QUANTITATIVE REWRITING LOGIC

Work in progress

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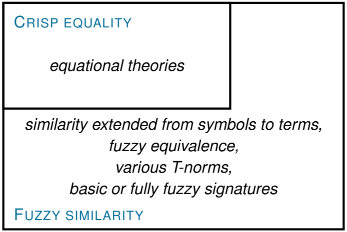
Quantitative approximations of equality

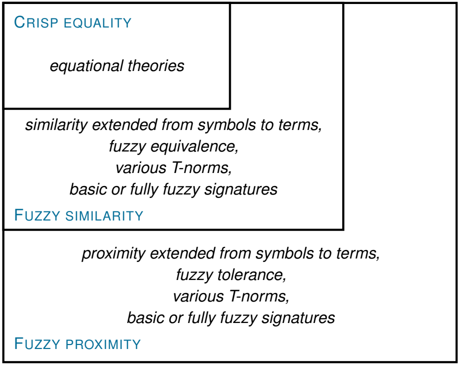
CRISP EQUALITY

equational theories

Quantitative approximations of equality



Quantitative approximations of equality



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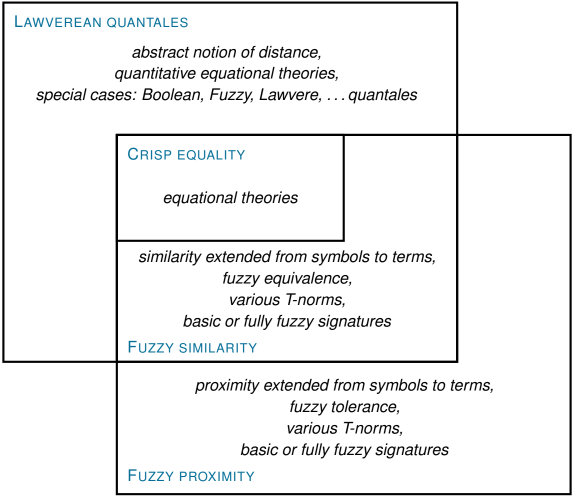

Rewriting Logic

A framework for modeling and reasoning about dynamic systems.

Treats computation and deduction in a uniform way.

Introduced by José Meseguer:

> J. Meseguer: Conditional rewriting logic as a united model of concurrency. *Theor. Comput. Sci*. 96(1), 73–155 (1992).

Rewriting Logic

The idea:

- system states are represented as terms,
- structural properties of states are described by equations,
- state transitions are specified by rewrite rules.

A rewriting step could be understood as

- a (concurrent) computational transformation, or
- an inference step in a logical system.

Rewriting Logic

Applications in modeling, specifying, and reasoning about systems:

- defining formal semantics of programming languages,
- specifying and verifying concurrent and distributed systems,
- modeling cyber-physical and real-time systems,
- analyzing security protocols,
- etc.

Maude: an executable realization of Rewriting Logic.

https://maude.cs.illinois.edu/

Towards Quantitative Rewriting Logic

Motivation: Make the power of Rewriting Logic available for modeling systems that exhibit resource-sensitive, approximate, probabilistic, or otherwise graded behavior.

Idea: Equip both equations and rewrite rules with quantitative information, and carry out deduction and computation relative to this quantitative structure.

Unified perspective on quantitative deduction and computation.

First steps towards this goal.

Towards Quantitative Rewriting Logic

We are given sets of quantitative equations E^{ϵ} and rewrite rules R^{ϵ} (simple case: no sorts, no conditions).

Development steps:

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1. **Computational view.** Define the notion of quantitative rewriting modulo quantitative equations: the $\to_{R^\epsilon/E^\epsilon}$ relation.

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1. **Computational view.** Define the notion of quantitative rewriting modulo quantitative equations: the $\rightarrow_{R^\epsilon/E^\epsilon}$ relation.
2. **Deduction view.** Define quantitative inference relation $\longrightarrow_{R^\epsilon/E^\epsilon}$ and relate it to the reflexive-transitive closure $\rightarrow^*_{R^\epsilon/E^\epsilon}$ of $\rightarrow_{R^\epsilon/E^\epsilon}$.

