Equational Tree Transformations[1]

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[1] Common work with Symeon Bozapalidis and Zoltán Fülöp

Motivation

Engelfriet & Schmidt 1977:

- . . . In theoretical computer science there are two basic ways of describing the meaning of a *syntactical object*: *operational* and *equational*. Operational semantics is defined by some effective (eventually nondeterministic) stepwise process which, from the syntactical object, generates its meaning. Equational semantics is defined by interpreting the syntactical object as a system of equations to be solved in some space of meanings. . . .

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- *OI* (*Outside-In*) interprets the *Call-by-Name* (or *Call-by-Reference*) method of "calling procedures, functions, etc." in programming languages

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- $L \subseteq T_\Sigma(X_n)$: *tree language*

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- *REL*: the class of all relabelings

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- define a mapping $\widetilde{\delta} : T_\Sigma \to Q$ inductively as follows:
  for every $t \in T_\Sigma$:

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  - $T \subseteq Q$ the *final state set*
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Example

$\sigma \in \Sigma_3,\ \delta \in \Delta_2,\ (s,t) = (\sigma(x_1, x_1, x_3), \delta(x_3, x_1)),$
$R_1 = \{(s_1, t_1), (s_1', t_1')\},\ R_2 = \emptyset,\ R_3 = \{(s_3, t_3)\}$

$$(s,t)\,[R_1, R_2, R_3]_{[IO]} = \{(\sigma(s_1, s_1, s_3), \delta(t_3, t_1)), (\sigma(s_1', s_1', s_3), \delta(t_3, t_1'))\}$$

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Systems of equations of tree transformations

Definition

A system of equations of tree transformations over Σ and Δ is a system

$$(\mathrm{E}) \qquad x_i = R_i,\ 1 \leq i \leq n,$$

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u-equational tree transformations

Definition

$S \subseteq T_\Sigma \times T_\Delta$ is *u-equational* (u-[IO], OI) if it is the union of some components of the least u-solution of a system of equations of tree transformations over Σ and Δ.

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- $lt\text{-}EQUT_u,\ \ vs\text{-}EQUT_u,\ \ rl\text{-}EQUT_u$
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Abstraction to algebras

- A Σ-*algebra*

$$\mathcal{A} = (A, \Sigma^{\mathcal{A}})$$

where A is a nonempty set, called the domain set of $\mathcal{A}$, and $\Sigma^{\mathcal{A}} = (\sigma^{\mathcal{A}} \mid \sigma \in \Sigma)$ such that $\forall k \geq 0$ and $\sigma \in \Sigma_k$, we have $\sigma^{\mathcal{A}} : A^k \to A$

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  - if $s = \sigma(s_1, \ldots, s_k)$ for $k \geq 0$ and $s_1, \ldots, s_k \in T_\Sigma(X_n)$, then let $|s_1|_{x_i} = \lambda_{1,i}, \ldots, |s_k|_{x_i} = \lambda_{k,i}$ and let $\mathbf{a}^{(1,i)}, \ldots, \mathbf{a}^{(k,i)}$ be the unique decomposition of the vector $\mathbf{a}^{(i)}$ into components of dimensional $\lambda_{1,i}, \ldots, \lambda_{k,i}$, respectively, $\forall 1 \leq i \leq n$, $(\lambda_i = \lambda_{1,i} + \ldots + \lambda_{k,i})$
    $s\left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\right]_{\mathcal{A}} =$
    $\sigma^{\mathcal{A}}\left(s_1\left[\mathbf{a}^{(1,1)}, \ldots, \mathbf{a}^{(1,n)}\right]_{\mathcal{A}}, \ldots, s_k\left[\mathbf{a}^{(k,1)}, \ldots, \mathbf{a}^{(k,n)}\right]_{\mathcal{A}}\right).$

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- *evaluation of (s, t) at $(v^{(1)}, \ldots, v^{(n)})$ in $(\mathcal{A}, \mathcal{B})$:*

Pairs of algebras: evaluations

- $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, $\mathcal{B} = (B, \Delta^{\mathcal{B}})$: arbitrary algebras
- $(s, t) \in T_\Sigma(X_n) \times T_\Delta(X_n)$ with
  $|s|_{x_i} = \lambda_i, \quad |t|_{x_i} = \mu_i, \quad m_i = \max\{\lambda_i, \mu_i\} \quad \forall 1 \leq i \leq n$
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  $\left(s \left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)} \right]_{\mathcal{A}}, t \left[\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(n)} \right]_{\mathcal{B}} \right)$

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- $R \subseteq T_{\Sigma}(X_n) \times T_{\Delta}(X_n)$, u=*[IO], OI*
- $R[U_1, \ldots, U_n]_{(\mathcal{A},\mathcal{B}),u} = \bigcup_{(s,t)\in R}(s, t)[U_1, \ldots, U_n]_{(\mathcal{A},\mathcal{B}),u}$

Systems of equations: solutions in pairs of algebras

A system of equations of tree transformations over Σ and Δ

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- Existence of the least u-solution of (E) in $(\mathcal{A}, \mathcal{B})$: as in tree transformations case
- $U \in \mathcal{P}(A \times B)$ is *u-equational* if it is the union of some components of the least u-solution in $(\mathcal{A}, \mathcal{B})$ of a system of equations of tree transformations

Mezei-Wright type result

Theorem

Let $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$ *and* $\mathcal{B} = (B, \Delta^{\mathcal{B}})$ *be arbitrary algebras and u=[IO], OI. A relation* $U \subseteq A \times B$ *is u-equational iff there exists a u-equational tree transformation* $S \subseteq T_\Sigma \times T_\Delta$ *such that* $H_{(\mathcal{A},\mathcal{B})}(S) = U$*, where* $H_{(\mathcal{A},\mathcal{B})}((s,t)) = (H_{\mathcal{A}}(s), H_{\mathcal{B}}(t))$ *for every* $(s,t) \in T_\Sigma \times T_\Delta$.

Thank you