Graded Quantitative Narrowing

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- Classic technique for solving equational problems.
- Like term rewriting, but variables of terms may be instantiated.
- Sound and complete method for unification w.r.t. complete TRSs (Hullot 1980).

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Computed Solution: $\{n \mapsto S(Z)\}$

Quantitative equational reasoning

Example

Consider the equation $n + 1 = 3n$.
It does not have an exact solution in $\mathbb{N}$, but $\{n \mapsto 0\}$ and $\{n \mapsto 1\}$ could be considered approximate solutions.

- How can this idea (approximate solutions) be formalized? How can rewrite systems be extended to include quantitative information?

  $\rightarrow$ *Quantitative equational reasoning* (Gavazzo & Di Florio 2023)

- How can narrowing be transferred to the quantitative scenario?

  $\rightarrow$ This work.

Quantitative equational reasoning

- Equip equations with degrees to measure similarity/proximity of terms rather than just equality: $\varepsilon \Vdash t \approx s$.
- Degrees could correspond to a probability, distance in a metric space, ...
- Main requirements: **compare** and **compose** degrees.

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Definition (Quantale)

Quantale: $\mathbb{\Omega} = (\Omega, \precsim, \otimes, \kappa)$, where

- $(\Omega, \kappa, \otimes)$ is a monoid;
- $(\Omega, \precsim)$ is a complete lattice (with join $\vee$ and meet $\wedge$);
- distributivity laws hold:

$$\delta \otimes \left(\bigvee_{i \in I} \varepsilon_i\right) = \bigvee_{i \in I} (\delta \otimes \varepsilon_i), \qquad \left(\bigvee_{i \in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i \in I} (\varepsilon_i \otimes \delta)$$

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- We assume that we are working with *Lawverean* quantales, i.e. that
  1. $\otimes$ is commutative, 2. $\kappa = \top$, 3. if $\varepsilon \otimes \delta = \bot$, then either $\varepsilon = \bot$ or $\delta = \bot$, 4. $\kappa \neq \bot$.

Examples of Lawverean quantales: $\mathbb{L}$ and

Example (Lawvere quantale)

- $\mathbb{L} = ([0, \infty], \geqslant, +, 0)$.
- Note the direction of the order: 0 is the top element, ∞ is the bottom element.
- View terms as elements of metric spaces, degrees as distances.
- Read $\varepsilon \Vdash t \approx s$ as "the distance between t and s is at most ε".
- Corresponds to "Quantitative algebraic reasoning" (Mardare, Panangaden & Plotkin 2016).

Examples of Lawverean quantales: $\mathbb{L}$ and $\mathbb{I}$

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Example (Fuzzy quantales)

- $\mathbb{I} = ([0, 1], \leqslant, \otimes, 1)$, where $\otimes$ is multiplication or minimum.
- View degrees as "truth values", similar to probabilities.
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Quantitative equational reasoning (Gavazzo & Di Florio 2023) covers these (and more) frameworks.

Graded signatures (Gavazzo & Di Florio 2023)

Definition (Change of base endofunctor)

A monotone map $h\colon \Omega \to \Omega$ is a CBE if it preserves the unit, products and joins: $h(\kappa) = \kappa$, $h(\varepsilon) \otimes h(\delta) = h(\varepsilon \otimes \delta)$, and $h\left(\bigvee_{i \in I} \varepsilon_i\right) = \bigvee_{i \in I} h(\varepsilon_i)$.

CBEs can be used to describe how degrees are transformed under function applications.

Definition (Graded signature)

Graded signature $\mathcal{F}$: A set of function symbols, each endowed with a tuple $(\phi_1, \ldots, \phi_n)$ of CBEs called modal arities.
Notation: $f : (\phi_1, \ldots, \phi_n) \in \mathcal{F}$.

Definition (Grade of a term)

The grade of a position p of a term t is defined inductively via

- $\partial_\lambda(t) := \mathbb{1}$, ($\lambda$: top position, $\mathbb{1}$: identity map)
- $\partial_{i.p}(f(t_1, \ldots, t_n)) := \phi_i \circ \partial_p(t_i)$ (where $f : (\phi_1, \ldots, \phi_n) \in \mathcal{F}$).