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1. **Computational view.** Define the notion of quantitative rewriting modulo quantitative equations: the $\to_{R^\epsilon/E^\epsilon}$ relation.
2. **Deduction view.** Define quantitative inference relation $\longrightarrow_{R^\epsilon/E^\epsilon}$ and relate it to the reflexive-transitive closure $\to^*_{R^\epsilon/E^\epsilon}$ of $\to_{R^\epsilon/E^\epsilon}$.
3. **Practical consideration.** Relax the impractical $\to_{R^\epsilon/E^\epsilon}$ to a more feasible $\to_{R^\epsilon,E^\epsilon}$ relation, involving matching modulo E^ϵ. Study conditions under which $\to^*_{R^\epsilon,E^\epsilon}$ can “imitate” $\longrightarrow_{R^\epsilon/E^\epsilon}$.

Towards Quantitative Rewriting Logic

We are given sets of quantitative equations E^ϵ and rewrite rules R^ϵ (simple case: no sorts, no conditions).

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1. **Computational view.** Define the notion of quantitative rewriting modulo quantitative equations: the $\to_{R^\epsilon/E^\epsilon}$ relation.
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3. **Practical consideration.** Relax the impractical $\to_{R^\epsilon/E^\epsilon}$ to a more feasible $\to_{R^\epsilon,E^\epsilon}$ relation, involving matching modulo E^ϵ. Study conditions under which $\to^*_{R^\epsilon,E^\epsilon}$ can “imitate” $\longrightarrow_{R^\epsilon/E^\epsilon}$.
4. **Make $\to_{R^\epsilon,E^\epsilon}$ work.** Define matching modulo E^ϵ.

Towards Quantitative Rewriting Logic

Summary of the involved relations:

- Quantitative equality $=_{E^\epsilon}$
- Quantitative rewriting $\to_{R^\epsilon}$
- Quantitative rewriting modulo, the “slash” relation, $\to_{R^\epsilon/E^\epsilon}$
- Quantitative rewriting modulo, the “comma” relation, $\to_{R^\epsilon,E^\epsilon}$
- Quantitative provability $\longrightarrow_{R^\epsilon/E^\epsilon}$

Their crisp counterparts: without the superscript $^\epsilon$.

Towards Quantitative Rewriting Logic

A general picture, involving $=_{E^\epsilon}$, $\to_{R^\epsilon}$, $\to_{R^\epsilon/E^\epsilon}$, and their crisp counterparts:

Relation	**Trivial** R	**Crisp** R	**Arbitrary** R^ϵ
Trivial E	syntactic equality $=$	standard rewriting $\to_R$	quantitative rewriting $\to_{R^\epsilon}$
Crisp E	equality modulo standard theory $=_E$	standard rewriting modulo standard theory $\to_{R/E}$	quantitative rewriting modulo standard theory $\to_{R^\epsilon/E}$
Arbitrary E^ϵ	equality modulo quantitative theory $=_{E^\epsilon}$	standard rewriting modulo quantitative theory $\to_{R/E^\epsilon}$	quantitative rewriting modulo quantitative theory $\to_{R^\epsilon/E^\epsilon}$

DETAILED VIEW

Quantitative information

Quantitative information is represented at an abstract level by means of quantales.

The approach to quantitative equational theories introduced in

> F. Gavazzo, C. Di Florio: Elements of Quantitative Rewriting. *Proc. ACM Program. Lang. 7(POPL)*, 1832–1863 (2023).

Quantales

A quantale $\mathbb{\Omega} = (\Omega, \precsim, \otimes, \kappa)$: an algebraic structure, where

- $(\Omega, \kappa, \otimes)$ is a monoid,
- $(\Omega, \precsim)$ is a complete lattice (with join $\vee$ and meet $\wedge$),
- the following distributivity laws are satisfied:
  $\delta \otimes \left(\bigvee_{i \in I} \varepsilon_i\right) = \bigvee_{i \in I}(\delta \otimes \varepsilon_i)$ and
  $\left(\bigvee_{i \in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i \in I}(\varepsilon_i \otimes \delta)$.

Terminology:

- Ω: carrier set
- $\precsim$: order
- κ: unit
- $\otimes$: tensor

Quantale examples

Correspondence between quantales Ω (generic), $\mathbb{2}$ (Boolean), $\mathbb{I}$ (fuzzy), $\mathbb{L}$ (Lawvere), $\mathbb{L}^{\max}$ (strong Lawvere).

