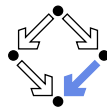


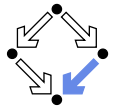
Term Algebras

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Term Algebra

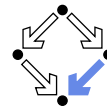


Take signature $\Sigma = (S, \Omega)$.

- **Term algebra** $T(\Sigma)$:
 - Σ -algebra whose carriers are Σ -terms.
 - $T(\Sigma)(s) = T_{\Sigma, s}$, for every $s \in S$.
 - $T(\Sigma)(\omega) = n$
 - for every $\omega = (n : \rightarrow s) \in \Omega$.
 - $T(\Sigma)(\omega)(t_1, \dots, t_k) = n(t_1, \dots, t_k)$
 - for every $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega, t_i \in T(\Sigma)(s_i)$.

$T(\Sigma)$ is the algebra of (well-typed) ground terms of Σ .

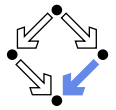
Term Algebras



- Example: $\text{NAT} = (\{\text{nat}\}, \{0 : \rightarrow \text{nat}, \text{Succ} : \text{nat} \rightarrow \text{nat}\})$.
 - $T(\text{NAT})(\text{nat}) = \{0, \text{Succ}(0), \text{Succ}(\text{Succ}(0)), \dots\}$.
 - $T(\text{NAT})(0) = 0$.
 - $T(\text{NAT})(\text{Succ})(t) = \text{Succ}(t)$, for every $t \in T(\text{NAT})(\text{nat})$.
- Term value $T(\Sigma)(t) = t$, for every ground term $t \in T(\Sigma)$.
 - A ground term denotes itself.
- $T(\Sigma)$ is freely generated.
 - Generated: every carrier is denoted by itself.
 - Free: two different ground terms denote two different carriers.

In a term algebra, a ground term and its interpretation coincide.

Initiality



Take signature Σ , class $\mathcal{C} \subseteq \text{Alg}(\Sigma)$ of Σ -algebras, and Σ -algebra $A \in \mathcal{C}$.

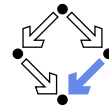
- A is **initial** in \mathcal{C} if
 - for every $B \in \mathcal{C}$, there exists exactly one homomorphism $h : A \rightarrow B$.
 - A distinguishes most among all algebras of \mathcal{C} .
- Initial algebras are unique up to isomorphism:
 - If A is initial in \mathcal{C} , then B is initial in \mathcal{C} iff $A \simeq B$.
- **Theorem:** $T(\Sigma)$ is initial in $\text{Alg}(\Sigma)$.
 - For every $A \in \text{Alg}(\Sigma)$, there exists the unique **evaluation homomorphism**:

$$h : T(\Sigma) \rightarrow A$$

$$h(t) := A(t), \text{ for every ground term } t \in T_{\Sigma}.$$

The term algebra $T(\Sigma)$ distinguishes most among all Σ -algebras.

Congruence Relation

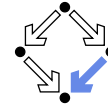


Take signature $\Sigma = (S, \Omega)$, Σ -algebra A .

- Congruence relation $Q = (Q_s)_{s \in S}$ on A :
 - Q_s is an equivalence relation on $A(s)$ for every $s \in S$.
 - $(a_1, a'_1) \in Q_{s_1} \wedge \dots \wedge (a_k, a'_k) \in Q_{s_k} \Rightarrow (A(\omega)(a_1, \dots, a_k), A(\omega)(a'_1, \dots, a'_k)) \in Q_s$
 - for every $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega$, and
 - for every $a_1, a'_1 \in A(s_1), \dots, a_k, a'_k \in A(s_k)$.
 - Equivalent arguments yield equivalent results.

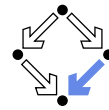
A congruence relation preserves equivalence across function applications.

Example



- BOOL-algebra D :
 - $D(\text{bool}) = \mathbb{N}$
 - $D(\neg)(n) = \begin{cases} n + 1, & \text{if } n \text{ is even} \\ n - 1, & \text{otherwise} \end{cases}$
 - $D(\wedge)(n, m) = n * m$
- Q is a congruence relation on D .
 - $(m, n) \in Q_{\text{bool}} :\Leftrightarrow m + n \text{ is even.}$
- Take $\omega = \neg : \text{bool} \rightarrow \text{bool}$:
 - Take $n, n' \in D(\text{bool})$ with $(n, n') \in Q_{\text{bool}}$.
 - We have to show $(D(\neg)(n), D(\neg)(n')) \in Q_{\text{bool}}$.
 - $n + n'$ is even. Thus n and n' are either both even or both odd.
 - Case 1: we have to show $(n + 1, n' + 1) \in Q_{\text{bool}}$, i.e., $(n + 1) + (n' + 1) = (n + n') + 2$ is even. ...
 - Case 2: we have to show $(n - 1, n' - 1) \in Q_{\text{bool}}$, i.e., $(n - 1) + (n' - 1) = (n + n') - 2$ is even. ...
- Take $\omega = \wedge : \text{bool} \times \text{bool} \rightarrow \text{bool}$:
 - ...

Quotient Algebra

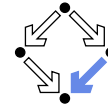


Take signature $\Sigma = (S, \Omega)$, Σ -algebra A , congruence relation Q on A .

