Logic, Checking, and Proving

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1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

4. The RISC Theorem Proving Interface

The Language of Logic

Two kinds of syntactic phrases.

- Term T denoting an object.
  - Variable x
  - Object constant c
  - Function application $f(T_1, \ldots, T_n)$ (may be written infix)

    n-ary function constant f
- Formula F denoting a truth value.
  - Atomic formula $p(T_1, \ldots, T_n)$ (may be written infix)

    n-ary predicate constant p.
  - Negation $\neg F$ ("not F")
  - Conjunction $F_1 \land F_2$ ("F_1 and F_2")
  - Disjunction $F_1 \lor F_2$ ("F_1 or F_2")
  - Implication $F_1 \Rightarrow F_2$ ("if F_1, then F_2")
  - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1, then F_2, and vice versa")
  - Universal quantification $\forall x : F$ ("for all x, F")
  - Existential quantification $\exists x : F$ ("for some x, F")

Syntactic Shortcuts

- $\forall x_1, \ldots, x_n : F$
  - $\forall x_1 : \ldots : \forall x_n : F$
- $\exists x_1, \ldots, x_n : F$
  - $\exists x_1 : \ldots : \exists x_n : F$
- $\forall x \in S : F$
  - $\forall x : x \in S \Rightarrow F$ (not: $\wedge$)
- $\exists x \in S : F$
  - $\exists x : x \in S \wedge F$ (not: $\Rightarrow$)

Help to make formulas more readable.

Examples

Terms and formulas may appear in various syntactic forms.

- Terms:

$$\exp(x)$$
$$a \cdot b + 1$$
$$a[i] \cdot b$$
$$\sqrt{\frac{x^2+2x+1}{(y+1)^2}}$$

- Formulas:

$$a^2 + b^2 = c^2$$
$$n \mid 2n$$
$$\forall x \in \mathbb{N} : x \geq 0$$
$$\forall x \in \mathbb{N} : 2|x \vee 2|(x+1)$$
$$\begin{aligned}&\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow \\ &\qquad \exists z \in \mathbb{N} : x + z = y\end{aligned}$$

Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas

- Atomic formula $p(T_1, \ldots, T_n)$
  - True if the predicate denoted by p holds for the values of $T_1, \ldots, T_n$.
- Negation $\neg F$
  - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ (“F_1 and F_2”)
  - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ (“F_1 or F_2”)
  - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ (“if F_1, then F_2”)
  - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ (“if F_1, then F_2, and vice versa”)
  - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ (“for all x, F”)
  - True if and only if F is true for every possible value assignment of x.
- Existential quantification $\exists x : F$ (“for some x, F”)
  - True if and only if F is true for at least one value assignment of x.

Example

We assume the domain of natural numbers and the "classical" interpretation of constants 1, 2, $+$, $=$, $<$.

- $1+1=2$
  - True.
- $1+1=2 \vee 2+2=2$
  - True.
- $1+1=2 \wedge 2+2=2$
  - False.
- $1+1=2 \Rightarrow 2=1+1$
  - True.
- $1+1=1 \Rightarrow 2+2=2$
  - True.
- $1+1=2 \Rightarrow 2+2=2$
  - False.
- $1+1=1 \Leftrightarrow 2+2=2$
  - True.

Example

- $x + 1 = 1 + x$
  - True, for every assignment of a number a to variable x.
- $\forall x : x + 1 = 1 + x$
  - True (because for every assignment a to x, $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
  - If x is assigned “one”, the formula is true.
  - If x is assigned “two”, the formula is false.
- $\exists x : x + 1 = 2$
  - True (because $x + 1 = 2$ is true for assignment “one” to x).
- $\forall x : x + 1 = 2$
  - False (because $x + 1 = 2$ is false for assignment “two” to x).
- $\forall x : \exists y : x < y$
  - True (because for every assignment a to x, there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
  - False (because for every assignment a to y, there is the assignment $a + 1$ to x which makes $x < y$ false).

Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \leftrightsquigarrow F_1$
- $\neg(F_1 \land F_2) \leftrightsquigarrow \neg F_1 \lor \neg F_2$
- $\neg(F_1 \lor F_2) \leftrightsquigarrow \neg F_1 \land \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \leftrightsquigarrow F_1 \land \neg F_2$
- $\neg\forall x : F \leftrightsquigarrow \exists x : \neg F$
- $\neg\exists x : F \leftrightsquigarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_1 \lor F_2$
- $F_1 \Leftrightarrow F_2 \leftrightsquigarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.

Example

- "All swans are white or black."
  - $\forall x : \mathit{swan}(x) \Rightarrow \mathit{white}(x) \vee \mathit{black}(x)$
- "There exists a black swan."
  - $\exists x : \mathit{swan}(x) \wedge \mathit{black}(x)$.
- "A swan is white, unless it is black."
  - $\forall x : \mathit{swan}(x) \wedge \neg \mathit{black}(x) \Rightarrow \mathit{white}(x)$
  - $\forall x : \mathit{swan}(x) \wedge \neg \mathit{white}(x) \Rightarrow \mathit{black}(x)$
  - $\forall x : \mathit{swan}(x) \Rightarrow \mathit{white}(x) \vee \mathit{black}(x)$
- "Not everything that is white or black is a swan."
  - $\neg \forall x : \mathit{white}(x) \vee \mathit{black}(x) \Rightarrow \mathit{swan}(x)$.
  - $\exists x : (\mathit{white}(x) \vee \mathit{black}(x)) \wedge \neg \mathit{swan}(x)$.
- "Black swans have at least one black parent".
  - $\forall x : \mathit{swan}(x) \wedge \mathit{black}(x) \Rightarrow \exists y : \mathit{swan}(y) \wedge \mathit{black}(y) \wedge \mathit{parent}(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.

The Usage of Formulas

Precise formulation of statements describing object relationships.

- Statement:

  If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y.

- Formula:

  $$\begin{array}{l} x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow \\ \quad q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y \end{array}$$

- Problem specification:

  Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y.

  - Inputs: x, y
  - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
  - Output: q
  - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y$

Problem Specifications

- The specification of a computation problem:
  - Input: variables $x_1 \in S_1, \dots, x_n \in S_n$
  - Input condition ("precondition"): formula $I(x_1, \dots, x_n)$.
  - Output: variables $y_1 \in T_1, \dots, y_m \in T_n$
  - Output condition ("postcondition"): $O(x_1, \dots, x_n, y_1, \dots, y_m)$.
    - $F(x_1, \dots, x_n)$: only $x_1, \dots, x_n$ are free in formula F.
    - x is *free* in F, if not every occurrence of x is inside the scope of a quantifier (such as $\forall$ or $\exists$) that binds x.
- An implementation of the specification:
  - A function (program) $f : S_1 \times \ldots \times S_n \to T_1 \times \ldots \times T_m$ such that

$$\begin{array}{l} \forall x_1 \in S_1, \dots, x_n \in S_n : I(x_1, \dots, x_n) \Rightarrow \\ \qquad \textbf{let } (y_1, \dots, y_m) = f(x_1, \dots, x_n) \textbf{ in} \\ \qquad O(x_1, \dots, x_n, y_1, \dots, y_m) \end{array}$$

  - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.

Example: A Problem Specification

Given an integer array a, a position p in a, and a length l, return the array b derived from a by removing $a[p], \ldots, a[p+l-1]$.

- Input: $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- Input condition:

  $$p + l \leq \text{length}(a)$$

- Output: $b \in \mathbb{Z}^*$
- Output condition:

  $$\begin{array}{l} \textbf{let } n = \text{length}(a) \textbf{ in} \\ \text{length}(b) = n - l \wedge \\ (\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge \\ (\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i+l]) \end{array}$$

Mathematical theory:

$$\begin{array}{l} T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \to T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\} \\ \text{length} : T^* \to \mathbb{N}, \text{length}(a) = \textbf{such } i \in \mathbb{N} : a \in T^i \end{array}$$

Validating Problem Specifications

Do formal input condition $I(x)$ and output condition $O(x, y)$ really capture our informal intentions?

- Do concrete inputs/output satisfy/violate these conditions?
  - $I(a_1)$, $\neg I(a_2)$, $O(a_1, b_1)$, $\neg O(a_1, b_2)$.
- Is input condition satisfiable?
  - $\exists x : I(x)$.
- Is input condition not trivial?
  - $\exists x : \neg I(x)$.
- Is output condition satisfiable for every input?
  - $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- Is output condition for all (at least some) inputs not trivial?
  - $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.
  - $\exists x : I(x) \wedge \exists y : \neg O(x, y)$.
- Is for every legal input at most one output legal?
  - $\forall x : I(x) \Rightarrow \forall y_1, y_2 : O(x, y_1) \wedge O(x, y_2) \Rightarrow y_1 = y_2$.

Validate specification to increase our confidence in its meaning!