	Ω	$\mathbb{2}$	$\mathbb{I}$	$\mathbb{L}$	$\mathbb{L}^{\max}$
Carrier	Ω	$\{0,1\}$	$[0,1]$	$[0,\infty]$	$[0,\infty]$
Order	$\precsim$	$\leqslant$	$\leqslant$	$\geqslant$	$\geqslant$
Unit	κ	1	1	0	0
Tensor	$\otimes$	$\wedge$	left-cont. T-norm	$+$	$\max$
Join	$\vee$	$\sup$	$\sup$	$\inf$	$\inf$
Meet	$\wedge$	$\inf$	$\inf$	$\sup$	$\sup$

Quantales: terminology

$\top$, $\bot$: the top and bottom elements of a quantale.

- integral quantale: $\kappa = \top$
- commutative quantale: $\otimes$ is commutative
- nontrivial quantale: $\kappa \neq \bot$
- cointegral quantale: $\varepsilon \otimes \delta = \bot$ implies $\varepsilon = \bot$ or $\delta = \bot$
- idempotent element: $\varepsilon \in \Omega$ such that $\varepsilon \otimes \varepsilon = \varepsilon$
- idempotent quantale: every element is idempotent

Quantales: terminology

Lawvereian quantales: integral, commutative, nontrivial, and cointegral quantales.

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$\mathbb{L}$ is not an idempotent quantale.

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$\mathbb{I}$ is idempotent for the $\min$ t-norm, but not for the product and Łukasiewicz t-norms.

$\mathbb{L}$ is not an idempotent quantale.

Only Lawvereian quantales are considered.

Quantitative equational theory: $=_{E^\epsilon}$

Given:

- a quantale $\mathbb{\Omega} = (\Omega, \precsim, \kappa, \otimes)$,
- a set of terms $\mathcal{T}$,
- a set of triples $E^\epsilon \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ ($\mathbb{\Omega}$-equalities).

Notation: $\varepsilon \Vdash t \approx_{E^\epsilon} s$ for $(t, \varepsilon, s) \in E^\epsilon$.

Intuition: E^ϵ is a set of axioms that induces an equational theory.

Quantitative equational theory: $=_{E^\epsilon}$

Given:

- a quantale $\Omega = (\Omega, \precsim, \kappa, \otimes)$,
- a set of terms $\mathcal{T}$,
- a set of triples $E^\epsilon \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ (Ω-equalities).

Notation: $\varepsilon \Vdash t \approx_{E^\epsilon} s$ for $(t, \varepsilon, s) \in E^\epsilon$.

Intuition: E^ϵ is a set of axioms that induces an equational theory.

Quantitative equational theory induced by E^ϵ (wrt Ω): a ternary relation $=_{E^\epsilon} \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ defined by the rules on the next slide.

Informally, $\varepsilon \Vdash t =_{E^\epsilon} s$ is read as

- "t and s are at most ε-apart modulo E^ϵ" or
- "t and s are equal modulo E^ϵ with degree ε".

Quantitative equational theory: $=_{E^\epsilon}$

The rules define (a non-expansive version of) the quantitative equational theory $=_{E^\epsilon}$.

$$(\mathsf{Ax}^\epsilon_=)\ \frac{\varepsilon \Vdash t \approx s \in E^\epsilon}{\varepsilon \Vdash t =_{E^\epsilon} s} \quad (\mathsf{Refl}^\epsilon_=)\ \frac{}{\kappa \Vdash t =_{E^\epsilon} t} \quad (\mathsf{Symm}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash s =_{E^\epsilon} t}$$

$$(\mathsf{Trans}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s \quad \delta \Vdash s =_{E^\epsilon} r}{\varepsilon \otimes \delta \Vdash t =_{E^\epsilon} r} \qquad (\mathsf{Subst}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash t\sigma =_{E^\epsilon} s\sigma}$$

$$(\mathsf{NExp}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t_1 =_{E^\epsilon} s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_{E^\epsilon} s_n}{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \Vdash f(t_1, \ldots, t_n) =_{E^\epsilon} f(s_1, \ldots, s_n)}$$

$$(\mathsf{Ord}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t =_{E^\epsilon} s}$$

$$(\mathsf{Join}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t =_{E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_{E^\epsilon} s}$$

Quantitative equality

Example

Take the Lawvere quantale $\mathbb{L}$.

