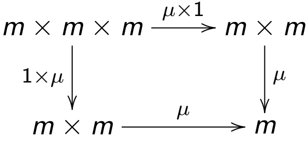
1st Generalization: Monoid in a category C

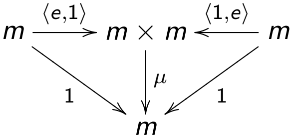
$$(m, m \times m \xrightarrow{\mu} m, \star \xrightarrow{e} m)$$

m	an object
$m \times m \xrightarrow{\mu} m$	arrow
$\star \xrightarrow{e} m$	arrow

where $\star$ is a terminal object

subject to





The same game can be played with groups, rings, algebras etc.

Examples

Monoid (group, ring, ...) in cat of set = *ordinary monoid (group, ring, ...)*

group in cat of topological spaces = *topological group*

group in cat of groups = *abelian group*

group in cat of C^∞-manifolds = *Lie group*

2nd generalization: Monoidal Categories

A category C is **strict monoidal** if it is equipped with

a functor: $\otimes \colon C \times C \longrightarrow C$ (the **tensor product**)
an object $e \in C$ (the **unit object**)

subject to

1. Associativity

$$\begin{array}{ccc} C \times C \times C & \xrightarrow{\otimes \times 1} & C \times C \\ {\scriptstyle 1 \times \otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\ C \times C & \xrightarrow{\otimes} & C \end{array}$$

2. Unit

$$\begin{array}{ccccc} C & \xrightarrow{\langle e,1 \rangle} & C \times C & \xleftarrow{\langle 1,e \rangle} & C \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \otimes} & \swarrow{\scriptstyle 1} & \\ & & C & & \end{array}$$

So, in order that $(C, C \times C \xrightarrow{\otimes} C, e)$ be strictly monoidal, it has to obey the rules

$$
\begin{aligned}
g_1 f_1 \otimes g_2 f_2 &= (g_1 \otimes g_2) \circ (f_1 \otimes f_2) \\
1_a \otimes 1_b &= 1_{a \otimes b} \\
\\
a \otimes (b \otimes c) &= (a \otimes b) \otimes c \\
f \otimes (g \otimes h) &= (f \otimes g) \otimes h \\
\\
e \otimes a &= a = a \otimes e \\
1_e \otimes f &= f = f \otimes 1_e
\end{aligned}
$$

Here a, b, c are objects, f, g, h, f_i, g_i are morphisms.

Examples

1. $(M, \cdot, e)$ *ordinary monoid as discrete category.*
   Multiplication $M \times M \overset{\cdot}{\longrightarrow} M$ *is a functor*
   $e \in M$ *the unit object.*
2. $(M, \cdot, e)$ *commutative monoid as one-object category* $(\star, M)$.
   Multiplication is a functor $(\star, M) \times (\star, M) \longrightarrow (\star, M)$
   $\star$ *is the unit object.*
3. $(\mathrm{End}\ A, \otimes, 1_A)$ *where* A *is an arbitrary category.*

$$\left\{\begin{matrix} F \otimes G = F \circ G & \textit{for functors } A \longrightarrow A \\ \sigma \otimes \tau = \sigma \star \tau & \textit{for transformations} \end{matrix}\right\}$$

$1_A \colon A \longrightarrow A$ *is the unit object.*

General monoidal categories

This may exhausted even more in that commutativity of the fundamental diagrams is relaxed to natural isomorphism.
Then we have

Examples (Monoidal categories)

1. *Any category* $(A, \times, t)$ *where* $\times$ *is the product,* t *a terminal object*
2. *Any category* $(A, \sqcap, i)$ *where* $\sqcup$ *is the coproduct,* i *an initial object*
3. $(\Lambda\Lambda$*-bimodules,* $\otimes_\Lambda, \Lambda)$ *where* Λ *is a ring;*
4. $(\text{vectorspaces}, \otimes_k, k)$, $(\text{abelian groups}, \otimes_{\mathbb{Z}}, \mathbb{Z})$*;*
5. $(\mathcal{ALG}_R, \otimes_R, R)$ *where* R *is a commutative ring;*
6. *(pointed spaces, smash product, pointed 0-sphere);*
7. *(bounded semilattices, meet, 1).*

Furthermore: graded modules, chain complexes, ...