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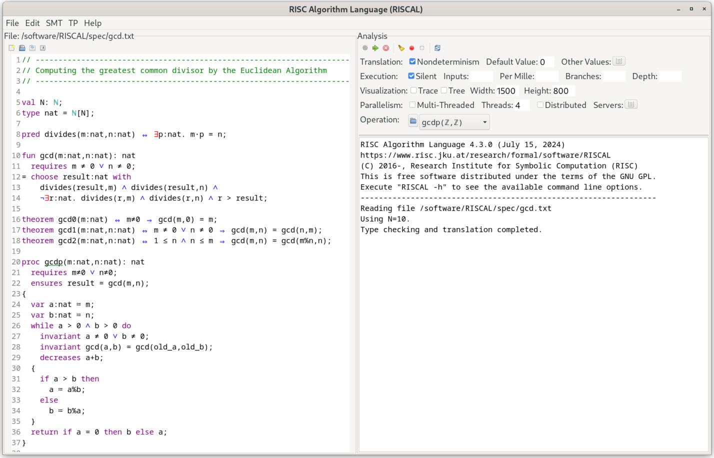
The RISC Algorithm Language (RISCAL)

- A system for formally specifying and checking algorithms.
  - Research Institute for Symbolic Computation (RISC), 2016–.
    https://www.risc.jku.at/research/formal/software/RISCAL.
  - Implemented in Java with the Eclipse SWT library for the GUI.
    - Tested under Linux only; freely available as open source (GPL3).
- A language for the defining mathematical theories and algorithms.
  - A static type system with only finite types (of parameterized sizes).
  - Predicates, explicitly (also recursively) and implicitly def.d functions.
  - Theorems (universally quantified predicates expected to be true).
  - Procedures (also recursively defined).
  - Pre- and post-conditions, invariants, termination measures.
- A framework for evaluating/executing all definitions.
  - Model checking: predicates, functions, theorems, procedures, annotations may be evaluated/executed for all possible inputs.
  - All paths of a non-deterministic execution may be elaborated.
  - The execution/evaluation may be visualized.

Validating algorithms by automatically verifying finite approximations.

The RISC Algorithm Language (RISCAL)

```
RISCAL divide.txt &
```



File: /software/RISCAL/spec/gcd.txt

```
// ----------------------------------------------------------------------
// Computing the greatest common divisor by the Euclidean Algorithm
// ----------------------------------------------------------------------

val N: ℕ;
type nat = ℕ[N];

pred divides(m:nat,n:nat) ⇔ ∃p:nat. m·p = n;

fun gcd(m:nat,n:nat): nat
  requires m ≠ 0 ∨ n ≠ 0;
= choose result:nat with
    divides(result,m) ∧ divides(result,n) ∧
    ¬∃r:nat. divides(r,m) ∧ divides(r,n) ∧ r > result;

theorem gcd0(m:nat) ⇔ m≠0 ⇒ gcd(m,0) = m;
theorem gcd1(m:nat,n:nat) ⇔ m ≠ 0 ∨ n ≠ 0 ⇒ gcd(m,n) = gcd(n,m);
theorem gcd2(m:nat,n:nat) ⇔ 1 ≤ n ∧ n ≤ m ⇒ gcd(m,n) = gcd(m%n,n);

proc gcdp(m:nat,n:nat): nat
  requires m≠0 ∨ n≠0;
  ensures result = gcd(m,n);
{
  var a:nat = m;
  var b:nat = n;
  while a > 0 ∧ b > 0 do
    invariant a ≠ 0 ∨ b ≠ 0;
    invariant gcd(a,b) = gcd(old_a,old_b);
    decreases a+b;
  {
    if a > b then
      a = a%b;
    else
      b = b%a;
  }
  return if a = 0 then b else a;
}
```

Translation: ☑ Nondeterminism Default Value: 0 Other Values:

Execution: ☑ Silent Inputs: Per Mille: Branches: Depth:

Visualization: □ Trace □ Tree Width: 1500 Height: 800

Parallelism: □ Multi-Threaded Threads: 4 □ Distributed Servers:

Operation: gcdp(ℤ,ℤ)

```
RISC Algorithm Language 4.3.0 (July 15, 2024)
https://www.risc.jku.at/research/formal/software/RISCAL
(C) 2016-, Research Institute for Symbolic Computation (RISC)
This is free software distributed under the terms of the GNU GPL.
Execute "RISCAL -h" to see the available command line options.
-----------------------------------------------------------------
Reading file /software/RISCAL/spec/gcd.txt
Using N=10.
Type checking and translation completed.
```

Using RISCAL

See also the (printed/online) “Tutorial and Reference Manual”.

- Press button 💾 (or <Ctrl>-s) to save specification.
  - Automatically processes (parses and type-checks) specification.
  - Press button ⚙ to re-process specification.
- Choose values for undefined constants in specification.
  - Natural number for `val` const: $\mathbb{N}$.
  - *Default Value*: used if no other value is specified.
  - *Other Values*: specific values for individual constants.
- Select *Operation* from menu and then press button ➡.
  - Executes operation for chosen constant values and all possible inputs.
  - Option *Silent*: result of operation is not printed.
  - Option *Nondeterminism*: all execution paths are taken.
  - Option *Multi-threaded*: multiple threads execute different inputs.
  - Press buttton ❌ to abort execution.

During evaluation all annotations (pre/postconditions, etc.) are checked.

Typing Mathematical Symbols

ASCII String	Unicode Character
Int	ℤ
Nat	ℕ
:=	:=
true	⊤
false	⊥
~	¬
/\	∧
\/	∨
=>	⇒
<=>	⇔
forall	∀
exists	∃
sum	∑
product	∏

ASCII String	Unicode Character
~=	≠
<=	≤
>=	≥
*	·
times	×
{}	∅
intersect	∩
union	∪
Intersect	⋂
Union	⋃
isin	∈
subseteq	⊆
<<	⟨
>>	⟩

Type the ASCII string and press <Ctrl>-# to get the Unicode character.

Example: Quotient and Remainder

Given natural numbers *n* and *m*, we want to compute the quotient *q* and remainder *r* of *n* divided by *m*.

```
// the type of natural numbers less than equal N
val N: ℕ;
type Num = ℕ[N];

// the precondition of the computation
pred pre(n:Num, m:Num) ⇔ m ≠ 0;

// the postcondition, first formulation
pred post1(n:Num, m:Num, q:Num, r:Num) ⇔
  n = m·q + r ∧
  ∀q0:Num, r0:Num.
    n = m·q0 + r0 ⇒ r ≤ r0;

// the postcondition, second formulation
pred post2(n:Num, m:Num, q:Num, r:Num) ⇔
  n = m·q + r ∧ r < m;
```

We will investigate this specification.

Example: Quotient and Remainder

```
// for all inputs that satisfy the precondition
// both formulations are equivalent:
// ∀n:Num, m:Num, q:Num, r:Num.
//   pre(n, m) ⇒ (post1(n, m, q, r) ⇔ post2(n, m, q, r));
theorem postEquiv(n:Num, m:Num, q:Num, r:Num)
  requires pre(n, m);
⇔ post1(n, m, q, r) ⇔ post2(n, m, q, r);

// we will thus use the simpler formulation from now on
pred post(n:Num, m:Num, q:Num, r:Num) ⇔ post2(n, m, q, r);
```

Check equivalence for all values that satisfy the precondition.

Example: Quotient and Remainder

Choose e.g. value 5 for N.

- Switch option *Silent* off:

```
Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function postEquiv(0,1,0,0):
Result (0 ms): true
Run 7 of deterministic function postEquiv(1,1,0,0):
Result (0 ms): true
...
Run 1295 of deterministic function postEquiv(5,5,5,5):
Result (0 ms): true
Execution completed for ALL inputs (6314 ms, 1080 checked, 216 inadmissible).
```

- Switch option *Silent* on:

```
Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Execution completed for ALL inputs (244 ms, 1080 checked, 216 inadmissible).
```

If theorem is false for some input, an error message is displayed.

Example: Quotient and Remainder

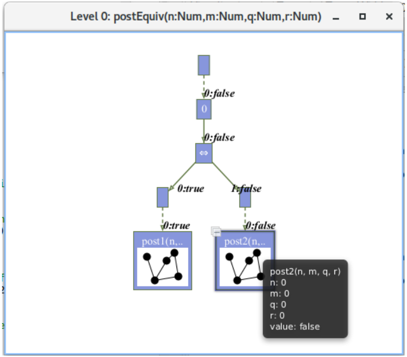
Drop precondition from theorem.

```
theorem postEquiv(n:Num, m:Num, q:Num, r:Num) ⇔
  // requires pre(n, m);
  post1(n, m, q, r) ⇔ post2(n, m, q, r);

Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Run 0 of deterministic function postEquiv(0,0,0,0):
ERROR in execution of postEquiv(0,0,0,0): evaluation of
  postEquiv
at line 25 in file divide.txt:
  theorem is not true
ERROR encountered in execution.
```

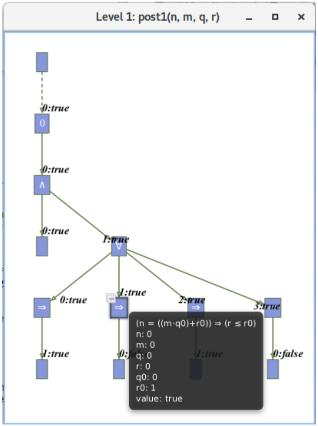
For $n = 0, m = 0, q = 0, r = 0$, the modified theorem is not true.

Visualizing the Formula Evaluation

Select $N = 1$ and visualization option "Tree".



```
post2(n, m, q, r)
n: 0
m: 0
q: 0
r: 0
value: false
```



```
(n = ((m·q0)+r0)) ⇒ (r ≤ r0)
n: 0
m: 0
q: 0
r: 0
q0: 0
r0: 1
value: true
```

Investigate the (pruned) evaluation tree to determine how the truth value of a formula was derived (double click to zoom into/out of predicates).

Example: Quotient and Remainder

Switch option “Nondeterminism” on.

```
// 1. investigate whether the specified input/output combinations are as desired
fun quotremFun(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
= choose q:Num, r:Num with post(n, m, q, r);

Executing quotremFun(ℤ,ℤ) with all 36 inputs.
Ignoring inadmissible inputs...
Branch 0:6 of nondeterministic function quotremFun(0,1):
Result (0 ms): [0,0]
Branch 1:6 of nondeterministic function quotremFun(0,1):
No more results (8 ms).
...
Branch 0:35 of nondeterministic function quotremFun(5,5):
Result (0 ms): [1,0]
Branch 1:35 of nondeterministic function quotremFun(5,5):
No more results (14 ms).
Execution completed for ALL inputs (413 ms, 30 checked, 6 inadmissible).
```

First validation by inspecting the values determined by output condition (nondeterminism may produce for some inputs multiple outputs).