Graded quantitative equational theories (Gavazzo & Di Florio 2023)

- Quantitative ternary relation E: finite set of triples (t, s, ε) (where t, s are terms, $\varepsilon \in \Omega$).
- View elements as quantitative equations: write $\varepsilon \Vdash t \approx s$.

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- View elements as quantitative equations: write $\varepsilon \Vdash t \approx s$.
- Quantitative equational theory induced by E is obtained by the following inference rules:

$$\text{(Ax)}\ \frac{\varepsilon \Vdash t \approx s \in E}{\varepsilon \Vdash t =_E s} \qquad \text{(Refl)}\ \frac{}{\varepsilon \Vdash t =_E t} \qquad \text{(Sym)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash s =_E t}$$

$$\text{(Trans)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \Vdash s =_E r}{\varepsilon \otimes \delta \Vdash t =_E r} \qquad \text{(Ord)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \precsim \varepsilon}{\delta \Vdash t =_E s}$$

$$\text{(Ampl)}\ \frac{\varepsilon_1 \Vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_E s_n \quad f : (\phi_1, \ldots, \phi_n) \in \mathcal{F}}{\phi_1(\varepsilon_1) \otimes \cdots \otimes \phi_n(\varepsilon_n) \Vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)}$$

$$\text{(Subst)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash t\sigma =_E s\sigma} \qquad \text{(Join)}\ \frac{\varepsilon_1 \Vdash t =_E s \quad \cdots \quad \varepsilon_n \Vdash t =_E s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_E s}$$

Quantitative rewriting and narrowing

- Let R be a quantitative ternary relation.
- View elements of R as quantitative rewrite rules: write $\varepsilon \Vdash t \mapsto_R s$

Definition (Quantitative rewrite relation $\rightarrow_R$)

$\rightarrow_R$ is obtained by closing R under

$$\frac{\varepsilon \Vdash l \mapsto_R r}{\partial_p(s)(\varepsilon) \Vdash s[l\sigma]_p \rightarrow_R s[r\sigma]_p}.$$

Definition (Quantitative narrowing relation $\rightsquigarrow_R$)

$\rightsquigarrow_R$ is obtained by closing R under

$$\frac{\varepsilon \Vdash l \mapsto_R r}{\partial_p(s)(\varepsilon) \Vdash s \rightsquigarrow_R (s[r\rho]_p)\sigma},$$

where ρ is a variable renaming and $\sigma = \mathrm{mgu}_\emptyset(s|_p, l\rho)$.

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n + S(Z) =^? (n + n) + n & \text{Substitution} \\
\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n & \{x \mapsto n, y \mapsto Z\} \\
\rightsquigarrow_0\ \underline{S(n)} =^? (n + n) + n & \{x \mapsto n\} \\
 & \{x \mapsto n\}
\end{array}
$$

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ \underline{1 \Vdash S(x) \mapsto x}\}$

Quantitative Narrowing derivation:

	Substitution
$n + S(Z) =^? (n + n) + n$	
$\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n$	$\{x \mapsto n, y \mapsto Z\}$
$\rightsquigarrow_0\ \underline{S(n)} =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? (n + n) + n$	$\{x \mapsto n\}$

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$\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n$	$\{x \mapsto n, y \mapsto Z\}$
$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
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\rightsquigarrow_1\ n =^? \underline{(n + n) + n} & \{x \mapsto n\}
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$$

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Quantitative Narrowing derivation:

Derivation	Substitution
$n + S(Z) =^? (n + n) + n$	
$\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n$	$\{x \mapsto n, y \mapsto Z\}$
$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? \underline{(n + n) + n}$	$\{x \mapsto n\}$
	$\{n \mapsto Z\}$

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Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

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\rightsquigarrow_0\ S(n) =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_1\ n =^? \underline{(n + n) + n} & \{x \mapsto n\} \\
\rightsquigarrow_0\ Z =^? Z + Z & \{n \mapsto Z\}
\end{array}
$$

