Specifying and Verifying Programs (Part 1)

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Specifying and Verifying Programs

We will discuss three (closely interrelated) calculi.

- Hoare Calculus: $\{P\}\ c\ \{Q\}$
  - If command c is executed in a pre-state with property P and terminates, it yields a post-state with property Q.

    $\{x = a \wedge y = b\} x := x + y \{x = a + y \wedge y = b\}$
- Predicate Transformers: $\text{wp}(c, Q) = P$
  - If the execution of command c shall yield a post-state with property Q, it must be executed in a pre-state with property P.

    $\text{wp}(c, x = a + y \wedge y = b) = (x + y = a + y \wedge y = b)$
- State Relations: $c : [P \Rightarrow Q]^{x,\dots}$
  - The post-state generated by the execution of command c is related to the pre-state by $P \Rightarrow Q$ (where only variables $x, \dots$ have changed).

    $x = x + y : [\text{var } x = \text{old } x + \text{old } y]^x$

1. The Hoare Calculus

2. Predicate Transformers

3. Proving Verification Conditions

4. Termination

5. Abortion

6. Procedures

The Hoare Calculus

First and best-known calculus for program reasoning (C.A.R. Hoare).

- "Hoare triple": $\{P\}\ c\ \{Q\}$
  - Logical propositions P and Q, program command c.
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.
- Partial correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If c is executed in a state in which P holds, then it terminates in a state in which Q holds unless it aborts or runs forever."
  - Program does not produce wrong result.
  - But program also need not produce any result.
    - Abortion and non-termination are not (yet) ruled out.
- Total correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If c is executed in a state in which P holds, then it terminates in a state in which Q holds."
  - Program produces the correct result.

We will use the partial correctness interpretation for the moment.

Weakening and Strengthening

$$\frac{P \Rightarrow P' \quad \{P'\}\ c\ \{Q'\} \quad Q' \Rightarrow Q}{\{P\}\ c\ \{Q\}}$$

- Logical derivation: $\dfrac{A_1 \ A_2}{B}$
  - Forward: If we have shown A_1 and A_2, then we have also shown B.
  - Backward: To show B, it suffices to show A_1 and A_2.
- Interpretation of above sentence:
  - To show that, if P holds, then Q holds after executing c, it suffices to show this for a P' weaker than P and a Q' stronger than Q.

Precondition may be weakened, postcondition may be strengthened.

Special Commands

$$\{P\}\ \textbf{skip}\ \{P\} \qquad \{\text{true}\}\ \textbf{abort}\ \{\text{false}\}$$

- The **skip** command does not change the state; if P holds before its execution, then P thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
  - Axiom implies $\{P\}$ **abort** $\{Q\}$ for arbitrary P, Q.

Useful commands for reasoning and program transformations.

Scalar Assignments

$$\{Q[e/x]\}\ x := e\ \{Q\}$$

- Syntax
  - Variable x, expression e.
  - $Q[e/x]$ ... Q where every free occurrence of x is replaced by e.
- Interpretation
  - To make sure that Q holds for x after the assignment of e to x, it suffices to make sure that Q holds for e before the assignment.
- Partial correctness
  - Evaluation of e may abort.

$$\begin{array}{rcl} \{x+3<5\} & x := x+3 & \{x<5\} \\ \{x<2\} & x := x+3 & \{x<5\} \end{array}$$

Array Assignments

$$\{Q[a[i \mapsto e]/a]\}\ a[i] := e\ \{Q\}$$

- An array is modelled as a function $a : I \to V$.
  - Index set I, value set V.
  - $a[i] = e$ ... array a contains at index i the value e.
- Term $a[i \mapsto e]$ ("array a updated by assigning value e to index i")
  - A new array that contains at index i the value e.
  - All other elements of the array are the same as in a.
- Thus array assignment becomes a special case of scalar assignment.
  - Think of "$a[i] := e$" as "$a := a[i \mapsto e]$".

$$\{\underline{a[i \mapsto x]}[1] > 0\} \quad a[i] := x \quad \{a[1] > 0\}$$

Arrays are here considered as basic values (no pointer semantics).