Example: Quotient and Remainder

```
// 2. check that some but not all inputs are allowed
theorem someInput() ⇔ ∃n:Num, m:Num. pre(n, m);
theorem notEveryInput() ⇔ ∃n:Num, m:Num. ¬pre(n, m);
```

```
Executing someInput().
Execution completed (0 ms).
Executing notEveryInput().
Execution completed (0 ms).
```

A very rough validation of the input condition.

Example: Quotient and Remainder

```
// 3. check whether for all inputs that satisfy the precondition
// there are some outputs that satisfy the postcondition
theorem someOutput(n:Num, m:Num)
  requires pre(n, m);
⇔ ∃q:Num, r:Num. post(n, m, q, r);

// 4. check that not every output satisfies the postcondition
theorem notEveryOutput(n:Num, m:Num)
  requires pre(n, m);
⇔ ∃q:Num, r:Num. ¬post(n, m, q, r);
```

```
Executing someOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).
Executing notEveryOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).
```

A very rough validation of the output condition.

Example: Quotient and Remainder

```
// 5. check that the output is uniquely defined
// (optional, need not generally be the case)
theorem uniqueOutput(n:Num, m:Num)
  requires pre(n, m);
⇔
  ∀q:Num, r:Num. post(n, m, q, r) ⇒
  ∀q0:Num, r0:Num. post(n, m, q0, r0) ⇒
    q = q0 ∧ r = r0;
```

```
Executing uniqueOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (18 ms, 30 checked, 6 inadmissible).
```

The output condition indeed determines the outputs uniquely.

Example: Quotient and Remainder

```
// 6. check whether the algorithm satisfies the specification
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures let q=result.1, r=result.2 in post(n, m, q, r);
{
  var q: Num = 0;
  var r: Num = n;
  while r ≥ m do
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

Check whether the algorithm satisfies the specification.

Example: Quotient and Remainder

```
Executing quotRemProc(ℤ,ℤ) with all 36 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function quotRemProc(0,1):
Result (0 ms): [0,0]
Run 7 of deterministic function quotRemProc(1,1):
Result (0 ms): [1,0]
...
Run 31 of deterministic function quotRemProc(1,5):
Result (1 ms): [0,1]
Run 32 of deterministic function quotRemProc(2,5):
Result (0 ms): [0,2]
Run 33 of deterministic function quotRemProc(3,5):
Result (0 ms): [0,3]
Run 34 of deterministic function quotRemProc(4,5):
Result (0 ms): [0,4]
Run 35 of deterministic function quotRemProc(5,5):
Result (1 ms): [1,0]
Execution completed for ALL inputs (161 ms, 30 checked, 6 inadmissible).
```

A verification of the algorithm by checking all possible executions.

Example: Quotient and Remainder

```
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures post(n, m, result.1, result.2);
{
  var q: Num = 0;
  var r: Num = n;
  while r > m do // error!
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

```
Executing quotRemProc(ℤ,ℤ) with all 36 inputs.
ERROR in execution of quotRemProc(1,1): evaluation of
  ensures let q = result.1, r = result.2 in post(n, m, q, r);
at line 65 in file divide.txt:
  postcondition is violated by result [0,1]
ERROR encountered in execution.
```

A falsification of an incorrect algorithm.

Example: Sorting an Array

```
val N:ℕ; val M:ℕ;
type elem = ℕ[M]; type array = Array[N,elem]; type index = ℤ[-1,N];

proc sort(a:array): array
  ...
{
  var b:array = a;
  for var i:index := 1; i < N; i := i+1 do
  {
    var x:elem := b[i];
    var j:index := i-1;
    while j ≥ 0 ∧ b[j] > x do
    {
      b[j+1] := b[j];
      j := j-1;
    }
    b[j+1] := x;
  }
  return b;
}
```

Example: Sorting an Array

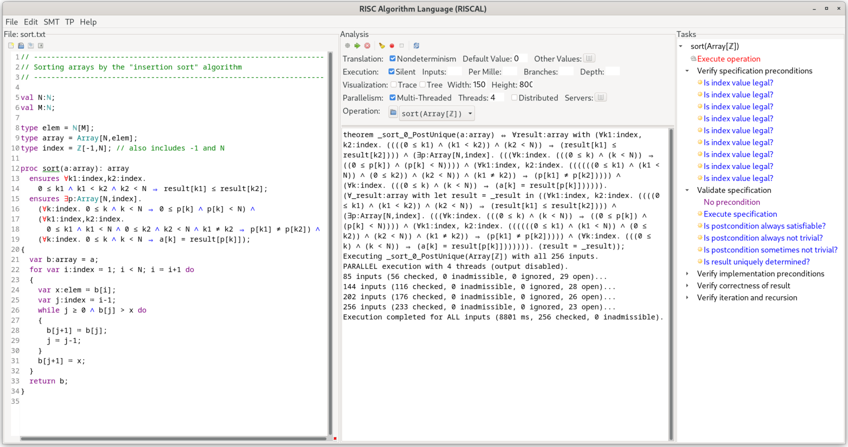
```
proc sort(a:array): array
  ensures ∀k1:index,k2:index.
    0 ≤ k1 ∧ k1 < k2 ∧ k2 < N ⇒ result[k1] ≤ result[k2];
  ensures ∃p:Array[N,index].
    (∀k:index. 0 ≤ k ∧ k < N ⇒ 0 ≤ p[k] ∧ p[k] < N) ∧
    (∀k1:index,k2:index.
      0 ≤ k1 ∧ k1 < N ∧ 0 ≤ k2 ∧ k2 < N ∧ k1 ≠ k2 ⇒ p[k1] ≠ p[k2]) ∧
    (∀k:index. 0 ≤ k ∧ k < N ⇒ a[k] = result[p[k]]);
```

```
Using N=4.
Using M=3.
Computing the value of _tbound_0...
Type checking and translation completed.
Executing sort(Array[ℤ]) with all 256 inputs.
Execution completed for ALL inputs (278 ms, 256 checked, 0 inadmissible).
```

Also this algorithm can be automatically checked.

Example: Sorting an Array

Select operation `sort` and press the button “Show/Hide Tasks”.



```
// ----------------------------------------------------------------
// Sorting arrays by the "insertion sort" algorithm
// ----------------------------------------------------------------

val N:ℕ;
val M:ℕ;

type elem = ℕ[M];
type array = Array[N,elem];
type index = ℤ[-1,N]; // also includes -1 and N

proc sort(a:array): array
  ensures ∀k1:index,k2:index.
    0 ≤ k1 ∧ k1 < k2 ∧ k2 < N ⇒ result[k1] ≤ result[k2];
  ensures ∃p:Array[N,index].
    (∀k:index. 0 ≤ k ∧ k < N → 0 ≤ p[k] ∧ p[k] < N) ∧
    (∀k1:index,k2:index.
      0 ≤ k1 ∧ k1 < N ∧ 0 ≤ k2 ∧ k2 < N ∧ k1 ≠ k2 ⇒ p[k1] ≠ p[k2]) ∧
    (∀k:index. 0 ≤ k ∧ k < N → a[k] = result[p[k]]);
{
  var b:array = a;
  for var i:index = 1; i < N; i = i+1 do
  {
    var x:elem = b[i];
    var j:index = i-1;
    while j ≥ 0 ∧ b[j] > x do
    {
      b[j+1] = b[j];
      j = j-1;
    }
    b[j+1] = x;
  }
  return b;
}
```

Translation: ☑ Nondeterminism Default Value: 0 Other Values:

Execution: ☑ Silent Inputs: Per Mille: Branches: Depth:

Visualization: □ Trace □ Tree Width: 150 Height: 800

Parallelism: ☑ Multi-Threaded Threads: 4 □ Distributed Servers:

Operation: sort(Array[ℤ])

```
theorem _sort_0_PostUnique(a:array) ⇔ ∀result:array with (∀k1:index,
k2:index. ((((0 ≤ k1) ∧ (k1 < k2)) ∧ (k2 < N)) → (result[k1] ≤
result[k2]))) ∧ (∃p:Array[N,index]. (((∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒
((0 ≤ p[k]) ∧ (p[k] < N)))) ∧ (∀k1:index, k2:index. ((((((0 ≤ k1) ∧ (k1 <
N)) ∧ (0 ≤ k2)) ∧ (k2 < N)) ∧ (k1 ≠ k2)) ⇒ (p[k1] ≠ p[k2])))) ∧
(∀k:index. (((0 ≤ k) ∧ (k < N)) → (a[k] = result[p[k]]))))).
(∀_result:array with let result = _result in ((∀k1:index, k2:index. ((((0
≤ k1) ∧ (k1 < k2)) ∧ (k2 < N)) ⇒ (result[k1] ≤ result[k2]))) ∧
(∃p:Array[N,index]. (((∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒ ((0 ≤ p[k]) ∧
(p[k] < N)))) ∧ (∀k1:index, k2:index. ((((((0 ≤ k1) ∧ (k1 < N)) ∧ (0 ≤
k2)) ∧ (k2 < N)) ∧ (k1 ≠ k2)) ⇒ (p[k1] ≠ p[k2])))) ∧ (∀k:index. (((0 ≤
k) ∧ (k < N)) ⇒ (a[k] = result[p[k]]))))))). (result = _result));
Executing _sort_0_PostUnique(Array[ℤ]) with all 256 inputs.
PARALLEL execution with 4 threads (output disabled).
85 inputs (56 checked, 0 inadmissible, 0 ignored, 29 open)...
144 inputs (116 checked, 0 inadmissible, 0 ignored, 28 open)...
202 inputs (176 checked, 0 inadmissible, 0 ignored, 26 open)...
256 inputs (233 checked, 0 inadmissible, 0 ignored, 23 open)...
Execution completed for ALL inputs (8801 ms, 256 checked, 0 inadmissible).
```

- sort(Array[ℤ])
  - Execute operation
  - Verify specification preconditions
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
    - Is index value legal?
  - Validate specification
    - No precondition
    - Execute specification
    - Is postcondition always satisfiable?
    - Is postcondition always not trivial?
    - Is postcondition sometimes not trivial?
    - Is result uniquely determined?
  - Verify implementation preconditions
  - Verify correctness of result
  - Verify iteration and recursion

Automatically generated formulas to validate procedure specifications.