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$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? (n + n) + n$	$\{x \mapsto n\}$
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$\rightsquigarrow_0\ S(n+Z) =^? (n+n) + n$	$\{x \mapsto n, y \mapsto Z\}$
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Quantitative Narrowing derivation:

$$
\begin{array}{llr}
 & n + S(Z) =^? (n + n) + n & \text{Substitution} \\
\rightsquigarrow_0 & S(n + Z) =^? (n + n) + n & \{x \mapsto n, y \mapsto Z\} \\
\rightsquigarrow_0 & S(n) =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_1 & n =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_0 & Z =^? \underline{Z + Z} & \{n \mapsto Z\} \\
 & & \mathrm{Id}
\end{array}
$$

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

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\rightsquigarrow_1\ n =^? (n+n) + n & \{x \mapsto n\} \\
\rightsquigarrow_0\ Z =^? \underline{Z + Z} & \{n \mapsto Z\} \\
\rightsquigarrow_0\ Z =^? Z & \mathrm{Id}
\end{array}
$$

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.
QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ 1 \Vdash S(x) \mapsto x\}$
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\begin{array}{llr}
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\rightsquigarrow_0 & S(n + Z) =^? (n + n) + n & \{x \mapsto n, y \mapsto Z\} \\
\rightsquigarrow_0 & S(n) =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_1 & n =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_0 & Z =^? Z + Z & \{n \mapsto Z\} \\
\rightsquigarrow_0 & Z =^? Z & \mathrm{Id}
\end{array}
$$

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

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\begin{array}{lr}
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\rightsquigarrow_1\ n =^? (n + n) + n & \{x \mapsto n\} \\
\rightsquigarrow_0\ Z =^? Z + Z & \{n \mapsto Z\} \\
\rightsquigarrow_0\ \underline{Z =^? Z} & \mathrm{Id}
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	Substitution
$n + S(Z) =^? (n + n) + n$	
$\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n$	$\{x \mapsto n, y \mapsto Z\}$
$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_0\ Z =^? Z + Z$	$\{n \mapsto Z\}$
$\rightsquigarrow_0\ Z =^? Z$	Id

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Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

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$n + S(Z) =^? (n + n) + n$	
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$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_0\ Z =^? Z + Z$	$\{n \mapsto Z\}$
$\rightsquigarrow_0\ Z =^? Z$	Id
$\rightsquigarrow_0\ \mathrm{TRUE}$	

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.
QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ 1 \Vdash S(x) \mapsto x\}$
Quantitative Narrowing derivation:

	Substitution
$n + S(Z) =^? (n + n) + n$	
$\rightsquigarrow_0\ S(n + Z) =^? (n + n) + n$	$\{x \mapsto n, y \mapsto Z\}$
$\rightsquigarrow_0\ S(n) =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_1\ n =^? (n + n) + n$	$\{x \mapsto n\}$
$\rightsquigarrow_0\ Z =^? Z + Z$	$\{n \mapsto Z\}$
$\rightsquigarrow_0\ Z =^? Z$	Id
$\rightsquigarrow_0\ \mathrm{TRUE}$	

Computed approximate solution: $\{n \mapsto Z\}$ with degree 1

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ 1 \Vdash S(x) \mapsto x\}$

Alternative derivation:

$$
\begin{array}{llr}
& \underline{n + S(Z)} =^? (n + n) + n & \\
\rightsquigarrow_0 & S(\underline{n + Z}) =^? (n + n) + n & (\text{by } r_2) \\
\rightsquigarrow_0 & S(n) =^? \underline{(n + n) + n} & (\text{by } r_1) \\
\rightsquigarrow_0 & S(S(y)) =^? S(\underline{(S(y) + S(y)) + y}) & (\text{by } r_2) \\
\rightsquigarrow_0 & S(S(Z)) =^? S(\underline{S(Z) + S(Z)}) & (\text{by } r_1) \\
\rightsquigarrow_0 & S(S(Z)) =^? S(S(\underline{S(Z) + Z})) & (\text{by } r_2) \\
\rightsquigarrow_0 & S(S(Z)) =^? S(S(\underline{S(Z)})) & (\text{by } r_1) \\
\rightsquigarrow_1 & \underline{S(S(Z)) =^? S(S(Z))} & (\text{by } r_3) \\
\rightsquigarrow_0 & \mathrm{TRUE} &
\end{array}
$$

Computed approximate solution: $\{n \mapsto S(Z)\}$ with degree 1.