Array Assignments

How to reason about $a[i \mapsto e]$?

$$\frac{Q[\underline{a[i \mapsto e]}[j]]}{\rightsquigarrow}$$

$$(i = j \Rightarrow Q[e]) \land (i \neq j \Rightarrow Q[a[j]])$$

- Array Axioms

$$i = j \Rightarrow \underline{a[i \mapsto e]}[j] = e$$
$$i \neq j \Rightarrow \underline{a[i \mapsto e]}[j] = a[j]$$

$$\begin{array}{rcl} \{a[i \mapsto x][1] > 0\} & a[i] := x & \{a[1] > 0\} \\ \{(i = 1 \Rightarrow x > 0) \land (i \neq 1 \Rightarrow a[1] > 0)\} & a[i] := x & \{a[1] > 0\} \end{array}$$

Get rid of "array update terms" when applied to indices.

Command Sequences

$$\frac{\{P\}\ c_1\ \{R\} \quad \{R\}\ c_2\ \{Q\}}{\{P\}\ c_1; c_2\ \{Q\}}$$

- Interpretation
  - To show that, if P holds before the execution of $c_1; c_2$, then Q holds afterwards, it suffices to show for some R that
    - if P holds before c_1, that R holds afterwards, and that
    - if R holds before c_2, then Q holds afterwards.
- Problem: find suitable R.
  - Easy in many cases (see later).

$$\frac{\{x+y-1>0\}\ y := y-1\ \{x+y>0\} \quad \{x+y>0\}\ x := x+y\ \{x>0\}}{\{x+y-1>0\}\ y := y-1; x := x+y\ \{x>0\}}$$

The calculus itself does not indicate how to find intermediate property.

Conditionals

$$\frac{\{P \land b\}\ c_1\ \{Q\} \quad \{P \land \neg b\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{\{P \land b\}\ c\ \{Q\} \quad (P \land \neg b) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

- Interpretation
  - To show that, if P holds before the execution of the conditional, then Q holds afterwards,
  - it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

$$\frac{\{x \neq 0 \land x \geq 0\}\ y := x\ \{y > 0\} \quad \{x \neq 0 \land x \not\geq 0\}\ y := -x\ \{y > 0\}}{\{x \neq 0\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ y := x\ \textbf{else}\ y := -x\ \{y > 0\}}$$

Loops

$$\{\text{true}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{\{I \land b\}\ c\ \{I\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \land \neg b\}}$$

- Interpretation:
  - The **loop** command does not terminate and thus trivially satisfies partial correctness.
    - Axiom implies $\{P\}$ **loop** $\{Q\}$ for arbitrary P, Q.
  - If it is the case that
    - I holds before the execution of the **while**-loop and
    - I also holds after every iteration of the loop body,

    then I holds also after the execution of the loop (together with the negation of the loop condition b).
    - I is a loop invariant.
- Problem:
  - Rule for **while**-loop does not have arbitrary pre/post-conditions P, Q.

In practice, we combine this rule with the strengthening/weakening-rule.

Loops (Generalized)

$$\frac{P \Rightarrow I \quad \{I \wedge b\}\ c\ \{I\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

- Interpretation:
  - To show that, if before the execution of a **while**-loop the property P holds, after its termination the property Q holds, it suffices to show for some property I (the loop invariant) that
    - I holds before the loop is executed (i.e. that P implies I),
    - if I holds when the loop body is entered (i.e. if also b holds), that after the execution of the loop body I still holds,
    - when the loop terminates (i.e. if b does not hold), I implies Q.
- Problem: find appropriate loop invariant I.
  - Strongest relationship between all variables modified in loop body.

The calculus itself does not indicate how to find suitable loop invariant.

Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$

$$\frac{\begin{array}{c}(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I \\ \{I \wedge i \leq n\}\ s := s + i; i := i + 1\ \{I\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{n \geq 0 \wedge i = 1 \wedge s = 0\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s + i; i := i + 1)\ \{s = \sum_{j=1}^{n} j\}}$$

The invariant captures the "essence" of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.

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Backward Reasoning

Implication of rule for command sequences and rule for assignments:

$$\frac{\{P\}\ c\ \{Q[e/x]\}}{\{P\}\ c; x := e\ \{Q\}}$$

- Interpretation
  - If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
  - By multiple application, assignment sequences can be removed from the back to the front.

$\{P\}$	$\{P\}$	$\{P\}$	$\{P\}$	$P \Rightarrow x = 4$
x := x+1;	x := x+1;	x := x+1;	$\{x + 1 = 5\}$	
y := 2*x;	y := 2*x;	$\{x + 2x = 15\}$	$(\Leftrightarrow x = 4)$	
z := x+y	$\{x + y = 15\}$	$(\Leftrightarrow 3x = 15)$		
$\{z = 15\}$		$(\Leftrightarrow x = 5)$		

Weakest Preconditions

A calculus for "backward reasoning" (E.W. Dijkstra).