Example: Sorting an Array

Right-click to print definition of a formula, double-click to check it.

```
For every input, is postcondition true for only one output?
theorem _sort_0_PostUnique(a:array) ⇔
  ∀result:array with (∀k1:index, k2:index.
    ((((0 ≤ k1) ∧ (k1 < k2)) ∧ (k2 < N)) ⇒ (result[k1] ≤ result[k2]))) ∧
    (∃p:Array[N,index].
      (((∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒ ((0 ≤ p[k]) ∧ (p[k] < N)))) ∧
      (∀k1:index, k2:index. ((((((0 ≤ k1) ∧ (k1 < N)) ∧ (0 ≤ k2)) ∧ (k2 < N)) ∧ (k1 ≠ k2)) ⇒
        (p[k1] ≠ p[k2])))) ∧
      (∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒ (a[k] = result[p[k]]))))).
    (∀_result:array with let result = _result in ((∀k1:index, k2:index.
      ((((0 ≤ k1) ∧ (k1 < k2)) ∧ (k2 < N)) ⇒ (result[k1] ≤ result[k2]))) ∧
      (∃p:Array[N,index].
        (((∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒ ((0 ≤ p[k]) ∧ (p[k] < N)))) ∧
        (∀k1:index, k2:index. ((((((0 ≤ k1) ∧ (k1 < N)) ∧ (0 ≤ k2)) ∧ (k2 < N)) ∧ (k1 ≠ k2)) ⇒
          (p[k1] ≠ p[k2])))) ∧
        (∀k:index. (((0 ≤ k) ∧ (k < N)) ⇒ (a[k] = result[p[k]])))))).
          (result = _result));

Executing _sort_0_PostUnique(Array[ℤ]) with all 256 inputs.
PARALLEL execution with 4 threads (output disabled).
85 inputs (56 checked, 0 inadmissible, 0 ignored, 29 open)...
144 inputs (116 checked, 0 inadmissible, 0 ignored, 28 open)...
202 inputs (176 checked, 0 inadmissible, 0 ignored, 26 open)...
256 inputs (233 checked, 0 inadmissible, 0 ignored, 23 open)...
Execution completed for ALL inputs (8801 ms, 256 checked, 0 inadmissible).
```

The output is indeed uniquely defined by the output condition.

Model Checking versus Proving

Two fundamental techniques for the verification of computer programs.

- Checking Program Executions
  - Enumeration of all possible executions and evaluation of formulas (e.g. postconditions) on the resulting states.
  - Fully automatic, no human interaction is required.
  - Only possible if there are only finitely many executions (and finitely many values for the quantified variables in the formulas).
  - State space explosion: "finitely many" means "not too many".
- Proving Verification Conditions
  - Logic formulas that are valid if and only if program is correct with respect to its specification.
  - Also possible if there are infinitely many excutions and infinitely many values for the quantified variables.
  - Many conditions can be automatically proved (automated reasoners); in general interaction with human is required (proof assistants).

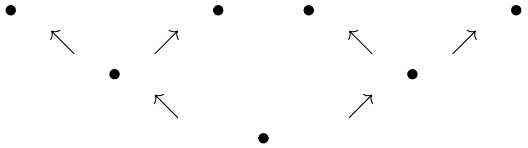
General verification requires the proving of logic formulas.

1. The Language of Logic
2. The RISC Algorithm Language
3. **The Art of Proving**
4. The RISC Theorem Proving Interface

Proofs

A proof is a structured argument that a formula is true.

- A tree whose nodes represent proof situations (states).



- Each proof situation consists of knowledge and a goal.
    - $K_1, \ldots, K_n \vdash G$
    - Knowledge $K_1, \ldots, K_n$: formulas assumed to be true.
    - Goal G: formula to be proved relative to knowledge.
- The root of the tree is the initial proof situation.
    - $K_1, \ldots, K_n$: axioms of mathematical background theories.
    - G: formula to be proved.

Proof Rules

A proof rules describes how a proof situation can be reduced to zero, one, or more “subsituations”.

$$\frac{\ldots \vdash \ldots \qquad\qquad \ldots \vdash \ldots}{K_1, \ldots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
  - Zero subsituations: G has been proved, (sub)proof is closed.
  - One or more subsituations: G is proved, if all subgoals are proved.
- Top-down rules: focus on G.
  - G is decomposed into simpler goals $G_1, G_2, \ldots$
- Bottom-up rules: focus on $K_1, \ldots, K_n$.
  - Knowledge is extended to $K_1, \ldots, K_n, K_{n+1}$.

In each proof situation, we aim at showing that the goal is “apparently” true with respect to the given knowledge.

Conjunction $F_1 \wedge F_2$

$$\frac{K \vdash G_1 \qquad K \vdash G_2}{K \vdash G_1 \wedge G_2} \qquad\qquad \frac{\ldots, K_1 \wedge K_2, K_1, K_2 \vdash G}{\ldots, K_1 \wedge K_2 \vdash G}$$

- Goal $G_1 \wedge G_2$.
  - Create two subsituations with goals G_1 and G_2.

    We have to show $G_1 \wedge G_2$.
    - *We show G_1: ... (proof continues with goal G_1)*
    - *We show G_2: ... (proof continues with goal G_2)*
- Knowledge $K_1 \wedge K_2$.
  - Create one subsituation with K_1 and K_2 in knowledge.

    We know $K_1 \wedge K_2$. We thus also know K_1 and K_2.
    (proof continues with current goal and additional knowledge K_1 and K_2)

Disjunction $F_1 \lor F_2$

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \lor G_2} \qquad \frac{\ldots, K_1 \vdash G \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \lor K_2 \vdash G}$$

- Goal $G_1 \lor G_2$.
  - Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

    We have to show $G_1 \lor G_2$. We assume $\neg G_1$ and show G_2. (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- Knowledge $K_1 \lor K_2$.
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \lor K_2$. We thus proceed by case distinction:
    - *Case K_1: ... (proof continues with current goal and additional knowledge K_1).*
    - *Case K_2: ... (proof continues with current goal and additional knowledge K_2).*

Implication $F_1 \Rightarrow F_2$

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$.
    - *We show K_1: …(proof continues with goal K_1)*
    - *We know K_2: …(proof continues with current goal and additional knowledge K_2).*

Equivalence $F_1 \Leftrightarrow F_2$

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots, (\neg)K_2 \vdash G}{\ldots, K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$.

    - *We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$)*
    - *We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)*
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$.

    - *We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$)*
    - *We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)*

Universal Quantification $\forall x : F$

$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G}\ (x_0 \text{ new for } K, G) \qquad \frac{\ldots, \forall x : K, K[T/x] \vdash G}{\ldots, \forall x : K \vdash G}$$

- Goal $\forall x : G$
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x_0/x]$.

    We have to show $\forall x : G$. Take arbitrary x_0.
    We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)
- Knowledge $\forall x : K$
  - Choose term T to create one subsituation with formula $K[T/x]$ added to the knowledge.

    We know $\forall x : K$ and thus also $K[T/x]$.
    (proof continues with current goal and additional knowledge $K[T/x]$)

Existential Quantification $\exists x : F$

$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \qquad \frac{\ldots, K[x_0/x] \vdash G}{\ldots, \exists x : K \vdash G} \ (x_0 \text{ new for } K, G)$$

- Goal $\exists x : G$
  - Choose term T to create one subsituation with goal $G[T/x]$.

    We have to show $\exists x : G$. It suffices to show $G[T/x]$.
    (proof continues with goal $G[T/x]$)

- Knowledge $\exists x : K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K[x_0/x]$.

    We know $\exists x : K$. Let x_0 be such that $K[x_0/x]$.
    (proof continues with current goal and additional knowledge $K[x_0/x]$)

Example

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

Take arbitrary y_0. We show

(c) $\exists x : P(x, y_0)$

From (1) we know for some x_0

(2) $\forall y : P(x_0, y)$

From (2) we know

(3) $P(x_0, y_0)$

From (3), we know (c). QED.

Example

We show

(a) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y)) \Rightarrow (\exists x, y : q(x,y))$

We assume

(1) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y))$

and show

(b) $\exists x, y : q(x,y)$

From (1), we know

(2) $\exists x : p(x)$
(3) $\forall x : p(x) \Rightarrow \exists y : q(x,y)$

From (2) we know for some x_0

(4) $p(x_0)$

...

Example (Contd)

...

From (3), we know

$$(5)\ p(x_0) \Rightarrow \exists y : q(x_0, y)$$

From (4) and (5), we know

$$(6)\ \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7)\ q(x_0, y_0)$$

From (7), we know (b). QED.

1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

4. **The RISC Theorem Proving Interface**

The RISC Theorem Proving Interface

- RISCTP: an interface to various theorem proving methods.
  - Research Institute for Symbolic Computation (RISC), 2022–.
    https://www.risc.jku.at/research/formal/software/RISCTP
- Proof Method SMT:
  - Translation to a proof problem in the SMT-LIB language.
  - Application of external provers/SMT solvers Z3, cvc5, Vampire.
  - Fast and effective for problems of moderate complexity.
  - Black box: no human-readable/understandable proofs.
- Proof Method MESON:
  - First proof decomposition/simplification by logical/arithmetical rules.
  - Then application of “Model Elimination, Subgoal-Oriented”.
  - (Optional) support by external SMT solvers for larger efficiency.
  - Transparent: human-readable/understandable proofs.