3rd Generalization: Monoids in strict monoidal categories

Let $A = (A, \otimes, e)$ be a strict monoidal category

$$A = (A, A \times A \xrightarrow{\otimes} A, e \in A).$$

A **monoid** in A is a triple (m, μ, η), where

$m \in A$	is an object
$m \otimes m \xrightarrow{\mu} m$	is an arrow
$e \xrightarrow{\eta} m$	is an arrow

subject to

$$\begin{array}{ccc} m \otimes m \otimes m & \xrightarrow{\mu \otimes 1} & m \otimes m \\ {\scriptstyle 1 \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ m \otimes m & \xrightarrow{\mu} & m \end{array}$$

$$\begin{array}{ccccc} e \otimes m & \xrightarrow{\eta \otimes 1} & m \times m & \xleftarrow{1 \otimes \eta} & m \otimes e \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & m & & \end{array}$$

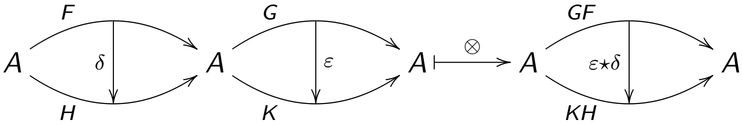
A morphism of monoids $(m, \mu, \eta) \longrightarrow (m', \mu', \eta')$ is an A-arrow $f : m \longrightarrow m'$ such that

$$\begin{CD} m \otimes m @>{f \otimes f}>> m \\ @V{\mu}VV @VV{\mu'}V \\ m @>{f}>> m \end{CD} \qquad \begin{array}{ccc} & e & \\ {}^{\eta}\swarrow & & \searrow^{\eta'} \\ m & \xrightarrow{\quad f \quad} & m' \end{array}$$

For arbitrary category A, $(\mathrm{End}\, A,\ \otimes, 1_A)$ is monoidal.

$$\mathrm{End}\, A \times \mathrm{End}\, A \xrightarrow{\ \otimes\ } \mathrm{End}\, A$$

$$(G, F) \xrightarrow{(\varepsilon,\delta)} (K, H) \longmapsto^{\otimes} GF \xrightarrow{\varepsilon\star\delta} KH$$



where $\varepsilon \star \delta \colon GF \longrightarrow KH$ is given by the dotted arrow below

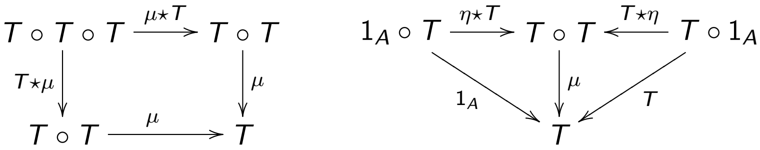
$$a \in A \quad\Longrightarrow\quad Fa \xrightarrow{\delta_a} Ha \quad\Longrightarrow\quad \begin{array}{ccc} GFa & \xrightarrow{G(\delta_a)} & GHa \\ {\scriptstyle \varepsilon_{Fa}}\downarrow & \overset{\varepsilon\star\delta}{\searrow} & \downarrow{\scriptstyle \varepsilon_{Ha}} \\ KFa & \xrightarrow[K(\delta_a)]{} & KHa \end{array}$$

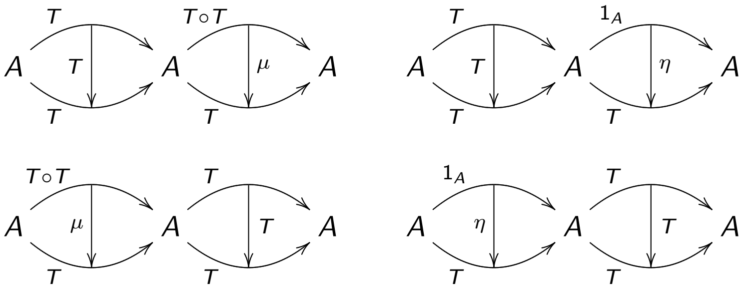
Thus, $(\varepsilon \star \delta)_a = \varepsilon_{Ha} \circ G(\delta_a) = K(\delta_a) \circ \varepsilon_{Fa}$.

Definition

Let A be a category. A **monad** *in A is a monoid in $(\mathrm{End}\, A, \otimes, 1_A)$.*

Thus, a monad is a triple (T, μ, η) where $T \circ T \xrightarrow{\mu} T$, $1_A \xrightarrow{\eta} T$ such that





Examples

1. $G = (G, \cdot, {}^{-1}, e)$ *ordinary group.*

$$\begin{array}{ccc} \mathcal{SET} \xrightarrow{T} \mathcal{SET} & T \circ T \xrightarrow{\quad\mu\quad} T & 1_{\mathcal{SET}} \xrightarrow{\eta} T \\ X \longmapsto G \times X & G \times G \times X \xrightarrow{\mu_X} G \times X & X \xrightarrow{\eta_X} G \times X \\ & (g_1, g_2, x) \longmapsto (g_1 \cdot g_2, x) & x \longmapsto (e, x) \end{array}$$

defines a monad in $\mathcal{SET}$.