Let $E^\epsilon = \{0 \Vdash f(x,y) \approx f(y,x),\ 1 \Vdash a \approx b,\ 2 \Vdash h(x) \approx p(x)\}$.

Then $4 \Vdash f(a,h(a)) =_{E^\epsilon} f(p(b),b)$, because

(1)	$0 \Vdash f(x,y) =_{E^\epsilon} f(y,x)$		by $(\mathsf{Ax}^\epsilon_=)$.
(2)	$1 \Vdash a =_{E^\epsilon} b$		by $(\mathsf{Ax}^\epsilon_=)$.
(3)	$2 \Vdash h(x) =_{E^\epsilon} p(x)$		by $(\mathsf{Ax}^\epsilon_=)$.
(4)	$1 \Vdash h(a) =_{E^\epsilon} h(b)$	from (2)	by $(\mathsf{NExp}^\epsilon_=)$.
(5)	$2 \Vdash h(b) =_{E^\epsilon} p(b)$	from (3)	by $(\mathsf{Subst}^\epsilon_=)$.
(6)	$3 \Vdash h(a) =_{E^\epsilon} p(b)$	from (4), (5)	by $(\mathsf{Trans}^\epsilon_=)$.
(7)	$4 \Vdash f(h(a),a) =_{E^\epsilon} f(p(b),b)$	from (6), (2)	by $(\mathsf{NExp}^\epsilon_=)$.
(8)	$0 \Vdash f(a,h(a)) =_{E^\epsilon} f(h(a),a)$	from (1)	by $(\mathsf{Subst}^\epsilon_=)$.
(9)	$4 \Vdash f(a,h(a)) =_{E^\epsilon} f(p(b),b)$	from (8), (7)	by $(\mathsf{Trans}^\epsilon_=)$.

Quantitative rewriting relation: $\to_{R^\epsilon}$

A quantitative rewrite rule: an oriented quantitative equation, written $\varepsilon \Vdash t \mapsto s$.

The rules define (a non-expansive version of) the quantitative rewrite relation $\to_{R^\epsilon}$, generated by a set R^ϵ of quantitative rewrite rules:

$$\frac{(\varepsilon \Vdash t \mapsto s) \in R^\epsilon}{\varepsilon \Vdash C[t\sigma] \to_{R^\epsilon} C[s\sigma]} \qquad \frac{\varepsilon \Vdash t \to_{R^\epsilon} s \qquad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon} s}$$

$$\frac{\varepsilon_1 \Vdash t \to_{R^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t \to_{R^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t \to_{R^\epsilon} s}$$

The symbol C stands for an arbitrary context.

Quantitative rewriting

Example

Take the Lawvere quantale $\mathbb{L}$.

Let $R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x), h(b)) \mapsto g(x, x)\}$.

Then $10 \Vdash f(h(a), h(b)) \to^*_{R^\epsilon} g(c, c)$, because

$4 \Vdash f(h(a), h(b)) \to_{R^\epsilon} g(a, a)$ by $4 \Vdash f(h(x), h(b)) \mapsto g(x, x)$.

$3 \Vdash g(a, a) \to_{R^\epsilon} g(a, c)$ by $3 \Vdash a \mapsto c$.

$3 \Vdash g(a, c) \to_{R^\epsilon} g(c, c)$ by $3 \Vdash a \mapsto c$.

But also $7 \Vdash f(h(a), h(b)) \to^*_{R^\epsilon} g(c, c)$, because

$3 \Vdash f(h(a), h(b)) \to_{R^\epsilon} f(h(c), h(b))$ by $3 \Vdash a \mapsto c$.