- Predicate transformer wp
  - Function "wp" that takes a command c and a postcondition Q and returns a precondition.
  - Read wp(c, Q) as "the weakest precondition of c w.r.t. Q".
- wp(c, Q) is a precondition for c that ensures Q as a postcondition.
  - Must satisfy $\{\text{wp}(c, Q)\}\ c\ \{Q\}$.
- wp(c, Q) is the weakest such precondition.
  - Take any P such that $\{P\}\ c\ \{Q\}$.
  - Then $P \Rightarrow \text{wp}(c, Q)$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $P \Rightarrow \text{wp}(c, Q)$ instead.

Verification is reduced to the calculation of weakest preconditions.

Weakest Preconditions

The weakest precondition of each program construct.

$$
\begin{aligned}
&\text{wp}(\textbf{skip}, Q) = Q\\
&\text{wp}(\textbf{abort}, Q) = \text{true}\\
&\text{wp}(x := e, Q) = Q[e/x]\\
&\text{wp}(c_1; c_2, Q) = \text{wp}(c_1, \text{wp}(c_2, Q))\\
&\text{wp}(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, Q) = (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q))\\
&\text{wp}(\textbf{if } b \textbf{ then } c, Q) \Leftrightarrow (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)\\
&\text{wp}(\textbf{while } b \textbf{ do } c, Q) = \ldots
\end{aligned}
$$

Loops represent a special problem (see later).

Forward Reasoning

Sometimes, we want to derive a postcondition from a given precondition.

$$\{P\}\ x := e\ \{\exists x_0 : P[x_0/x] \wedge x = e[x_0/x]\}$$

- Forward Reasoning
  - What is the maximum we know about the post-state of an assignment $x := e$, if the pre-state satisfies P?
  - We know that P holds for some value x_0 (the value of x in the pre-state) and that x equals $e[x_0/x]$.

$$
\begin{gathered}
\{x \geq 0 \wedge y = a\}\\
x := x + 1\\
\{\exists x_0 : x_0 \geq 0 \wedge y = a \wedge x = x_0 + 1\}\\
(\Leftrightarrow (\exists x_0 : x_0 \geq 0 \wedge x = x_0 + 1) \wedge y = a)\\
(\Leftrightarrow x > 0 \wedge y = a)
\end{gathered}
$$

Strongest Postcondition

A calculus for forward reasoning.

- Predicate transformer sp
  - Function "sp" that takes a precondition P and a command c and returns a postcondition.
  - Read sp(c, P) as "the strongest postcondition of c w.r.t. P".
- sp(c, P) is a postcondition for c that is ensured by precondition P.
  - Must satisfy $\{P\}\ c\ \{\text{sp}(c, P)\}$.
- sp(c, P) is the strongest such postcondition.
  - Take any P, Q such that $\{P\}\ c\ \{Q\}$.
  - Then $\text{sp}(c, P) \Rightarrow Q$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(\text{sp}(c, P) \Rightarrow Q)$.
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $\text{sp}(c, P) \Rightarrow Q$ instead.

Verification is reduced to the calculation of strongest postconditions.

Strongest Postconditions

The strongest postcondition of each program construct.

$$
\begin{aligned}
&\text{sp}(\textbf{skip}, P) = P\\
&\text{sp}(\textbf{abort}, P) = \text{false}\\
&\text{sp}(x := e, P) = \exists x_0 : P[x_0/x] \wedge x = e[x_0/x]\\
&\text{sp}(c_1; c_2, P) = sp(c_2, sp(c_1, P))\\
&\text{sp}(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, P) \Leftrightarrow \text{sp}(c_1, P \wedge b) \vee \text{sp}(c_2, P \wedge \neg b)\\
&\text{sp}(\textbf{if } b \textbf{ then } c, P) = \text{sp}(c, P \wedge b) \vee (P \wedge \neg b)\\
&\text{sp}(\textbf{while } b \textbf{ do } c, P) = \ldots
\end{aligned}
$$

Forward reasoning as a (less-known) alternative to backward-reasoning.

Hoare Calculus and Predicate Transformers

In practice, often a combination of the calculi is applied.

$$\{P\}\ c_1; \textbf{while } b \textbf{ do } c; c_2\ \{Q\}$$

- Assume c_1 and c_2 do not contain loop commands.
- It suffices to prove

$$\{\text{sp}(P, c_1)\}\ \textbf{while } b \textbf{ do } c\ \{\text{wp}(c_2, Q)\}$$

Predicate transformers are applied to reduce the verification of a program to the Hoare-style verification of loops.