Developed to provide RISCAL with theorem proving capabilities.

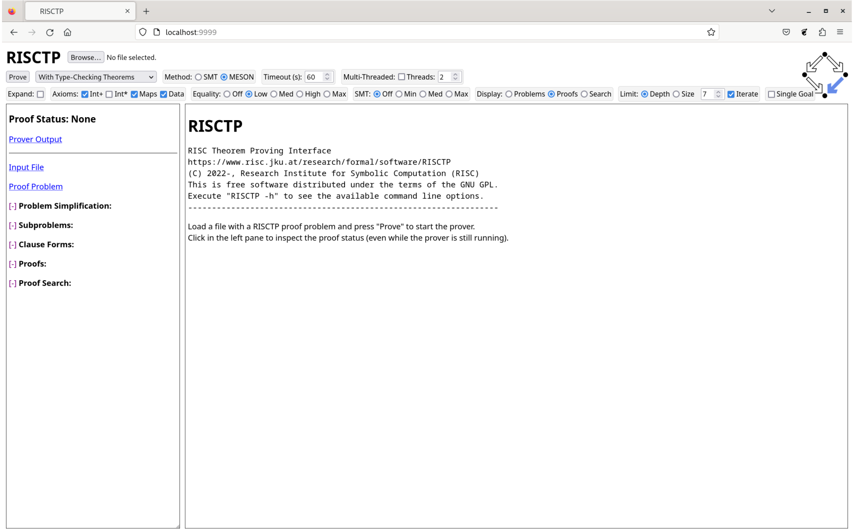
RISCTP as a Standalone Prover

```
> RISCTP -solver z3 -path /software/RISCTP/etc/z3 -web 9999 1
RISC Theorem Proving Interface 1.8.0 (July 15, 2024)
https://www.risc.jku.at/research/formal/software/RISCTP
(C) 2022-, Research Institute for Symbolic Computation (RISC)
This is free software distributed under the terms of the GNU GPL.
Execute "RISCTP -h" to see the available command line options.
-----------------------------------------------------------------
RISCTP GUI can be browsed at http://localhost:9999/
Press <Enter> to terminate the server.
```

- `-solver z3`: use SMT solver Z3 (default).
- `-path /software/...`: path to executable of SMT solver.
- `-web 9999 1`: show (full) GUI at `http://localhost:9999/`

The RISCTP GUI can be accessed by any web browser.

The RISCTP GUI



RISCTP Browse... No file selected.

Prove | With Type-Checking Theorems | Method: ○ SMT ● MESON | Timeout (s): 60 | Multi-Threaded: □ Threads: 2

Expand: □ | Axioms: ☑ Int+ □ Int* ☑ Maps ☑ Data | Equality: ○ Off ● Low ○ Med ○ High ○ Max | SMT: ● Off ○ Min ○ Med ○ Max | Display: ○ Problems ● Proofs ○ Search | Limit: ● Depth ○ Size 7 ☑ Iterate | □ Single Goal

Proof Status: None

Prover Output

Input File

Proof Problem

[-] **Problem Simplification:**

[-] **Subproblems:**

[-] **Clause Forms:**

[-] **Proofs:**

[-] **Proof Search:**

RISCTP

```
RISC Theorem Proving Interface
https://www.risc.jku.at/research/formal/software/RISCTP
(C) 2022-, Research Institute for Symbolic Computation (RISC)
This is free software distributed under the terms of the GNU GPL.
Execute "RISCTP -h" to see the available command line options.
-----------------------------------------------------------------
```

Load a file with a RISCTP proof problem and press "Prove" to start the prover.
Click in the left pane to inspect the proof status (even while the prover is still running).

Proof Method SMT

```
// problem file "fol2a.txt"
type T;
pred p(x:T,y:T);
theorem T ⇔ (∃x:T.∀y:T.p(x,y)) ⇒ (∀x:T.∃y:T.p(y,x));
```

- Button “Browse” `fol2a.txt`.
- Option “Method: SMT”, button “Prove” ⇝ “Proof Status: Success”.
- Link “Prover Output”.

```
=== SMT solving
SMT solver: Z3 version 4.13.0 - 64 bit
Proving theorem T...
SUCCESS: theorem was proved (11 ms).
=== SMT-LIB solver session
(set-logic ALL)
(set-option :produce-unsat-cores true)
(declare-sort T 0)
(declare-fun p ( T  T ) Bool)
(push 1)
(assert (not (=> (exists ((x T)) (forall ((y T)) (p x y))) (forall ((x T)) (exists ((y T)) (p y x))))))
(check-sat)
(pop 1)
(exit)
===
SUCCESS termination (26 ms).
```

SUCCESS: theorem was proved (however, claim is not substantiated).

Proof Method SMT

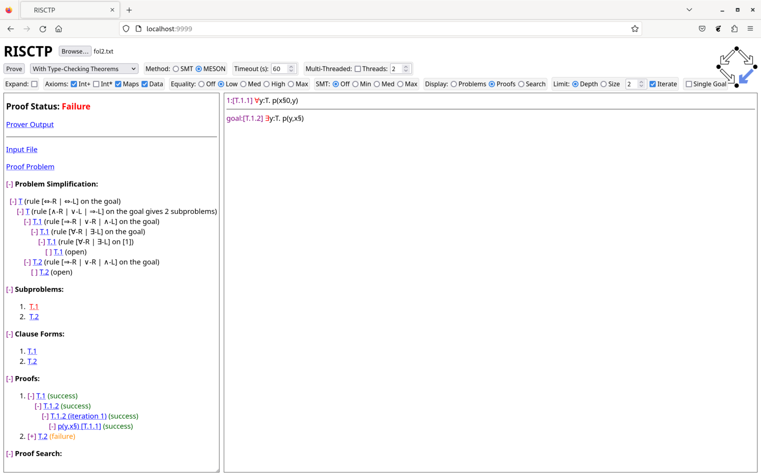
```
// problem file "fol2.txt"
type T;
pred p(x:T,y:T);
// actually, implication only holds from left to right
theorem T ⇔ (∃x:T.∀y:T.p(x,y)) ⇔ (∀x:T.∃y:T.p(y,x));
```

- Button "Browse" `fol2.txt` ⇝ "Proof Status: Failure".

```
=== SMT solving
SMT solver: Z3 version 4.13.0 - 64 bit
Proving theorem T...
FAILURE: theorem was not proved (13 ms).
theorem T ⇔ (∃x:T. (∀y:T. p(x,y))) ⇔ (∀x:T. (∃y:T. p(y,x)));
sat
=== SMT-LIB solver session
(set-logic ALL)
(set-option :produce-unsat-cores true)
(declare-sort T 0)
(declare-fun p ( T  T ) Bool)
(push 1)
(assert (not (= (exists ((x T)) (forall ((y T)) (p x y))) (forall ((x T)) (exists ((y T)) (p y x))))))
(check-sat)
(pop 1)
(exit)
===
FAILURE termination (31 ms).
```

FAILURE: theorem was not proved (however, no indication why this is so).

Proof Method MESON



RISCTP Browse... fol2.txt

Prove | With Type-Checking Theorems | Method: ○ SMT ● MESON | Timeout (s): 60 | Multi-Threaded: □ Threads: 2

Expand: □ | Axioms: ☑ Int+ □ Int* ☑ Maps ☑ Data | Equality: ○ Off ● Low ○ Med ○ High ○ Max | SMT: ● Off ○ Min ○ Med ○ Max | Display: ○ Problems ● Proofs ○ Search | Limit: ● Depth ○ Size 2 ☑ Iterate | □ Single Goal

Proof Status: Failure

Prover Output

Input File

Proof Problem

[-] **Problem Simplification:**

```
[-] T (rule [⇔-R | ⇔-L] on the goal)
  [-] T (rule [∧-R | ∨-L | ⇒-L] on the goal gives 2 subproblems)
    [-] T.1 (rule [⇒-R | ∨-R | ∧-L] on the goal)
      [-] T.1 (rule [∀-R | ∃-L] on the goal)
        [-] T.1 (rule [∀-R | ∃-L] on [1])
          [ ] T.1 (open)
    [-] T.2 (rule [⇒-R | ∨-R | ∧-L] on the goal)
      [ ] T.2 (open)
```

[-] **Subproblems:**

1. T.1
2. T.2

[-] **Clause Forms:**

1. T.1
2. T.2

[-] **Proofs:**

```
1. [-] T.1 (success)
     [-] T.1.2 (success)
       [-] T.1.2 (iteration 1) (success)
         [-] p(y,x§) [T.1.1] (success)
2. [+] T.2 (failure)
```

[-] **Proof Search:**

1:[T.1.1] ∀y:T. p(x§0,y)

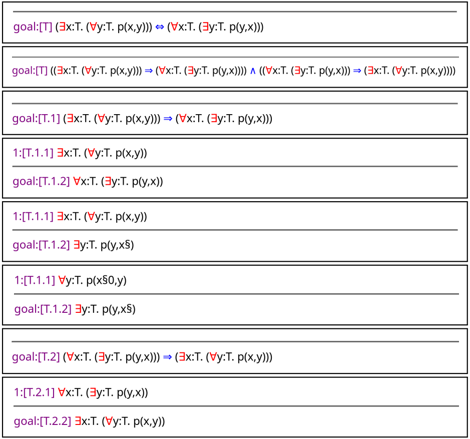
goal:[T.1.2] ∃y:T. p(y,x§)

Problem simplification yields two subproblems of which one can be proved.