$4 \Vdash f(h(c), h(b)) \mapsto g(c, c)$ by $4 \Vdash f(h(x), h(b)) \mapsto g(x, x)$

THE $\to_{R^\epsilon/E^\epsilon}$ RELATION: COMPUTATION VIA QUANTITATIVE REWRITING MODULO

Quantitative rewriting modulo: $\to_{R^\epsilon/E^\epsilon}$

Given E^ϵ and R^ϵ, the quantitative rewrite relation modulo quantitative equational theory $\to_{R^\epsilon/E^\epsilon}$ is defined by the rules:

$$\frac{\varepsilon_1 \Vdash t =_{E^\epsilon} t' \qquad \delta \Vdash t' \to_{R^\epsilon} s' \qquad \varepsilon_2 \Vdash s' =_{E^\epsilon} s}{\varepsilon_1 \otimes \delta \otimes \varepsilon_2 \Vdash t \to_{R^\epsilon/E^\epsilon} s}$$

$$\frac{\varepsilon \Vdash t \to_{R^\epsilon/E^\epsilon} s \qquad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon/E^\epsilon} s}$$

$$\frac{\varepsilon_1 \Vdash t \to_{R^\epsilon/E^\epsilon} s \qquad \cdots \qquad \varepsilon_n \Vdash t \to_{R^\epsilon/E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t \to_{R^\epsilon/E^\epsilon} s}$$

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$R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x),h(b)) \mapsto g(x,x)\}$.

Then $10 \Vdash f(p(b),h(a)) \to^*_{R^\epsilon/E^\epsilon} g(b,c)$, because it follows from (1), (2):

$(1)\quad 7 \Vdash f(p(b),h(a)) \to_{R^\epsilon/E^\epsilon} g(b,a),$ because

$$2 \Vdash f(p(b),h(a)) =_{E^\epsilon} f(h(a),h(b))$$
$$4 \Vdash f(h(a),h(b)) \to_{R^\epsilon} g(a,a)$$
$$1 \Vdash g(a,a) =_{E} g(b,a)$$

$(2)\quad 3 \Vdash g(b,a) \to_{R^\epsilon/E^\epsilon} g(b,c),$ because

$$0 \Vdash g(b,a) =_{E^\epsilon} g(b,a)$$
$$3 \Vdash g(b,a) \to_{R^\epsilon} g(b,c)$$
$$0 \Vdash g(b,c) =_{E^\epsilon} g(b,c)$$

DEDUCTION

Deduction in QRL: $\longrightarrow_{R^\epsilon/E^\epsilon}$

Simple version: no sorts, no conditions, no “rewriting-under-feet”.

$$(\mathsf{Refl}^{\epsilon}_{\mathsf{RL}})\ \frac{}{\kappa \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} t} \qquad (\mathsf{Repl}^{\epsilon}_{\mathsf{RL}})\ \frac{(\rho \Vdash t \mapsto s) \in R^\epsilon}{\rho \Vdash t\sigma \longrightarrow_{R^\epsilon/E^\epsilon} s\sigma}$$

$$(\mathsf{Eq}^{\epsilon}_{\mathsf{RL}})\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} t' \qquad \delta \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s' \qquad \varepsilon_2 \Vdash s' =_{E^\epsilon} s}{\varepsilon_1 \otimes \delta \otimes \varepsilon_2 \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'}$$

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$$(\mathsf{Trans}^\epsilon_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s \qquad \delta_2 \Vdash s \longrightarrow_{R^\epsilon/E^\epsilon} r}{\delta_1 \otimes \delta_2 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} r}$$

$$(\mathsf{NExp}^\epsilon_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t_1 \longrightarrow_{R^\epsilon/E^\epsilon} s_1 \quad \cdots \quad \delta_n \Vdash t_n \longrightarrow_{R^\epsilon/E^\epsilon} s_n}{\delta_1 \otimes \cdots \otimes \delta_n \Vdash f(t_1, \ldots, t_n) \longrightarrow_{R^\epsilon/E^\epsilon} f(s_1, \ldots, s_n)}$$

Deduction in QRL: $\longrightarrow_{R^\epsilon/E^\epsilon}$

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$$(\mathsf{Eq}^\epsilon_{\mathsf{RL}})\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} t' \qquad \delta \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s' \qquad \varepsilon_2 \Vdash s' =_{E^\epsilon} s}{\varepsilon_1 \otimes \delta \otimes \varepsilon_2 \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'}$$

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Deduction in QRL: $\longrightarrow_{R^\epsilon/E^\epsilon}$

Example

Take the Lawvere quantale $\mathbb{L}$. Let

$E^\epsilon = \{0 \Vdash f(x,y) \approx f(y,x),\ \ 1 \Vdash a \approx b,\ \ 2 \Vdash h(x) \approx p(x)\}$.
$R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x),h(b)) \mapsto g(x,x)\}$.