Weakest Liberal Preconditions for Loops

Why not apply predicate transformers to loops?

$$
\begin{aligned}
&\text{wp}(\textbf{loop}, Q) = \text{true}\\
&\text{wp}(\textbf{while } b \textbf{ do } c, Q) = L_0(Q) \wedge L_1(Q) \wedge L_2(Q) \wedge \ldots\\
\\
&L_0(Q) = \text{true}\\
&L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))
\end{aligned}
$$

- Interpretation
  - Weakest precondition that ensures that loops stops in a state satisfying Q, unless it aborts or runs forever.
- Infinite sequence of predicates $L_i(Q)$:
  - Weakest precondition that ensures that after less than i iterations the state satisfies Q, unless the loop aborts or does not yet terminate.
- Alternative view: $L_i(Q) = \text{wp}(\text{if}_i, Q)$

$$
\begin{aligned}
&\text{if}_0 = \textbf{loop}\\
&\text{if}_{i+1} = \textbf{if } b \textbf{ then } (c; \text{if}_i)
\end{aligned}
$$

Example

$$\text{wp}(\textbf{while } i < n \textbf{ do } i := i + 1, Q)$$

$$
\begin{aligned}
&L_0(Q) = \text{true}\\
&L_1(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, \text{true}))\\
&\quad \Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{true})\\
&\quad \Leftrightarrow (i \not< n \Rightarrow Q)\\
&L_2(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1, i \not< n \Rightarrow Q))\\
&\quad \Leftrightarrow (i \not< n \Rightarrow Q) \wedge\\
&\qquad\quad (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i]))\\
&L_3(Q) = (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{wp}(i := i + 1,\\
&\qquad\quad (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow (i + 1 \not< n \Rightarrow Q[i + 1/i]))))\\
&\quad \Leftrightarrow (i \not< n \Rightarrow Q) \wedge\\
&\qquad\quad (i < n \Rightarrow ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge\\
&\qquad\qquad (i + 1 < n \Rightarrow (i + 2 \not< n \Rightarrow Q[i + 2/i]))))
\end{aligned}
$$

Weakest Liberal Preconditions for Loops

- Sequence $L_i(Q)$ is monotonically increasing in strength:
  - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$.
- The weakest precondition is the "lowest upper bound":
  - $\forall i \in \mathbb{N} : \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : P \Rightarrow L_i(Q)) \Rightarrow (P \Rightarrow \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q))$.
- We can only compute weaker approximation $L_i(Q)$.
  - $\mathrm{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
- We want to prove $\{P\}$ **while** b **do** c $\{Q\}$.
  - This is equivalent to proving $P \Rightarrow \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - Thus $P \Rightarrow L_i(Q)$ must hold as well.
- If we can prove $\neg(P \Rightarrow L_i(Q))$, ...
  - $\{P\}$ **while** b **do** c $\{Q\}$ does not hold.
  - If we fail, we may try the easier proof $\neg(P \Rightarrow L_{i+1}(Q))$.

Falsification is possible by use of approximation L_i, but verification is not.

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A Constructive Definition of Arrays

```
% constructive array definition
newcontext "arrays2";

% the types
INDEX: TYPE = NAT;
ELEM:  TYPE;
ARR:   TYPE =
  [INDEX, ARRAY INDEX OF ELEM];

% error constants
any:      ARRAY INDEX OF ELEM;
anyelem:  ELEM;
anyarray: ARR;

% a selector operation
content:
  ARR -> (ARRAY INDEX OF ELEM) =
  LAMBDA(a:ARR): a.1;
```

```
% the array operations
length: ARR -> INDEX =
  LAMBDA(a:ARR): a.0;
new: INDEX -> ARR =
  LAMBDA(n:INDEX): (n, any);
put: (ARR, INDEX, ELEM) -> ARR =
 LAMBDA(a:ARR, i:INDEX, e:ELEM):
   IF i < length(a)
     THEN (length(a),
           content(a) WITH [i]:=e)
     ELSE anyarray
   ENDIF;
get: (ARR, INDEX) -> ELEM =
  LAMBDA(a:ARR, i:INDEX):
    IF i < length(a)
      THEN content(a)[i]
      ELSE anyelem ENDIF;
```