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ 1 \Vdash S(x) \mapsto x\}$

Alternative derivation:

$$
\begin{array}{lll}
& \underline{n + S(Z)} =^? (n + n) + n & \\
\rightsquigarrow_0 & S(\underline{n + Z}) =^? (n + n) + n & \text{(by } r_2) \\
\rightsquigarrow_0 & S(n) =^? \underline{(n + n)} + n & \text{(by } r_1) \\
\rightsquigarrow_0 & S(S(y)) =^? \underline{S(S(y) + y)} + S(y) & \text{(by } r_2) \\
\rightsquigarrow_1 & S(S(y)) =^? (S(y) + y) + \underline{S(y)} & \text{(by } r_3) \\
\rightsquigarrow_1 & S(S(y)) =^? (\underline{S(y)} + y) + y & \text{(by } r_3) \\
\rightsquigarrow_1 & S(S(y)) =^? (\underline{y + y}) + y & \text{(by } r_3) \\
\rightsquigarrow_0 & S(S(S(y'))) =^? S(\underline{S(y') + y'}) + S(y') & \text{(by } r_2) \\
\rightsquigarrow_0 & S(S(S(Z))) =^? \underline{S(S(Z)) + S(Z)} & \text{(by } r_1) \\
\rightsquigarrow_0 & S(S(S(Z))) =^? S(\underline{S(S(Z)) + Z}) & \text{(by } r_1) \\
\rightsquigarrow_0 & \underline{S(S(S(Z))) =^? S(S(S(Z)))} & \text{(by } r_1) \\
\rightsquigarrow_0 & \text{TRUE} &
\end{array}
$$

Computed approximate solution: $\{n \mapsto S(S(Z))\}$ with degree 3.

Example

We can also use degrees to keep track of the number of narrowing steps used:
QTRS: $R' = \{1 \Vdash x + Z \mapsto x,\ 1 \Vdash x + S(y) \mapsto S(x+y),\ 1 \Vdash S(x) \mapsto x\}$

Example

We can also use degrees to keep track of the number of narrowing steps used:
QTRS: $R' = \{1 \Vdash x + Z \mapsto x,\ 1 \Vdash x + S(y) \mapsto S(x+y),\ 1 \Vdash S(x) \mapsto x\}$
Quantitative Narrowing derivation:

$$
\begin{array}{lll}
& \underline{n + S(Z)} =^? (n+n) + n & \\
\rightsquigarrow_1 & S(\underline{n + Z}) =^? (n+n) + n & \text{(by } r_2) \\
\rightsquigarrow_1 & \underline{S(n)} =^? (n+n) + n & \text{(by } r_1) \\
\rightsquigarrow_1 & n =^? \underline{(n+n)} + n & \text{(by } r_3) \\
\rightsquigarrow_1 & Z =^? \underline{Z + Z} & \text{(by } r_1) \\
\rightsquigarrow_1 & Z =^? Z & \text{(by } r_1) \\
\rightsquigarrow_0 & \mathrm{TRUE} &
\end{array}
$$

Example

We can also use degrees to keep track of the number of narrowing steps used:
QTRS: $R' = \{1 \Vdash x + Z \mapsto x,\ 1 \Vdash x + S(y) \mapsto S(x+y),\ 1 \Vdash S(x) \mapsto x\}$
Quantitative Narrowing derivation:

$$
\begin{array}{lr}
\underline{n + S(Z)} =^? (n+n) + n & \\
\rightsquigarrow_1\ S(\underline{n + Z}) =^? (n+n) + n & (\text{by } r_2) \\
\rightsquigarrow_1\ \underline{S(n)} =^? (n+n) + n & (\text{by } r_1) \\
\rightsquigarrow_1\ n =^? \underline{(n+n)} + n & (\text{by } r_3) \\
\rightsquigarrow_1\ Z =^? \underline{Z + Z} & (\text{by } r_1) \\
\rightsquigarrow_1\ Z =^? Z & (\text{by } r_1) \\
\rightsquigarrow_0\ \mathrm{TRUE} &
\end{array}
$$

Computed approximate solution: $\{n \mapsto Z\}$ with degree 5 (i.e. 5 narrowing steps).