Then $10 \Vdash f(p(b),h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(b,c)$, because it follows from (1), (2) by $(\mathsf{Trans}^\epsilon_{\mathsf{RL}})$:

(1) $7 \Vdash f(p(b),h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(b,a)$ by $(\mathsf{Eq}^\epsilon_{\mathsf{RL}})$, because

$2 \Vdash f(p(b),h(a)) =_{E^\epsilon} f(h(a),h(b))$

$4 \Vdash f(h(a),h(b)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a,a)$ by $(\mathsf{Repl}^\epsilon_{\mathsf{RL}})$

$1 \Vdash g(a,a) =_E g(b,a)$

(2) $3 \Vdash g(b,a) \longrightarrow_{R^\epsilon/E^\epsilon} g(b,c)$ by $(\mathsf{NExp}^\epsilon_{\mathsf{RL}})$, because

$0 \Vdash b \longrightarrow_{R^\epsilon/E^\epsilon} b$ by $(\mathsf{Refl}^\epsilon_{\mathsf{RL}})$

$3 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} c$ by $(\mathsf{Repl}^\epsilon_{\mathsf{RL}})$

Deduction in QRL

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$R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x), h(b)) \mapsto g(x,x)\}$.

$\longrightarrow_{R^\epsilon/E^\epsilon} \neq \to^*_{R^\epsilon/E^\epsilon}$: $1 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} b$, but $\infty \Vdash a \to^*_{R^\epsilon/E^\epsilon} b$:

(1) $1 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} b$ by ($\mathsf{Eq}^\epsilon_{\mathsf{RL}}$), because

$1 \Vdash a =_{E^\epsilon} b$

$0 \Vdash b \longrightarrow_{R^\epsilon/E^\epsilon} b$ by ($\mathsf{Refl}^\epsilon_{\mathsf{RL}}$)

$0 \Vdash b =_E b$

(2) $\infty \Vdash a \to^*_{R^\epsilon/E^\epsilon} b$, because $\infty \Vdash a \to_{R^\epsilon/E^\epsilon} b$:

$1 \Vdash a =_{E^\epsilon} b$

$\infty \Vdash b \to_{R^\epsilon} b$ because $\bigvee \emptyset = \bot = \infty$

$0 \Vdash b =_{E^\epsilon} b$

Relating $\to_{R^\epsilon/E^\epsilon}$ and $\longrightarrow_{R^\epsilon/E^\epsilon}$

Relating computation and deduction:

$$\to^*_{R^\epsilon/E^\epsilon} \quad \subseteq \quad \longrightarrow_{R^\epsilon/E^\epsilon}.$$

$$\longrightarrow_{R^\epsilon/E^\epsilon} \quad \subseteq \quad \to^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}.$$

Relating $\rightarrow_{R^\epsilon/E^\epsilon}$ and $\longrightarrow_{R^\epsilon/E^\epsilon}$

Relating computation and deduction:

$$\rightarrow^*_{R^\epsilon/E^\epsilon} \quad \subseteq \quad \longrightarrow_{R^\epsilon/E^\epsilon}.$$

$$\longrightarrow_{R^\epsilon/E^\epsilon} \quad \subseteq \quad \rightarrow^*_{R^\epsilon/E^\epsilon} \cup =_{E^\epsilon}.$$

Hence, deduction in QRL is related to quantitative rewriting modulo quantitative equational theory in the same way as their crisp counterparts do.

As an computational counterpart, $\rightarrow_{R^\epsilon/E^\epsilon}$ has no practical advantage over $\longrightarrow_{R^\epsilon/E^\epsilon}$.

Its equational rewriting step is as difficult as $(\mathsf{Eq}^\epsilon_{\mathsf{RL}})$.