Proof of Fundamental Array Properties

```
% the classical array axioms as formulas to be proved
length1: FORMULA
  FORALL(n:INDEX): length(new(n)) = n;

length2: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => length(put(a, i, e)) = length(a);

get1: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => get(put(a, i, e), i) = e;

get2: FORMULA
  FORALL(a:ARR, i, j:INDEX, e:ELEM):
    i < length(a) AND j < length(a) AND
    i /= j =>
      get(put(a, i, e), j) = get(a, j);
```

[adu]: expand length, get, put, content
[c3b]: scatter
[qid]: proved (CVCL)

Proof of a Higher-Level Array Property

```
% extensionality on low-level arrays
extensionality: AXIOM
  FORALL(a, b:ARRAY INDEX OF ELEM):
    a=b <=> (FORALL(i:INDEX):a[i]=b[i]);

% unassigned parts hold identical values
unassigned: AXIOM
  FORALL(a:ARR, i:INT):
    (i >= length(a)) => content(a)[i

% extensionality on arrays to be prc
equality: FORMULA
  FORALL(a:ARR, b:ARR): a = b <=>
    length(a) = length(b) AND
    (FORALL(i:INDEX): i < length(a) => get(a,i) = get(b,i));
```

[adt]: expand length, get, content
  [cw2]: scatter
    [qey]: proved (CVCL)
    [rey]: assume b_0.1 = a_0.1
      [zpt]: proved (CVCL)
      [1pt]: instantiate a_0.1, b_0.1 in 1fm
        [y51]: scatter
          [ku2]: auto
            [iub]: proved (CVCL)

A Program Verification

Verification of the following Hoare triple:

$\{olda = a \wedge oldx = x \wedge n = |a| \wedge i = 0 \wedge r = -1\}$

while $i < n \wedge r = -1$ **do**
  if $a[i] = x$
    then $r := i$
    else $i := i + 1$

$\{a = olda \wedge x = oldx \wedge$
$((r = -1 \wedge \forall i : 0 \leq i < |a| \Rightarrow a[i] \neq x) \vee$
$(0 \leq r < |a| \wedge a[r] = x \wedge \forall i : 0 \leq i < r \Rightarrow a[i] \neq x))\}$

Find the smallest index r of an occurrence of value x in array a ($r = -1$, if x does not occur in a).

The Verification Conditions

$A :\Leftrightarrow Input \Rightarrow Invariant$
$B_1 :\Leftrightarrow Invariant \wedge i < n \wedge r = -1 \wedge a[i] = x \Rightarrow Invariant[i/r]$
$B_2 :\Leftrightarrow Invariant \wedge i < n \wedge r = -1 \wedge a[i] \neq x \Rightarrow Invariant[i+1/i]$
$C :\Leftrightarrow Invariant \wedge \neg(i < n \wedge r = -1) \Rightarrow Output$

$Input :\Leftrightarrow olda = a \wedge oldx = x \wedge n = length(a) \wedge i = 0 \wedge r = -1$

$Output :\Leftrightarrow a = olda \wedge x = oldx \wedge$
$((r = -1 \wedge \forall i : 0 \leq i < length(a) \Rightarrow a[i] \neq x) \vee$
$(0 \leq r < length(a) \wedge a[r] = x \wedge \forall i : 0 \leq i < r \Rightarrow a[i] \neq x))$

$Invariant :\Leftrightarrow olda = a \wedge oldx = x \wedge n = length(a) \wedge$
$0 \leq i \leq n \wedge \forall j : 0 \leq j < i \Rightarrow a[j] \neq x \wedge$
$(r = -1 \vee (r = i \wedge i < n \wedge a[r] = x))$

The verification conditions A, B_1, B_2, C have to be proved.

The Verification Conditions

```
newcontext
  "linsearch";

% declaration
% of arrays
...

a: ARR;
olda: ARR;
x: ELEM;
oldx: ELEM;
i: NAT;
n: NAT;
r: INT;
```

```
Input: BOOLEAN = olda = a AND oldx = x AND
  n = length(a) AND i = 0 AND r = -1;

Output: BOOLEAN = a = olda AND
  ((r = -1 AND
     (FORALL(j:NAT): j < length(a) =>
        get(a,j) /= x)) OR
   (0 <= r AND r < length(a) AND get(a,r) = x AND
     (FORALL(j:NAT):
       j < r => get(a,j) /= x)));

Invariant: (ARR, ELEM, NAT, NAT, INT) -> BOOLEAN =
  LAMBDA(a: ARR, x: ELEM, i: NAT, n: NAT, r: INT):
    olda = a AND oldx = x AND
    n = length(a) AND i <= n AND
    (FORALL(j:NAT): j < i => get(a,j) /= x) AND
    (r = -1 OR (r = i AND i < n AND get(a,r) = x));
...
```