Example

To account for the computational cost of finding the redex, one can add modal arities:
Assume that $\mathrm{arity}(S) = (d)$ and $\mathrm{arity}(+) = (d, d)$, where $d\colon [0, \infty] \to [0, \infty]$, $x \mapsto 2x$

Example

To account for the computational cost of finding the redex, one can add modal arities:
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QTRS: $R' = \{1 \Vdash x + Z \mapsto x,\ 1 \Vdash x + S(y) \mapsto S(x+y),\ 1 \Vdash S(x) \mapsto x\}$

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Assume that arity$(S) = (d)$ and arity$(+) = (d, d)$, where $d\colon [0,\infty] \to [0,\infty]$, $x \mapsto 2x$
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Quantitative Narrowing derivation:

$$
\begin{array}{lll}
 & \underline{n + S(Z)} =^? (n+n)+n & \\
\rightsquigarrow_1 & S(\underline{n+Z}) =^? (n+n)+n & \text{(by } r_2\text{)} \\
\rightsquigarrow_2 & \underline{S(n)} =^? (n+n)+n & \text{(by } r_1\text{)} \\
\rightsquigarrow_1 & n =^? \underline{(n+n)}+n & \text{(by } r_3\text{)} \\
\rightsquigarrow_2 & Z =^? \underline{Z+Z} & \text{(by } r_1\text{)} \\
\rightsquigarrow_1 & Z =^? Z & \text{(by } r_1\text{)} \\
\rightsquigarrow_0 & \mathrm{TRUE} &
\end{array}
$$

Example

To account for the computational cost of finding the redex, one can add modal arities:
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QTRS: $R' = \{1 \Vdash x + Z \mapsto x,\ 1 \Vdash x + S(y) \mapsto S(x+y),\ 1 \Vdash S(x) \mapsto x\}$
Quantitative Narrowing derivation:

$$
\begin{array}{lr}
\underline{n + S(Z)} =^? (n+n)+n & \\
\rightsquigarrow_1\ S(\underline{n+Z}) =^? (n+n)+n & \text{(by } r_2\text{)} \\
\rightsquigarrow_2\ \underline{S(n)} =^? (n+n)+n & \text{(by } r_1\text{)} \\
\rightsquigarrow_1\ n =^? \underline{(n+n)}+n & \text{(by } r_3\text{)} \\
\rightsquigarrow_2\ Z =^? \underline{Z+Z} & \text{(by } r_1\text{)} \\
\rightsquigarrow_1\ Z =^? Z & \text{(by } r_1\text{)} \\
\rightsquigarrow_0\ \mathrm{TRUE} &
\end{array}
$$

Computed approximate solution: $\{n \mapsto Z\}$ with degree 7 ($\sim$computational cost).

Calculus BQNARROW for basic quantitative narrowing

- BQNARROW: Rule-based calculus for quantitative narrowing
- Given a quantitative unification problem (equation to be solved), construct an initial configuration
- Apply rules until a terminal configuration is reached
- Failure or solution can be read off from terminal configuration
- Configurations: **F** (failure) or $\langle e; C; \sigma; \delta \rangle$, where
    - e: equation (or TRUE)
    - C: set of constraints
    - σ: substitution computed so far
    - δ: current degree of approximation
- *Basic* narrowing: Variables of the problem are only instantiated at the end.
  $\rightarrow$ *No instantiation of variables introduced by narrowing substitutions.*
  $\rightarrow$ Removes some sources of non-termination.

BQNarrow rules

LP: **Lazy Paramodulation**

$$\langle e[t]_p; C; \sigma; \delta\rangle \Longrightarrow_{\partial_p(e)(\varepsilon)} \langle e[r]_p; \{l\sigma = t\sigma\} \cup C; \sigma; \delta \otimes \partial_p(e)(\varepsilon)\rangle ,$$

where $e \neq \mathrm{TRUE}$, p is a non-variable position of e, and $\varepsilon \Vdash l \mapsto r$ is a fresh variant of a rule in R.