THE $\rightarrow_{R^\epsilon, E^\epsilon}$ RELATION: AN ALTERNATIVE APPROACH TO COMPUTATION

Quantitative rewriting modulo: $\rightarrow_{R^\epsilon, E^\epsilon}$

Given E^ϵ and R^ϵ, an alternative quantitative rewrite relation modulo quantitative equational theory $\rightarrow_{R^\epsilon, E^\epsilon}$ is defined by the rules:

$$\frac{(\rho \Vdash t \mapsto s) \in R^\epsilon \qquad \varepsilon \Vdash t\sigma =_{E^\epsilon} r}{\rho \otimes \varepsilon \Vdash C[r] \rightarrow_{R^\epsilon, E^\epsilon} C[s\sigma]}$$

$$\frac{\varepsilon \Vdash t \rightarrow_{R^\epsilon, E^\epsilon} s \quad \varepsilon \gtrsim \delta}{\delta \Vdash t \rightarrow_{R^\epsilon, E^\epsilon} s}$$

$$\frac{\varepsilon_1 \Vdash t \rightarrow_{R^\epsilon, E^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t \rightarrow_{R^\epsilon, E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t \rightarrow_{R^\epsilon, E^\epsilon} s}$$

Quantitative rewriting modulo: $\to_{R^\epsilon,E^\epsilon}$

Example

Take the Lawvere quantale $\mathbb{L}$. Let

$E^\epsilon = \{0 \Vdash f(x,y) \approx f(y,x),\ \ 1 \Vdash a \approx b,\ \ 2 \Vdash h(x) \approx p(x)\}$.
$R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x),h(b)) \mapsto g(x,x)\}$.

Then $9 \Vdash f(p(b),h(a)) \to^*_{R^\epsilon,E^\epsilon} g(a,c)$, because it follows from (1), (2):

$$(1)\quad 6 \Vdash f(p(b),h(a)) \to_{R^\epsilon,E^\epsilon} g(a,a),\ \text{because}$$

$$2 \Vdash f(p(b),h(a)) =_{E^\epsilon} f(h(a),h(b))$$

$$f(h(a),h(b)) = f(h(x),h(b))\{x \mapsto a\}$$

$$4 \Vdash f(h(a),h(b)) \to_{R^\epsilon} g(a,a)$$

$$(2)\quad 3 \Vdash g(a,a) \to_{R^\epsilon,E^\epsilon} g(a,c),\ \text{because}$$

$$3 \Vdash a \to_{R^\epsilon} c$$

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$R^\epsilon = \{3 \Vdash a \mapsto c,\ \ 4 \Vdash f(h(x), h(b)) \mapsto g(x,x)\}$.

$\to^*_{R^\epsilon, E^\epsilon}$ is weaker than $\to^*_{R^\epsilon / E^\epsilon}$: We have

$$10 \Vdash f(p(b), h(a)) \to^*_{R^\epsilon / E^\epsilon} g(b,c)$$

but

$$\infty \Vdash f(p(b), h(a)) \to^*_{R^\epsilon, E^\epsilon} g(b,c).$$

Consequently, $\to^*_{R^\epsilon, E^\epsilon}$ is weaker than $\twoheadrightarrow_{R^\epsilon / E^\epsilon}$.

Relating $\rightarrow_{R^\epsilon,E^\epsilon}$, $\rightarrow_{R^\epsilon/E^\epsilon}$, and $\longrightarrow_{R^\epsilon/E^\epsilon}$

Relating two ways of computation and deduction:

$$\rightarrow_{R^\epsilon,E^\epsilon} \quad \subseteq \quad \rightarrow_{R^\epsilon/E^\epsilon}.$$

$$\rightarrow^*_{R^\epsilon,E^\epsilon} \quad \subseteq \quad \rightarrow^*_{R^\epsilon/E^\epsilon} \quad \subseteq \quad \longrightarrow_{R^\epsilon/E^\epsilon}.$$

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What about the inverse direction?

Coherent relations.

Quantitative coherence

The relation $\to_{R^\epsilon,E^\epsilon}$ is called strictly E^ϵ-coherent iff

$$=_{E^\epsilon};\to_{R^\epsilon,E^\epsilon}\ \subseteq\ \to_{R^\epsilon,E^\epsilon};=_{E^\epsilon}$$

Quantitative coherence

The relation $\to_{R^\epsilon,E^\epsilon}$ is called strictly E^ϵ-coherent iff

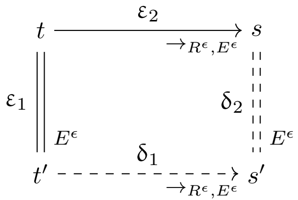
for all terms t, s, t' and for all $\varepsilon_1, \varepsilon_2 \in \Omega$, if

$$\varepsilon_1 \Vdash t' =_{E^\epsilon} t \text{ and } \varepsilon_2 \Vdash t \to_{R^\epsilon,E^\epsilon} s$$

then there exist a term s' and $\delta_1, \delta_2 \in \Omega$ such that

$$\delta_1 \Vdash t' \to_{R^\epsilon,E^\epsilon} s', \quad \delta_2 \Vdash s' =_{E^\epsilon} s, \text{ and } \varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$$