The Verification Conditions (Contd)

```
...

A: FORMULA
  Input => Invariant(a, x, i, n, r);

B1: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) = x
    => Invariant(a,x,i,n,i);

B2: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) /= x
    => Invariant(a,x,i+1,n,r);

C: FORMULA
  Invariant(a, x, i, n, r) AND NOT(i < n AND r = -1)
    => Output;
```


The Proofs

A:
```
[bca]: expand Input, Invariant
  [fuo]: scatter
    [bxg]: proved (CVCL)
```
(2 user actions)

B1:
```
[p1b]: expand Invariant
  [lf6]: proved (CVCL)
```
(1 user action)

B2:
```
[q1b]: expand Invariant in 6kv
  [slx]: scatter
    [a1y]: auto
      [cch]: proved (CVCL)
    [b1y]: proved (CVCL)
    [c1y]: proved (CVCL)
    [d1y]: proved (CVCL)
    [e1y]: proved (CVCL)
```
(3 user actions)

C:
```
[dca]: expand Invariant, Output in zfg
  [tvy]: scatter
    [dcu]: auto
      [t4c]: proved (CVCL)
    [ecu]: split pkg
      [kel]: proved (CVCL)
      [lel]: scatter
        [lvn]: auto
          [lap]: proved (CVCL)
    [fcu]: auto
      [blt]: proved (CVCL)
    [gcu]: proved (CVCL)
```
(6 user actions)

1. The Hoare Calculus

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Termination

Hoare rules for **loop** and **while** are replaced as follows:

$$\{\text{false}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{I \Rightarrow t \geq 0 \quad \{I \wedge b \wedge t = N\}\ c\ \{I \wedge t < N\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \wedge \neg b\}}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of c starts in a state where P holds, then execution terminates in a state where Q holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The **loop** command thus does not satisfy total correctness.
- Termination term t (type-checked to denote an integer).
  - Becomes smaller by every iteration of the loop.
  - But does not become negative.
  - Consequently, the loop must eventually terminate.

    The initial value of t limits the number of loop iterations.

Any *well-founded ordering* may be used for the domain of t.

Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$

$$\frac{\begin{array}{c}(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I \quad I \Rightarrow n - i + 1 \geq 0 \\ \{I \wedge i \leq n \wedge n - i + 1 = N\}\ s := s + i; i := i + 1\ \{I \wedge n - i + 1 < N\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{n \geq 0 \wedge i = 1 \wedge s = 0\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s + i; i := i + 1)\ \{s = \sum_{j=1}^{n} j\}}$$

In practice, termination is easy to show (compared to partial correctness).

Weakest Preconditions for Loops

$\text{wp}(\textbf{loop}, Q) = \text{false}$
$\text{wp}(\textbf{while}\ b\ \textbf{do}\ c, Q) = L_0(Q) \vee L_1(Q) \vee L_2(Q) \vee \ldots$

$L_0(Q) = \text{false}$
$L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$

- New interpretation
  - Weakest precondition that ensures that the loop terminates in a state in which Q holds, unless it aborts.
- New interpretation of $L_i(Q)$
  - Weakest precondition that ensures that the loop terminates after less than i iterations in a state in which Q holds, unless it aborts.
- Preserves property: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$
  - Now for total correctness interpretation of Hoare calculus.
- Preserves alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

  $\text{if}_0 = \textbf{loop}$
  $\text{if}_{i+1} = \textbf{if}\ b\ \textbf{then}\ (c; \text{if}_i)$

Example

$\text{wp}(\textbf{while}\ i < n\ \textbf{do}\ i := i + 1, Q)$

$$\begin{aligned}
L_0(Q) &= \text{false} \\
L_1(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ wp(i := i + 1, L_0(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ \text{false}) \\
&\Leftrightarrow i \not< n \wedge Q \\
L_2(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ wp(i := i + 1, L_1(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad i < n \Rightarrow (i + 1 \not< n \wedge Q[i + 1/i])) \\
L_3(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ \text{wp}(i := i + 1, L_2(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow \ ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge \\
&\qquad\qquad (i + 1 < n \Rightarrow (i + 2 \not< n \wedge Q[i + 2/i]))))
\end{aligned}$$

...