SU: **Syntactic Unification**

$$\langle e; C; \sigma; \delta\rangle \Longrightarrow_{\kappa} \langle e; \emptyset; \sigma\rho; \delta\rangle ,$$

where $C \neq \emptyset$ and ρ is a most general syntactic unifier of C.

Cla: **Clash**

$$\langle e; C; \sigma; \delta\rangle \Longrightarrow_{\kappa} \mathbf{F}, \quad \text{if } C \text{ is not unifiable.}$$

Con: **Constrain**

$$\langle e; C; \sigma; \delta\rangle \Longrightarrow_{\kappa} \langle \mathrm{TRUE}; C \cup \{e\sigma\}; \sigma; \delta\rangle , \quad \text{if } e \neq \mathrm{TRUE}.$$

Example

Find an approximate solution $n \in \mathbb{N}$ for the equation $n + 1 = 3n$.

QTRS: $R = \{0 \Vdash x + Z \mapsto x,\ 0 \Vdash x + S(y) \mapsto S(x + y),\ 1 \Vdash S(x) \mapsto x\}$

Derivation in $\mathrm{BQN}\textsc{arrow}$**:**

$$
\begin{aligned}
&\langle n + S(Z) =^? (n + n) + n;\ \emptyset;\ Id;\ 0\rangle \\
&\xRightarrow{LP}_0 \langle S(x + y) =^? (n + n) + n;\ \{x + S(y) = n + S(Z)\};\ Id;\ 0\rangle \\
&\xRightarrow{SU}_0 \langle S(x + y) =^? (n + n) + n;\ \emptyset;\ \{x, n \mapsto x_1; y \mapsto Z\};\ 0\rangle \\
&\xRightarrow{LP}_0 \langle S(x_2) =^? (n + n) + n;\ \{x_2 + Z = x_1 + Z\};\ \{x, n \mapsto x_1; y \mapsto Z\};\ 0\rangle \\
&\xRightarrow{LP}_0 \langle S(x_2) =^? x_3 + n;\ \{x_2 + Z = x_1 + Z,\ x_3 + Z = x_1 + x_1\};\ \{x, n \mapsto x_1; y \mapsto Z\};\ 0\rangle \\
&\xRightarrow{SU}_0 \langle S(x_2) =^? x_3 + n;\ \emptyset;\ \{x, x_1, x_2, x_3, y, n \mapsto Z\};\ 0\rangle \\
&\xRightarrow{LP}_1 \langle x_4 =^? x_3 + n;\ \{x_4 = Z\};\ \{x, x_1, x_2, x_3, y, n \mapsto Z\};\ 1\rangle \\
&\xRightarrow{LP}_0 \langle x_4 =^? x_5;\ \{x_4 = x_2,\ x_5 + Z = x_3 + Z\};\ \{x, x_1, x_2, x_3, y, n \mapsto Z\};\ 1\rangle \\
&\xRightarrow{Con}_0 \langle \mathrm{TRUE};\ \{x_4 = x_2,\ x_5 + Z = x_3 + n,\ x_4 = x_5\};\ \{x, x_1, x_2, x_3, y, n \mapsto Z\};\ 1\rangle \\
&\xRightarrow{SU}_0 \langle \mathrm{TRUE};\ \emptyset;\ \{x, x_1, x_2, x_3, x_4, x_5, y, n \mapsto Z\};\ 1\rangle
\end{aligned}
$$

Computed approximate solution: $\{n \mapsto Z\}$ with degree 1

Results

Theorem (Soundness of $\mathrm{BQN}\textsc{arrow}$)

If $\langle t =^? s; C; \sigma; \delta\rangle \Longrightarrow^+_\varepsilon \langle \mathrm{TRUE}; \emptyset; \sigma'; \delta'\rangle$ *is a derivation using the rules from* $\mathrm{BQN}\textsc{arrow}$, *then* $\varepsilon \Vdash t\sigma' =_R s\sigma'$.