2. Λ *ordinary ring,* $\mathcal{AB}$ *category of abelian groups*

$$\begin{array}{ccc} \mathcal{AB} \xrightarrow{T} \mathcal{AB} & T \circ T \xrightarrow{\quad\mu\quad} T & 1_{\mathcal{AB}} \xrightarrow{\eta} T \\ A \longmapsto \Lambda \otimes_{\mathbb{Z}} A & \Lambda \otimes_{\mathbb{Z}} \Lambda \otimes_{\mathbb{Z}} A \xrightarrow{\mu_A} \Lambda \otimes_{\mathbb{Z}} A & A \xrightarrow{\eta_A} \Lambda \otimes_{\mathbb{Z}} A \\ & \lambda_1 \otimes \lambda_2 \otimes a \longmapsto \lambda_1\lambda_2 \otimes a & a \longmapsto 1 \otimes a \end{array}$$

defines a monad in $\mathcal{AB}$.

Let (T, μ, η) be a monad in X. Because μ and η are natural transformations we get

$$x \xrightarrow{f} y \qquad \Longrightarrow \qquad \begin{array}{ccc} T^2x & \xrightarrow{T^2f} & T^2y \\ \downarrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle \mu_y} \\ Ty & \xrightarrow{Tf} & T^y \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ Tx & \xrightarrow{Tf} & Ty \end{array}$$

Therefore, together with the commutativity relations in the definition we obtain the following rules.

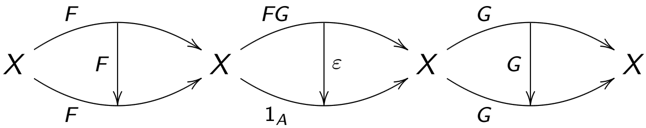
$$\begin{array}{rcll} x \in objects(X) & \Rightarrow & \mu_x \circ \mu_{Tx} = \mu_x \circ T(\mu_x) & \text{associativity} \\ x \in objects(X) & \Rightarrow & \mu_x \circ \eta_{Tx} = 1_{Tx} = \mu_x \circ T(\eta_x) & \text{unit} \\ x \xrightarrow{f} y & \Rightarrow & \mu_y \circ T^2 f = Tf \circ \mu_x & \text{naturalness of } \mu \\ x \xrightarrow{f} y & \Rightarrow & \eta_y \circ f = Tf \circ \eta_x & \text{naturalness of } \eta \end{array}$$

Every adjunction $A \underset{F}{\overset{G}{\rightleftarrows}} X$ defines a monad in X:

Together with F, G there come 2 transformations

$$1_X \xrightarrow{\eta} GF \text{ (unit)} \qquad \text{and} \qquad FG \xrightarrow{\varepsilon} 1_A \text{ (counit)}$$

We set $T := GF \in \mathrm{End}\, X$. Then the situation is



Define $\mu := G \star \varepsilon \star F$. Then

$$\underbrace{GFGF}_{T \circ T} \xrightarrow{\mu} \underbrace{GF}_{T}, \qquad 1_X \xrightarrow{\eta} \underbrace{GF}_{T}$$

and (T, μ, η) is a monad.

For a fixed category X the association defined by the last construction is a surjection

$$\left\{\text{adjunctions } A \underset{F}{\overset{G}{\rightleftarrows}} X \mid A \text{ cat}, F, G \text{ functors}\right\} \longrightarrow \left\{\text{monads in } X\right\}$$

For a given monad T in a category X there are plenty of adjunctions producing T. The minimal one is the **Kleisli-adjunction**.

The Kleisli-construction

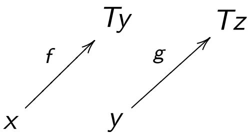
Given a monad (T, μ, η) in a category X.
Define new cat X_T:

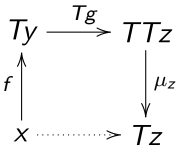
$$\mathrm{objects}(X_T) = \{x^\sharp \mid x \in \mathrm{objects}(X)\}$$
$$\mathrm{hom}_{X_T}(x^\sharp, y^\sharp) = \{f^\sharp \mid f \in \mathrm{hom}_X(x, Ty)\}.$$

$x \in X$	$\Rightarrow$	$x^\sharp$	$x \xrightarrow{f} Ty$	$\Rightarrow$	$x^\sharp \xrightarrow{f^\sharp} y^\sharp$
object in X		new object	arrow in X		new arrow

$$\mathrm{objects}(X_T) \cong \mathrm{objects}(X) \text{ and } \mathrm{hom}_X(x, Ty) \cong \mathrm{hom}_{X_T}(x^\sharp, y^\sharp).$$

Composition: Given $x^\sharp \xrightarrow{f^\sharp} y^\sharp \xrightarrow{g^\sharp} z^\sharp$.