Weakest Preconditions for Loops

- Sequence $L_i(Q)$ is now monotonically decreasing in strength:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q)$.
- The weakest precondition is the "greatest lower bound":
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow \text{wp}(\textbf{while}\ b\ \textbf{do}\ c, Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : L_i(Q) \Rightarrow P) \Rightarrow (\text{wp}(\textbf{while}\ b\ \textbf{do}\ c, Q) \Rightarrow P)$.
- We can only compute a stronger approximation $L_i(Q)$.
  - $L_i(Q) \Rightarrow \text{wp}(\textbf{while}\ b\ \textbf{do}\ c, Q)$.
- We want to prove $\{P\}\ c\ \{Q\}$.
  - It suffices to prove $P \Rightarrow \text{wp}(\textbf{while}\ b\ \textbf{do}\ c, Q)$.
  - It thus also suffices to prove $P \Rightarrow L_i(Q)$.
  - If proof fails, we may try the easier proof $P \Rightarrow L_{i+1}(Q)$

However, verifications are typically not successful with any finite approximation of the weakest precondition.

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Abortion

New rules to prevent abortion.

$$\{\text{false}\}\ \textbf{abort}\ \{\text{true}\}$$
$$\{Q[e/x] \wedge D(e)\}\ x := e\ \{Q\}$$
$$\{Q[a[i \mapsto e]/a] \wedge D(e) \wedge D(i) \wedge 0 \leq i < \text{length}(a)\}\ a[i] := e\ \{Q\}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of c starts in a state, in which property P holds, then it does not abort and eventually terminates in a state in which Q holds.
- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

$D(e)$ makes sure that every subexpression of e is well defined.

Definedness of Expressions

$D(0) = \text{true}.$
$D(1) = \text{true}.$
$D(x) = \text{true}.$
$D(a[i]) = D(i) \wedge 0 \leq i < \text{length}(a).$
$D(e_1 + e_2) = D(e_1) \wedge D(e_2).$
$D(e_1 * e_2) = D(e_1) \wedge D(e_2).$
$D(e_1/e_2) = D(e_1) \wedge D(e_2) \wedge e_2 \neq 0.$
$D(\text{true}) = \text{true}.$
$D(\text{false}) = \text{true}.$
$D(\neg b) = D(b).$
$D(b_1 \wedge b_2) = D(b_1) \wedge D(b_2).$
$D(b_1 \vee b_2) = D(b_1) \wedge D(b_2).$
$D(e_1 < e_2) = D(e_1) \wedge D(e_2).$
$D(e_1 \leq e_2) = D(e_1) \wedge D(e_2).$
$D(e_1 > e_2) = D(e_1) \wedge D(e_2).$
$D(e_1 \geq e_2) = D(e_1) \wedge D(e_2).$

Assumes that expressions have already been type-checked.

Abortion

Slight modification of existing rules.

$$\frac{P \Rightarrow D(b)\quad \{P \wedge b\}\ c_1\ \{Q\}\quad \{P \wedge \neg b\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{P \Rightarrow D(b)\quad \{P \wedge b\}\ c\ \{Q\}\quad (P \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

$$\frac{I \Rightarrow (t \geq 0 \wedge D(b))\quad \{I \wedge b \wedge t = N\}\ c\ \{I \wedge t < N\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \wedge \neg b\}}$$

Expressions must be defined in any context.

Abortion

Similar modifications of weakest preconditions.

$$
\begin{aligned}
&\mathrm{wp}(\mathbf{abort}, Q) = \text{false} \\
&\mathrm{wp}(x := e, Q) = Q[e/x] \land D(e) \\
&\mathrm{wp}(\mathbf{if}\ b\ \mathbf{then}\ c_1\ \mathbf{else}\ c_2, Q) = \\
&\quad D(b) \land (b \Rightarrow \mathrm{wp}(c_1, Q)) \land (\neg b \Rightarrow \mathrm{wp}(c_2, Q)) \\
&\mathrm{wp}(\mathbf{if}\ b\ \mathbf{then}\ c, Q) = D(b) \land (b \Rightarrow \mathrm{wp}(c, Q)) \land (\neg b \Rightarrow Q) \\
&\mathrm{wp}(\mathbf{while}\ b\ \mathbf{do}\ c, Q) = (L_0(Q) \lor L_1(Q) \lor L_2(Q) \lor \ldots)
\end{aligned}
$$

$$
\begin{aligned}
&L_0(Q) = \text{false} \\
&L_{i+1}(Q) = D(b) \land (\neg b \Rightarrow Q) \land (b \Rightarrow \mathrm{wp}(c, L_i(Q)))
\end{aligned}
$$

$\mathrm{wp}(c, Q)$ now makes sure that the execution of c does not abort but eventually terminates in a state in which Q holds.