Theorem (Weak completeness of $\mathrm{BQN}\textsc{arrow}$)

Suppose that Ω *is a Lawverean quantale whose order* $\precsim$ *is total. Let* $t =^? s$ *be a linear problem, and let* R *be a confluent, right-ground* (Ω, Φ)*-TRS. If* $\varepsilon \Vdash t\tau =_R s\tau$, *then* $\mathrm{BQN}\textsc{arrow}$ *admits a derivation* $\langle t =^? s; \emptyset; \mathrm{Id}; \kappa\rangle \Longrightarrow^* \langle \mathrm{TRUE}; \emptyset; \sigma; \delta\rangle$ *such that* $\delta \succsim \varepsilon$.

Results

Theorem (Soundness of $\mathrm{BQN{\small ARROW}}$)

If $\langle t =^? s; C; \sigma; \delta\rangle \Longrightarrow^+_\varepsilon \langle \mathrm{TRUE}; \emptyset; \sigma'; \delta'\rangle$ *is a derivation using the rules from* $\mathrm{BQN{\small ARROW}}$, *then* $\varepsilon \Vdash t\sigma' =_R s\sigma'$.

Theorem (Weak completeness of $\mathrm{BQN{\small ARROW}}$)

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Remark

- *Termination cannot be granted!*
- *Weak completeness: We do not necessarily compute the given* τ, *but some* σ *which solves the problem with a degree that is at least as good.*
- *Substantial improvement over previous results on quantitative unification (Ehling & Kutsia 2024).*

Steps of the completeness proof

$$\boxed{\varepsilon \Vdash t\tau =_R s\tau}$$

Steps of the completeness proof

$$
\begin{tikzcd}
\boxed{\varepsilon \Vdash t\tau =_R s\tau} \arrow[d, Rightarrow, "\precsim \text{ total}"] \\
\boxed{t\tau \leftrightarrow^*_{R,\varepsilon} s\tau}
\end{tikzcd}
$$

Steps of the completeness proof

$$\begin{tikzcd}
\varepsilon \Vdash t\tau =_R s\tau \arrow[d, Rightarrow, "\precsim \text{ total}"] \\
t\tau \leftrightarrow^*_{R,\varepsilon} s\tau \arrow[d, Rightarrow, "R \text{ confluent}"] \\
\begin{array}{c} t\tau \downarrow_{R,\zeta} s\tau \\ (\text{for some } \zeta \succsim \varepsilon) \end{array}
\end{tikzcd}$$

Steps of the completeness proof

$$
\begin{tikzcd}
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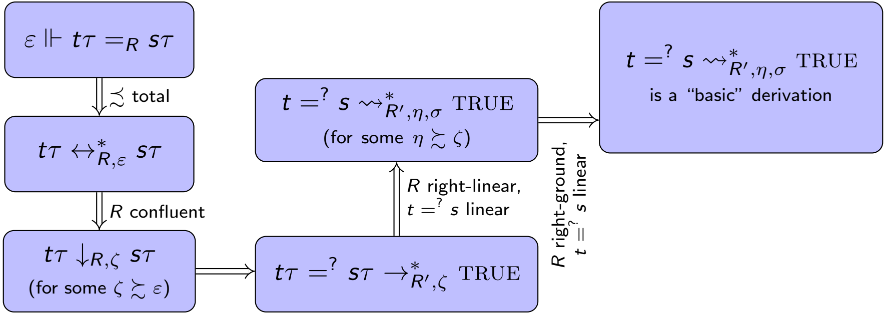
\boxed{\begin{array}{c} t\tau \downarrow_{R,\zeta} s\tau \\ (\text{for some } \zeta \succsim \varepsilon) \end{array}} \arrow[r, Rightarrow] & \boxed{t\tau =^? s\tau \rightarrow^*_{R',\zeta} \mathrm{TRUE}}
\end{tikzcd}
$$