Graphically:



$$\varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$$

Relating coherent $\to_{R^\epsilon,E^\epsilon}$ and $\to_{R^\epsilon/E^\epsilon}$

If $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then

$$\to_{R^\epsilon/E^\epsilon} = \to_{R^\epsilon,E^\epsilon} ; =_{E^\epsilon}$$

Relating coherent $\to_{R^\epsilon,E^\epsilon}$ and $\to_{R^\epsilon/E^\epsilon}$

If $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then

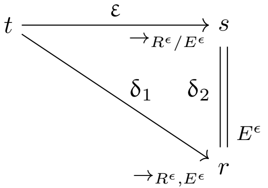
for all terms t and s and $\varepsilon \in \Omega$,

$$\varepsilon \Vdash t \to_{R^\epsilon/E^\epsilon} s,$$

iff there exist a term r and $\delta_1, \delta_2 \in \Omega$ such that

$$\delta_1 \Vdash t \to_{R^\epsilon,E^\epsilon} r, \quad \delta_2 \Vdash r =_{E^\epsilon} s, \text{ and } \varepsilon = \delta_1 \otimes \delta_2.$$

Graphically:



$$\varepsilon = \delta_1 \otimes \delta_2$$

Relating coherent $\to_{R^\epsilon,E^\epsilon}$ and $\to_{R^\epsilon/E^\epsilon}$

Example

Take the Lawvere quantale $\mathbb{L}$. Let

$$E^\epsilon = \{1 \Vdash x + y \approx y + x, \quad 2 \Vdash x + (y + z) \approx (x + y) + z\},$$
$$R^\epsilon = \{3 \Vdash a + b \mapsto a\}.$$

Then we have (since $\to_{R^\epsilon,E^\epsilon}$ is not E^ϵ-coherent):

$$3 \Vdash a + (a + b) \to_{R^\epsilon/E^\epsilon} a + a \quad \text{and} \quad 4 \Vdash a + (a + b) \to_{R^\epsilon,E^\epsilon} a + a,$$
$$6 \Vdash b + (a + a) \to_{R^\epsilon/E^\epsilon} a + a \quad \text{and} \quad \infty \Vdash b + (a + a) \to_{R^\epsilon,E^\epsilon} a + a.$$

Relating coherent $\to_{R^\epsilon,E^\epsilon}$ and $\to_{R^\epsilon/E^\epsilon}$

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However, if we extend R^ϵ with the rule $\delta \Vdash (a+b)+x \mapsto a+x$ for an arbitrary $\delta \leq 3$, then $\to_{R^\epsilon,E^\epsilon}$ will become E^ϵ-coherent and we get

$$3 + \delta \Vdash b + (a + a) \to_{R^\epsilon,E^\epsilon} a + a,$$

which, by weakening, gives

$$6 \Vdash b + (a + a) \to_{R^\epsilon,E^\epsilon} a + a.$$

Relating coherent $\rightarrow_{R^\epsilon,E^\epsilon}$, $\rightarrow_{R^\epsilon/E^\epsilon}$, and $\longrightarrow_{R^\epsilon/E^\epsilon}$

If rewrite rules can apply only on the top position (in implies strict coherence of $\rightarrow_{R^\epsilon,E^\epsilon}$), then

$$\rightarrow_{R^\epsilon/E^\epsilon} \equiv \rightarrow_{R^\epsilon,E^\epsilon};=_{E^\epsilon}.$$

If $\rightarrow_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then

$$\longrightarrow_{R^\epsilon/E^\epsilon} \equiv \rightarrow^*_{R^\epsilon,E^\epsilon};=_{E^\epsilon}.$$

Conclusion

First steps towards Quantitative Rewriting Logic.

Many good properties from the crisp variant transfer to the quantitative setting.

Challenges remain:

- generalizations:
  - from nonexpansive to graded systems, to model concurrent transformations,
  - from unconditional to conditional rewrite rules.