Problem Simplification

[-] **Problem Simplification:**

```
[-] T (rule [⇔-R | ⇔-L] on the goal)
    [-] T (rule [∧-R | ∨-L | ⇒-L] on the goal gives 2 subproblems)
        [-] T.1 (rule [⇒-R | ∨-R | ∧-L] on the goal)
            [-] T.1 (rule [∀-R | ∃-L] on the goal)
                [-] T.1 (rule [∀-R | ∃-L] on [1])
                    [ ] T.1 (open)
        [-] T.2 (rule [⇒-R | ∨-R | ∧-L] on the goal)
            [ ] T.2 (open)
```



```
goal:[T] (∃x:T. (∀y:T. p(x,y))) ⇔ (∀x:T. (∃y:T. p(y,x)))

goal:[T] ((∃x:T. (∀y:T. p(x,y))) ⇒ (∀x:T. (∃y:T. p(y,x)))) ∧ ((∀x:T. (∃y:T. p(y,x))) ⇒ (∃x:T. (∀y:T. p(x,y))))

goal:[T.1] (∃x:T. (∀y:T. p(x,y))) ⇒ (∀x:T. (∃y:T. p(y,x)))

1:[T.1.1] ∃x:T. (∀y:T. p(x,y))
goal:[T.1.2] ∀x:T. (∃y:T. p(y,x))

1:[T.1.1] ∃x:T. (∀y:T. p(x,y))
goal:[T.1.2] ∃y:T. p(y,x§)

1:[T.1.1] ∀y:T. p(x§0,y)
goal:[T.1.2] ∃y:T. p(y,x§)

goal:[T.2] (∀x:T. (∃y:T. p(y,x))) ⇒ (∃x:T. (∀y:T. p(x,y)))

1:[T.2.1] ∀x:T. (∃y:T. p(y,x))
goal:[T.2.2] ∃x:T. (∀y:T. p(x,y))
```

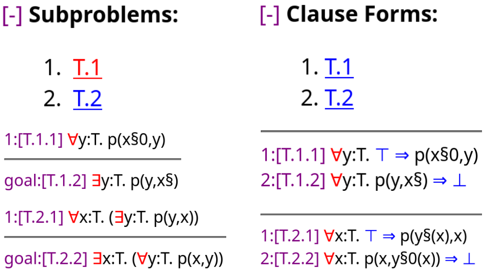
A step-by-step decomposition of the problem into simpler subproblems; each consists of “knowledge” formulas and a “goal” formula.

Clause Transformation

Each formula in a proof (sub)problem is transformed into a set of clauses.

- Clause $\forall x, \ldots. (A_1 \wedge \ldots \wedge A_a) \Rightarrow (B_1 \vee \ldots \vee B_b)$.
  - Closed formula with universally quantified variables $x, \ldots$.
  - The quantifier prefix $\forall x, \ldots$ is usually dropped.
  - Existential variables are replaced by Skolem constants/functions.
- Positive literals (atomic formulas) A_i and B_j.
  - Clause be written as disjunction $(\neg A_1 \vee \ldots \vee \neg A_a \vee B_1 \vee \ldots \vee B_b)$.
    - Negative literals $\neg A_i$, positive literals B_j.
  - Clause is true if some A_i is false or some B_j is true.
    - For some values of the quantified variables.
- Proof problem $K_1, \ldots, K_n \vdash G$:
  - Have to prove validity ("truth") of $(K_1 \wedge \ldots \wedge K_n \Rightarrow G)$.
  - Suffices to prove unsatisfiability ("falseness") of $(K_1 \wedge \ldots \wedge K_n \wedge \neg G)$.
  - Suffices to transform each K_i and $\neg G$ into clauses $\{C_1, \ldots, C_c\}$ and to prove the unsatisfiability of their conjunction $(C_1 \wedge \ldots \wedge C_c)$.
    - Suffices to prove the validity of $(C_1 \wedge \ldots \wedge C_{c-1}) \Rightarrow \neg C_c$.

Clause Transformation



[-] **Subproblems:**

1. T.1
2. T.2

1:[T.1.1] ∀y:T. p(x§0,y)

goal:[T.1.2] ∃y:T. p(y,x§)

1:[T.2.1] ∀x:T. (∃y:T. p(y,x))

goal:[T.2.2] ∃x:T. (∀y:T. p(x,y))

[-] **Clause Forms:**

1. T.1
2. T.2

1:[T.1.1] ∀y:T. ⊤ ⇒ p(x§0,y)
2:[T.1.2] ∀y:T. p(y,x§) ⇒ ⊥

1:[T.2.1] ∀x:T. ⊤ ⇒ p(y§(x),x)
2:[T.2.2] ∀x:T. p(x,y§0(x)) ⇒ ⊥

- Subproblems:
  - Above line: knowledge formulas.
  - Below line: goal.
- Clause Forms:
  - Above line: clauses from theory axioms (here none).
  - Below line: clauses from theorem (knowledge and negation of goal).

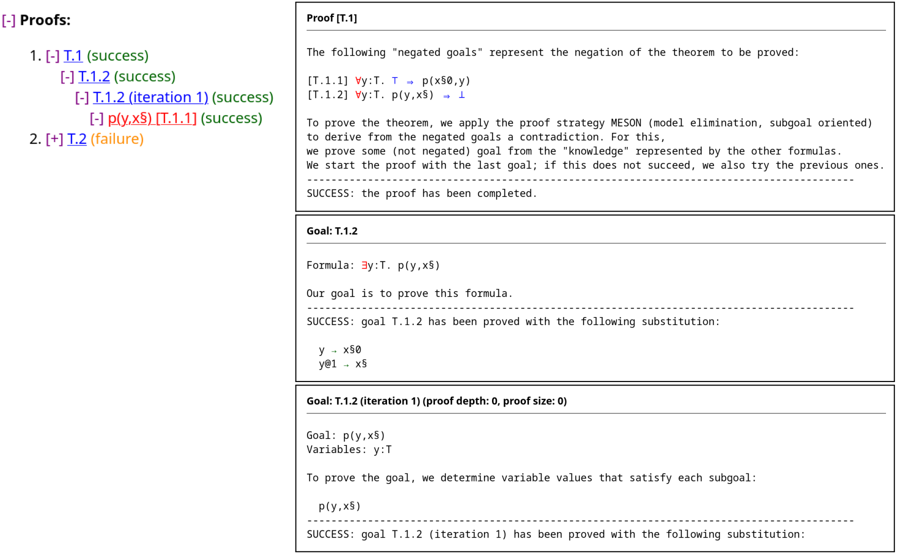
It suffices to prove the negation of the last clause from the other clauses.

Proof Method MESON

- MESON: Model Elimination, Subgoal-Oriented (Loveland, 1968).
  - A (Prolog-like) “backchaining strategy” for proving.
- Current goal: literal G (initially from the goal clause).
  - Current variable substitution σ.
- Pick clause $(L_1 \lor \ldots \lor L_i \lor \ldots \lor L_n)$.
  - Goal $G\sigma$ can be unified with $L_i\sigma$ by new substitution σ'.
  - Equivalent to $(\neg L_1 \land \ldots \land \neg L_{i-1} \land \neg L_{i+1} \land \ldots \land \neg L_n \Rightarrow L_i)$.
- New goals: $(\neg L_1, \ldots, \neg L_{i-1}, \neg L_{i+1}, \ldots, \neg L_l)$.
  - New variable substitution $\sigma\sigma'$.
  - Goal G is replaced by negations of clause literals other than L_i.
- Assumptions: literals $A_1, \ldots, A_a$.
  - G may be also proved from the current set of assumptions.
  - If not, we add $\neg G$ to the assumptions for the proof of the new goal.

During the proof search, the method must attempt every literal in every clause that can be unified with the goal literal; furthermore, the proof search must start from every clause arising from the theorem.

Proof Method MESON



[-] **Proofs:**

1. [-] T.1 (success)
   - [-] T.1.2 (success)
     - [-] T.1.2 (iteration 1) (success)
       - [-] p(y,x§) [T.1.1] (success)
2. [+] T.2 (failure)

Proof [T.1]

```
The following "negated goals" represent the negation of the theorem to be proved:

[T.1.1] ∀y:T. ⊤ ⇒ p(x§0,y)
[T.1.2] ∀y:T. p(y,x§) ⇒ ⊥

To prove the theorem, we apply the proof strategy MESON (model elimination, subgoal oriented)
to derive from the negated goals a contradiction. For this,
we prove some (not negated) goal from the "knowledge" represented by the other formulas.
We start the proof with the last goal; if this does not succeed, we also try the previous ones.
-----------------------------------------------------------------------------------------
SUCCESS: the proof has been completed.
```

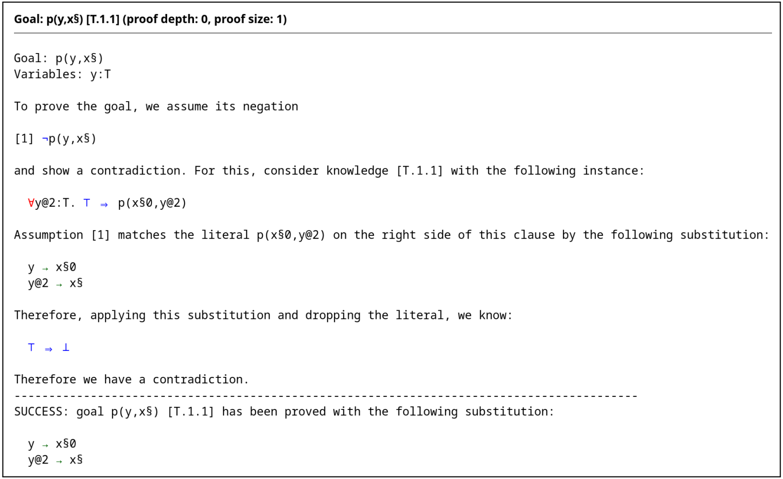
Goal: T.1.2

```
Formula: ∃y:T. p(y,x§)

Our goal is to prove this formula.
-----------------------------------------------------------------------------------------
SUCCESS: goal T.1.2 has been proved with the following substitution:

  y → x§0
  y@1 → x§
```