Composition is associative:

Let $x^\sharp \xrightarrow{f^\sharp} y^\sharp \xrightarrow{g^\sharp} z^\sharp \xrightarrow{h^\sharp} w^\sharp$. Then

$$\begin{aligned} & h^\sharp \circ (g^\sharp \circ f^\sharp) = h^\sharp \circ (\mu_z \circ Tg \circ f)^\sharp = (\mu_w \circ Th \circ \mu_z \circ Tg \circ f)^\sharp \\ = \; & (\mu_w \circ \mu_{Tw} \circ T^2 h \circ Tg \circ f)^\sharp = (\mu_w \circ T(\mu_w) \circ T^2 h \circ Tg \circ f)^\sharp \\ = \; & (h^\sharp \circ g^\sharp) \circ f^\sharp \end{aligned}$$

Every object has a unit: Let $x^\sharp \xrightarrow{f^\sharp} y^\sharp$. Then

$$\begin{aligned} \eta_y^\sharp \circ f^\sharp &= (\mu_y \circ T(\eta_y) \circ f)^\sharp = (1_{Ty} \circ f)^\sharp = f^\sharp \\ f^\sharp \circ \eta_x^\sharp &= (\mu_y \circ T(f) \circ \eta_x)^\sharp = (\mu_y \circ \eta_{Ty} \circ f)^\sharp = (1_{Ty} \circ f)^\sharp = f^\sharp \end{aligned}$$

Thus X_T is indeed a category.

Adjoint functors

$F_T \colon X \longrightarrow X_T$:

$$x \xrightarrow{u} y \qquad \Longrightarrow \qquad x \xrightarrow{u} y \xrightarrow{\eta_y} Ty$$

$$F_T(x \xrightarrow{u} y) := x^\sharp \xrightarrow{(\eta_y \circ u)^\sharp} y^\sharp$$

$G_T \colon X_T \longrightarrow X$:

$$x^\sharp \xrightarrow{f^\sharp} y^\sharp \qquad \Longrightarrow \qquad Tx \xrightarrow{Tf} T^2 y \xrightarrow{\mu_y} Ty$$

$$G_T(x^\sharp \xrightarrow{f^\sharp} y^\sharp) := \; Tx \xrightarrow{\mu_y \circ Tf} Ty$$

Thus $G_T(x^\sharp) = Tx$ on objects.

$X_T \underset{F_T}{\overset{G_T}{\rightleftarrows}} X$ is an adjunction which produces the given monad T.

Kleisli star

For the following construction we assume that the category X is concrete, i.e., the objects of X do have elements.

Let $(X \xrightarrow{T} X,\ T^2 \xrightarrow{\mu} T,\ 1_X \xrightarrow{\eta} T)$ be a monad in X such that $T\colon X \longrightarrow X$ is injective on objects. We define

$$\begin{array}{rl} \hom(x, Ty) \xrightarrow{\star} \hom(Tx, Ty) & \qquad Tx \times \hom(x, Ty) \xrightarrow{\ltimes} Ty \\ f \mapsto f^\star := \mu_y \circ Tf & \qquad (\xi, f) \mapsto f^\star(\xi) =: \xi \ltimes f \end{array}$$

The operation $Tx \times \hom(x, Ty) \xrightarrow{\ltimes} Ty$ - called the **bind operator** - is an obvious version of the operator $f \mapsto f^\star$.

The following three **monad laws** are easily verified:

$$\begin{array}{rcll} f^\star \circ \eta_x & = & f & \text{for } x \xrightarrow{f} Ty \\ \eta_x^\star & = & 1_{Tx} & \text{for } x \in \text{objects}(X) \\ (g^\star \circ f)^\star & = & g^\star \circ f^\star & \text{for } x \xrightarrow{f} Ty,\ y \xrightarrow{g} Tz. \end{array}$$

Bind operator

The three monad laws written in terms of the bind operator are:

$$\begin{array}{rcll} \eta_x(\xi) \ltimes f &=& f(\xi) & (\xi \in x) \\ \xi \ltimes \eta_x &=& \xi & (\xi \in Tx) \\ (\xi \ltimes f) \ltimes g &=& \xi \ltimes (f(\bullet) \ltimes g) & (\xi \in Tx). \end{array}$$

Basic category theory:

$www.risc.jku.at/education/courses/ss2012/alg - alggeo/$