- Quotient (algebra) A/Q of A by Q :
 - Σ -algebra whose carriers are congruence classes.
 - $[a]_Q = \{a' : (a, a') \in Q\}$.
 - Class of a with respect to congruence relation Q .
 - $A/Q(s) = \{[a]_{Q_s} \mid a \in A(s)\}$
 - for every $s \in S$.
 - $A/Q(\omega) = [A(\omega)]_{Q_s}$
 - for every $\omega = (n : \rightarrow s) \in \Omega$.
 - $A/Q(\omega)([a_1]_{Q_{s_1}}, \dots, [a_k]_{Q_{s_k}}) = [A(\omega)(a_1, \dots, a_k)]_{Q_s}$
 - for every $\omega = (n : s_1 \times \dots \times s_k \rightarrow s) \in \Omega$.

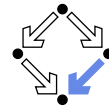
Congruent elements of A are combined to a single element of A/Q .

Example



- BOOL-algebra D and congruence relation Q on D (as before).
 - $(m, n) \in Q_{\text{bool}} :\Leftrightarrow m + n \text{ is even.}$
- Quotient algebra D/Q :
 - $[0] = \{n \in \mathbb{N} \mid 0 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is even}\}$
 - $[1] = \{n \in \mathbb{N} \mid 1 + n \text{ is even}\} = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
 - $(D/Q)(\text{bool}) = \{[0], [1]\}$.
 - $(D/Q)(\neg)(n) = \begin{cases} [1] & \text{if } n = [0] \\ [0] & \text{if } n = [1] \end{cases}$
 - $(D/Q)(\wedge)(n, m) = \begin{cases} [1] & \text{if } n = m = [1] \\ [0] & \text{else} \end{cases}$
- $(D/Q) \simeq C$
 - $C(\text{bool}) = \{0, 1\}$
 - $C(\text{True}) = 1$
 - $C(\text{False}) = 0$
 - $C(\neg)(n) = 1 - n$
 - $C(\wedge)(n, m) = n * m$

Quotient Term Algebra

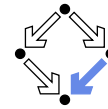


Take signature $\Sigma = (S, \Omega)$ and class of algebras $\mathcal{C} \subseteq Alg(\Sigma)$.

- **Congruence relation** $\equiv_{\mathcal{C}}$ of \mathcal{C} :
 - $\equiv_{\mathcal{C}} := (\equiv_{\mathcal{C},s})_{s \in S}$.
 - $\equiv_{\mathcal{C},s} := \{(t, u) \in T_{\Sigma,s} \times T_{\Sigma,s} \mid \forall A \in \mathcal{C} : A(t) = A(u)\}$.
 - All ground terms are congruent that have the same value in all algebras of \mathcal{C} .
- **Quotient Term Algebra** $T(\Sigma, \mathcal{C})$ of \mathcal{C} :
 - $T(\Sigma, \mathcal{C}) := T(\Sigma) / \equiv_{\mathcal{C}}$.
 - Σ -algebra whose carrier are congruence classes of ground terms of Σ .
- **Theorem:** If $T(\Sigma, \mathcal{C}) \in \mathcal{C}$, then $T(\Sigma, \mathcal{C})$ is initial in \mathcal{C} .
 - For every $A \in \mathcal{C}$, there exists the unique **evaluation homomorphism**:
$$h : T(\Sigma, \mathcal{C}) \rightarrow A$$
$$h([t]) := A(t), \text{ for every ground term } t \in T_{\Sigma}.$$

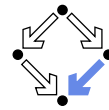
$T(\Sigma, \mathcal{C})$ relates similarly to \mathcal{C} as $T(\Sigma)$ relates to $Alg(\Sigma)$.

Examples



- $T(\Sigma, Alg(\Sigma)) \simeq T(\Sigma)$.
 - Carriers of $T(\Sigma, Alg(\Sigma))$ are singletons $[t] = \{t\}$ for every ground term $t \in T_{\Sigma}$.
- $T(\Sigma, \{A\}) \simeq A$, for every Σ -algebra A .
 - Carriers of $T(\Sigma, \{A\})$ are classes of all those terms that denote the same carrier in A .
- Let B be the “classical” NATBOOL-algebra.
 - Terms *True* and \neg *False* belong to the same carrier of $T(\Sigma, \{B\})$.
 - Terms 0 and $0 + 0$ belong to the same carrier of $T(\Sigma, \{B\})$.

Quotient Term Algebra of a Set of Formulas



Take logic L , signature Σ , set of formulas $\Phi \subseteq L(\Sigma)$.

- **Quotient term algebra** $T(\Sigma, \Phi)$ of Φ :
 - $T(\Sigma, \Phi) := T(\Sigma, Mod_{\Sigma}(\Phi)) (= T(\Sigma) / \equiv_{Mod_{\Sigma}(\Phi)})$.
 - $Mod_{\Sigma}(\Phi) = \{A \in Alg(\Sigma) \mid A \text{ is a model of } \Phi\}$.
 - $\equiv_{Mod_{\Sigma}(\Phi),s} := \{(t, u) \in T_{\Sigma,s} \times T_{\Sigma,s} \mid \forall A \in Mod_{\Sigma}(\Phi) : A(t) = A(u)\}$.
 - Σ -algebra whose carriers are classes of those terms that have the same value in all models of Φ .
- **Theorem:** If $T(\Sigma, \Phi)$ is model of Φ , $T(\Sigma, \Phi)$ is initial in $Mod_{\Sigma}(\Phi)$.
 - For every model A of Φ , there exists the unique **evaluation homomorphism**:
$$h : T(\Sigma, \Phi) \rightarrow A$$
$$h([t]) := A(t), \text{ for every ground term } t \in T_{\Sigma}.$$

Basis of initial specification semantics.