Goal: T.1.2 (iteration 1) (proof depth: 0, proof size: 0)

```
Goal: p(y,x§)
Variables: y:T

To prove the goal, we determine variable values that satisfy each subgoal:

  p(y,x§)
-----------------------------------------------------------------------------------------
SUCCESS: goal T.1.2 (iteration 1) has been proved with the following substitution:
```

Proof Method MESON



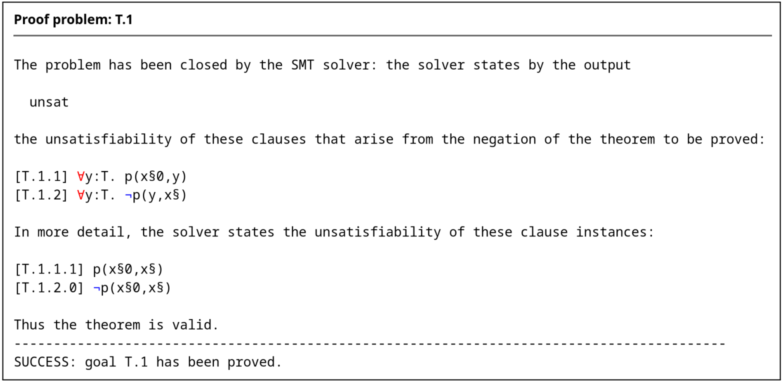
Goal: p(y,x§) [T.1.1] (proof depth: 0, proof size: 1)

```
Goal: p(y,x§)
Variables: y:T

To prove the goal, we assume its negation

[1] ¬p(y,x§)

and show a contradiction. For this, consider knowledge [T.1.1] with the following instance:

  ∀y@2:T. ⊤ ⇒ p(x§0,y@2)

Assumption [1] matches the literal p(x§0,y@2) on the right side of this clause by the following substitution:

  y → x§0
  y@2 → x§

Therefore, applying this substitution and dropping the literal, we know:

  ⊤ ⇒ ⊥

Therefore we have a contradiction.
------------------------------------------------------------------------------------------
SUCCESS: goal p(y,x§) [T.1.1] has been proved with the following substitution:

  y → x§0
  y@2 → x§
```

The problem is closed by substituting in the first clause variable y with constant $x\S 0$ and in the second clause variable y with constant $x\S$.

Proof Method MESON with “SMT: Max”

We attempt the proof with the help of the external SMT solver first.

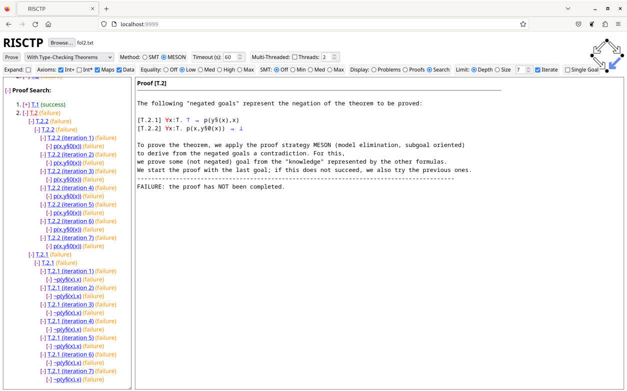


Proof problem: T.1

```
The problem has been closed by the SMT solver: the solver states by the output

  unsat

the unsatisfiability of these clauses that arise from the negation of the theorem to be proved:

[T.1.1] ∀y:T. p(x§0,y)
[T.1.2] ∀y:T. ¬p(y,x§)

In more detail, the solver states the unsatisfiability of these clause instances:

[T.1.1.1] p(x§0,x§)
[T.1.2.0] ¬p(x§0,x§)

Thus the theorem is valid.
-------------------------------------------------------------------------------------------
SUCCESS: goal T.1 has been proved.
```

First we determine the clauses needed to close the proof problem, then we determine the actual instances of the clauses needed.

Proof Method MESON: Failed Proofs



RISCTP Browse... fol2.txt

Prove | With Type-Checking Theorems | Method: ○ SMT ● MESON | Timeout (s): 60 | Multi-Threaded: □ Threads: 2

Expand: □ | Axioms: ☑ Int+ □ Int* ☑ Maps ☑ Data | Equality: ○ Off ● Low ○ Med ○ High ○ Max | SMT: ● Off ○ Min ○ Med ○ Max | Display: ○ Problems ○ Proofs ● Search | Limit: ● Depth ○ Size 7 ☑ Iterate | □ Single Goal

```
[-] Proof Search:

    1. [+] T.1 (success)
    2. [-] T.2 (failure)
       [-] T.2.2 (failure)
         [-] T.2.2 (failure)
           [-] T.2.2 (iteration 1) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 2) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 3) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 4) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 5) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 6) (failure)
             [-] p(x,y§0(x)) (failure)
           [-] T.2.2 (iteration 7) (failure)
             [-] p(x,y§0(x)) (failure)
       [-] T.2.1 (failure)
         [-] T.2.1 (failure)
           [-] T.2.1 (iteration 1) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 2) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 3) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 4) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 5) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 6) (failure)
             [-] ¬p(y§(x),x) (failure)
           [-] T.2.1 (iteration 7) (failure)
             [-] ¬p(y§(x),x) (failure)
```

Proof [T.2]

```
The following "negated goals" represent the negation of the theorem to be proved:

[T.2.1] ∀x:T. ⊤ → p(y§(x),x)
[T.2.2] ∀x:T. p(x,y§0(x)) ⇒ ⊥

To prove the theorem, we apply the proof strategy MESON (model elimination, subgoal oriented)
to derive from the negated goals a contradiction. For this,
we prove some (not negated) goal from the "knowledge" represented by the other formulas.
We start the proof with the last goal; if this does not succeed, we also try the previous ones.
-----------------------------------------------------------------------------------
FAILURE: the proof has NOT been completed.
```

- Limit: Depth *D* Iterate: iteratively search for a proof up to depth *D*.
- Display: Search: generate a proof tree also for a failed search.

We may also investigate failed proof attempts.

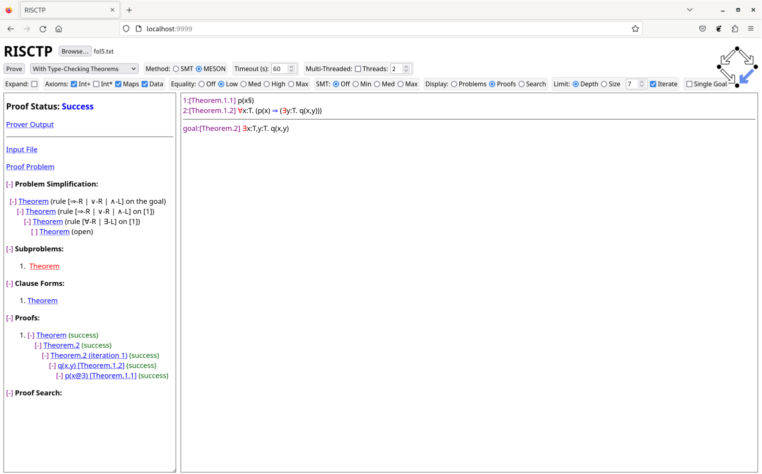
Another Proof Problem

```
// problem file "fol5.txt"
type T;
pred p(x:T);
pred q(x:T,y:T);
theorem Theorem ⇔
  (∃x:T. p(x)) ∧ (∀x:T. p(x) ⇒ ∃y:T. q(x,y)) ⇒ (∃x:T,y:T. q(x,y));
```

```
=== SMT solving
SMT solver: Z3 version 4.13.0 - 64 bit
Proving theorem Theorem...
SUCCESS: theorem was proved (9 ms).
=== SMT-LIB solver session
(set-logic ALL)
(set-option :produce-unsat-cores true)
(declare-sort T 0)
(declare-fun p ( T ) Bool)
(declare-fun q ( T  T ) Bool)
(push 1)
(assert (not (=> (and (exists ((x T)) (p x)) (forall ((x T))
  (=> (p x) (exists ((y T)) (q x y))))) (exists ((x T)) (exists ((y T)) (q x y))))))
(check-sat)
(pop 1)
(exit)
===
SUCCESS termination (15 ms).
```

Proof succeeds with Method SMT.