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Procedure Specifications

global g;
requires Pre;
ensures $Post$;
$o := p(i)\ \{\ c\ \}$

- Specification of a procedure p implemented by a command c.
  - Input parameter i, output parameter o, global variable g.
    - Command c may read/write i, o, and g.
  - Precondition Pre (may refer to i, g).
  - Postcondition $Post$ (may refer to i, o, g, g_0).
    - g_0 denotes the value of g before the execution of p.
- Proof obligation

$$\{Pre \land i_0 = i \land g_0 = g\}\ c\ \{Post[i_0/i]\}$$

Proof of the correctness of the implementation of a procedure with respect to its specification.

Example

- Procedure specification:

  global g
  requires $g \geq 0 \land i > 0$
  ensures $g_0 = g \cdot i + o \land 0 \leq o < i$
  $o := p(i)\ \{\ o := g\%i;\ g := g/i\ \}$

- Proof obligation:

$$
\begin{aligned}
&\{g \geq 0 \land i > 0 \land i_0 = i \land g_0 = g\} \\
&o := g\%i;\ g := g/i \\
&\{g_0 = g \cdot i_0 + o \land 0 \leq o < i_0\}
\end{aligned}
$$

A procedure that divides g by i and returns the remainder.

Procedure Calls

A call of p provides actual input argument e and output variable x.

$$x := p(e)$$

Similar to assignment statement; we thus first give an alternative (equivalent) version of the assignment rule.

- Original:

$$\begin{array}{c}\{D(e) \wedge Q[e/x]\} \\ x := e \\ \{Q\}\end{array}$$

- Alternative:

$$\begin{array}{c}\{D(e) \wedge \forall x' : x' = e \Rightarrow Q[x'/x]\} \\ x := e \\ \{Q\}\end{array}$$

The new value of x is given name x' in the precondition.

Procedure Calls

From this, we can derive a rule for the correctness of procedure calls.

$$\begin{array}{c}\{D(e) \wedge Pre[e/i] \wedge \\ \forall x', g' : Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]\} \\ x := p(e) \\ \{Q\}\end{array}$$

- $Pre[e/i]$ refers to the values of the actual argument e (rather than to the formal parameter i).
- x' and g' denote the values of the vars x and g after the call.
- $Post[\ldots]$ refers to the argument values before and after the call.
- $Q[x'/x, g'/g]$ refers to the argument values after the call.

Modular reasoning: rule only relies on the *specification* of p, not on its implementation.

Corresponding Predicate Transformers

$$\begin{array}{l}\mathrm{wp}(x = p(e), Q) = \\ \quad D(e) \wedge Pre[e/i] \wedge \\ \quad \forall x', g' : \\ \qquad Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]\end{array}$$

$$\begin{array}{l}\mathrm{sp}(P, x = p(e)) = \\ \quad \exists x_0, g_0 : \\ \qquad P[x_0/y, g_0/g] \wedge \\ \qquad (Pre[e[x_0/x, g_0/g]/i, g_0/g] \Rightarrow Post[e[x_0/x, g_0/g]/i, x/o])\end{array}$$

Explicit naming of old/new values required.

Example

- Procedure specification:

$$\begin{array}{l}\text{global } g \\ \text{requires } g \geq 0 \wedge i > 0 \\ \text{ensures } g_0 = g \cdot i + o \wedge 0 \leq o < i \\ o = p(i)\ \{\ o := g\%i;\ \ g := g/i\ \}\end{array}$$

- Procedure call:

$$\begin{array}{l}\{g \geq 0 \wedge g = N \wedge b \geq 0\} \\ x = p(b+1) \\ \{g \cdot (b+1) \leq N < (g+1) \cdot (b+1)\}\end{array}$$

- To be proved:

$$\begin{array}{l}g \geq 0 \wedge g = N \wedge b \geq 0 \Rightarrow \\ \quad D(b+1) \wedge g \geq 0 \wedge b+1 > 0 \wedge \\ \quad \forall x', g' : \\ \qquad g = g' \cdot (b+1) + x' \wedge 0 \leq x' < b+1 \Rightarrow \\ \qquad\quad g' \cdot (b+1) \leq N < (g'+1) \cdot (b+1)\end{array}$$