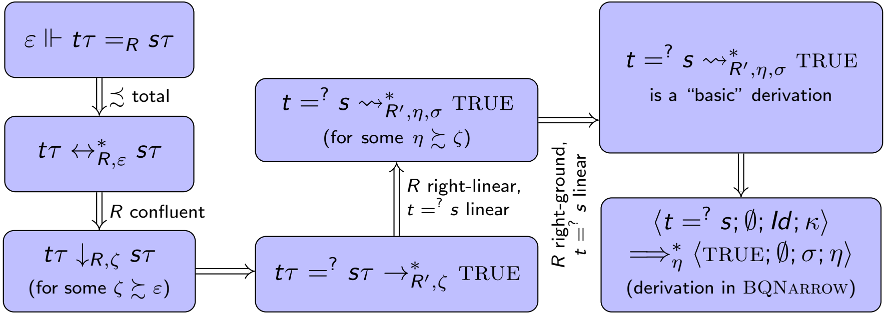
Steps of the completeness proof

$$
\begin{tikzcd}
\boxed{\varepsilon \Vdash t\tau =_R s\tau} \arrow[d, Rightarrow, "\precsim \text{ total}"] & \boxed{\begin{array}{c} t =^? s \rightsquigarrow^*_{R',\eta,\sigma} \mathrm{TRUE} \\ \text{(for some } \eta \succsim \zeta\text{)} \end{array}} \\
\boxed{t\tau \leftrightarrow^*_{R,\varepsilon} s\tau} \arrow[d, Rightarrow, "R \text{ confluent}"] & \\
\boxed{\begin{array}{c} t\tau \downarrow_{R,\zeta} s\tau \\ \text{(for some } \zeta \succsim \varepsilon\text{)} \end{array}} \arrow[r, Rightarrow] & \boxed{t\tau =^? s\tau \to^*_{R',\zeta} \mathrm{TRUE}} \arrow[uu, Rightarrow, "\begin{array}{l} R \text{ right-linear,} \\ t =^? s \text{ linear} \end{array}"']
\end{tikzcd}
$$

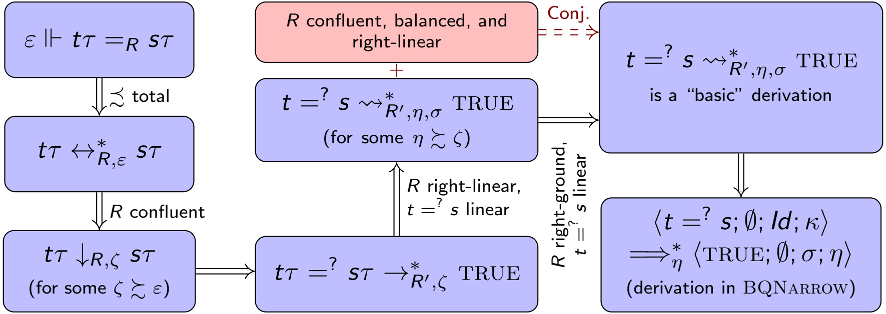
Steps of the completeness proof



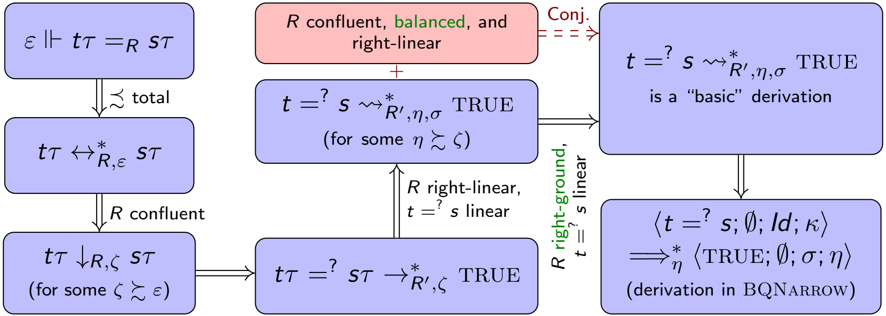
Steps of the completeness proof



Steps of the completeness proof



Steps of the completeness proof



Conclusion and future work

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- Quantitative equational theories (Gavazzo & Di Florio 2023) cover various approaches of reasoning with quantitative information.
- Adapted narrowing to the quantitative setting.
- Established a rule-based narrowing calculus for quantitative unification and proved its soundness and (weak) completeness.
- Improved on previous results for quantitative unification.

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- Improved on previous results for quantitative unification.

Future work

- Under which conditions can we guarantee termination?
- Stronger results might be possible if we restrict to certain types of quantales: totally ordered, idempotent, divisible, . . .
- Investigate other classic (equational) problems in the quantitative setting: matching, anti-unification, resolution,. . .

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