Another Proof Problem (Continued)



RISCTP | localhost:9999

RISCTP Browse… fol5.txt

Prove | With Type-Checking Theorems | Method: ○ SMT ● MESON | Timeout (s): 60 | Multi-Threaded: □ Threads: 2

Expand: □ | Axioms: ☑ Int+ □ Int* ☑ Maps ☑ Data | Equality: ○ Off ● Low ○ Med ○ High ○ Max | SMT: ● Off ○ Min ○ Med ○ Max | Display: ○ Problems ● Proofs ○ Search | Limit: ● Depth ○ Size 7 ☑ Iterate | □ Single Goal

Proof Status: Success

Prover Output

Input File

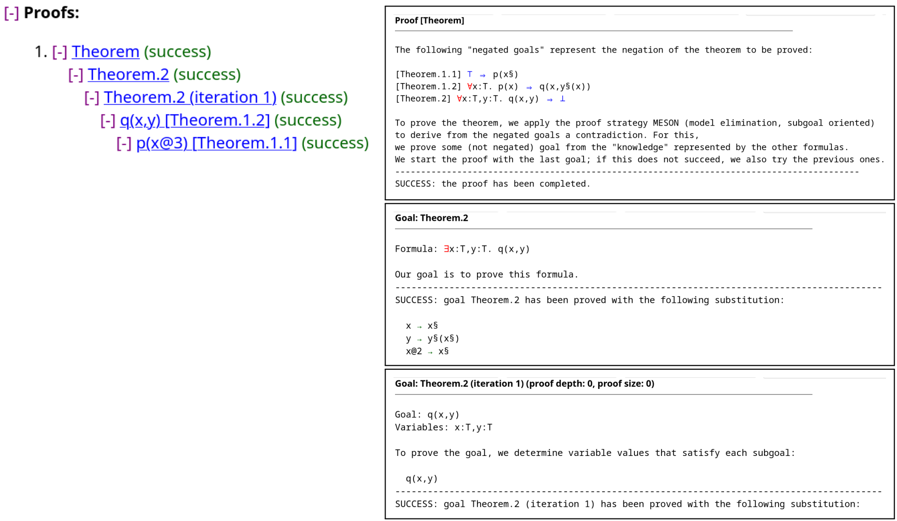
Proof Problem

```
[-] Problem Simplification:

 [-] Theorem (rule [⇒-R | ∨-R | ∧-L] on the goal)
   [-] Theorem (rule [⇒-R | ∨-R | ∧-L] on [1])
     [-] Theorem (rule [∀-R | ∃-L] on [1])
       [ ] Theorem (open)

[-] Subproblems:

    1. Theorem

[-] Clause Forms:

    1. Theorem

[-] Proofs:

    1. [-] Theorem (success)
         [-] Theorem.2 (success)
           [-] Theorem.2 (iteration 1) (success)
             [-] q(x,y) [Theorem.1.2] (success)
               [-] p(x@3) [Theorem.1.1] (success)

[-] Proof Search:
```

```
1:[Theorem.1.1] p(x§)
2:[Theorem.1.2] ∀x:T. (p(x) ⇒ (∃y:T. q(x,y)))

goal:[Theorem.2] ∃x:T,y:T. q(x,y)
```

Proof succeeds with Method MESON.

Another Proof Problem (Continued)



[-] **Proofs:**

1. [-] Theorem (success)
   - [-] Theorem.2 (success)
     - [-] Theorem.2 (iteration 1) (success)
       - [-] q(x,y) [Theorem.1.2] (success)
         - [-] p(x@3) [Theorem.1.1] (success)

Proof [Theorem]

```
The following "negated goals" represent the negation of the theorem to be proved:

[Theorem.1.1] ⊤ ⇒ p(x§)
[Theorem.1.2] ∀x:T. p(x) ⇒ q(x,y§(x))
[Theorem.2] ∀x:T,y:T. q(x,y) ⇒ ⊥

To prove the theorem, we apply the proof strategy MESON (model elimination, subgoal oriented)
to derive from the negated goals a contradiction. For this,
we prove some (not negated) goal from the "knowledge" represented by the other formulas.
We start the proof with the last goal; if this does not succeed, we also try the previous ones.
--------------------------------------------------------------------------------------
SUCCESS: the proof has been completed.
```

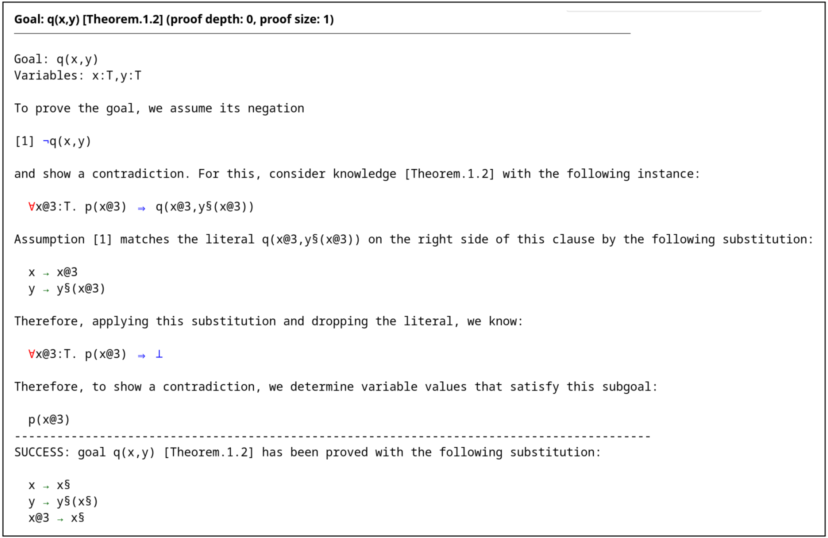
Goal: Theorem.2

```
Formula: ∃x:T,y:T. q(x,y)

Our goal is to prove this formula.
------------------------------------------------------------------------------------------
SUCCESS: goal Theorem.2 has been proved with the following substitution:

  x → x§
  y → y§(x§)
  x@2 → x§
```

Goal: Theorem.2 (iteration 1) (proof depth: 0, proof size: 0)

```
Goal: q(x,y)
Variables: x:T,y:T

To prove the goal, we determine variable values that satisfy each subgoal:

  q(x,y)
------------------------------------------------------------------------------------------
SUCCESS: goal Theorem.2 (iteration 1) has been proved with the following substitution:
```

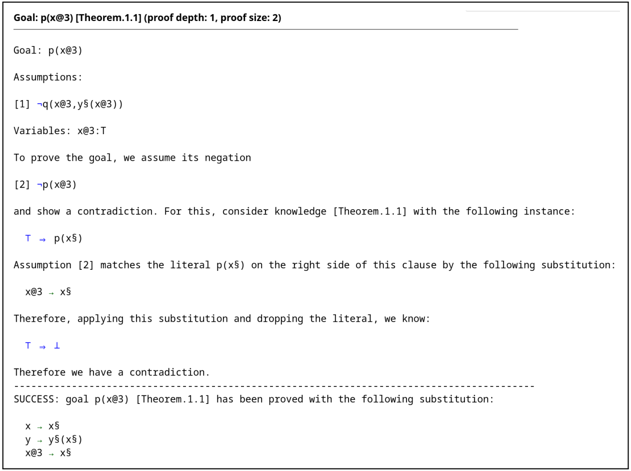
Another Proof Problem (Continued)



Goal: q(x,y) [Theorem.1.2] (proof depth: 0, proof size: 1)

```
Goal: q(x,y)
Variables: x:T,y:T

To prove the goal, we assume its negation

[1] ¬q(x,y)

and show a contradiction. For this, consider knowledge [Theorem.1.2] with the following instance:

  ∀x@3:T. p(x@3) ⇒ q(x@3,y§(x@3))

Assumption [1] matches the literal q(x@3,y§(x@3)) on the right side of this clause by the following substitution:

  x → x@3
  y → y§(x@3)

Therefore, applying this substitution and dropping the literal, we know:

  ∀x@3:T. p(x@3) ⇒ ⊥

Therefore, to show a contradiction, we determine variable values that satisfy this subgoal:

  p(x@3)
-----------------------------------------------------------------------------------------
SUCCESS: goal q(x,y) [Theorem.1.2] has been proved with the following substitution:

  x → x§
  y → y§(x§)
  x@3 → x§
```

Another Proof Problem (Continued)



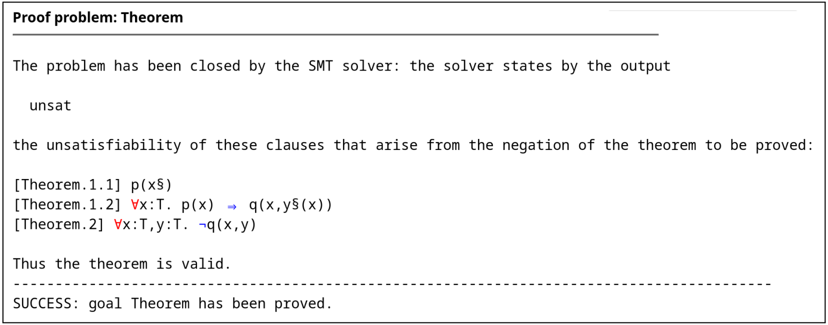
Goal: p(x@3) [Theorem.1.1] (proof depth: 1, proof size: 2)

```
Goal: p(x@3)

Assumptions:

[1] ¬q(x@3,y§(x@3))

Variables: x@3:T

To prove the goal, we assume its negation

[2] ¬p(x@3)

and show a contradiction. For this, consider knowledge [Theorem.1.1] with the following instance:

  ⊤ ⇒ p(x§)

Assumption [2] matches the literal p(x§) on the right side of this clause by the following substitution:

  x@3 → x§

Therefore, applying this substitution and dropping the literal, we know:

  ⊤ ⇒ ⊥

Therefore we have a contradiction.
------------------------------------------------------------------------------------------
SUCCESS: goal p(x@3) [Theorem.1.1] has been proved with the following substitution:

  x → x§
  y → y§(x§)
  x@3 → x§
```

Proof succeeds by instantiating in clause 2 variable x with constant $x§$ and in clause 3 variables x and y with constants $x§$ and $y§(x§)$, respectively.

Anoter Proof Problem (Continued)

Set option “SMT: Max”.



Proof problem: Theorem

```
The problem has been closed by the SMT solver: the solver states by the output

  unsat

the unsatisfiability of these clauses that arise from the negation of the theorem to be proved:

[Theorem.1.1] p(x§)
[Theorem.1.2] ∀x:T. p(x) ⇒ q(x,y§(x))
[Theorem.2] ∀x:T,y:T. ¬q(x,y)

Thus the theorem is valid.
-------------------------------------------------------------------------------------------
SUCCESS: goal Theorem has been proved.
```

Here the actual clause instances could not be determined (a simple strategy is applied that attempts only instantiations with variable-free terms that appear in the proof problem).