COMBINING GENERALIZATION ALGORITHMS

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Generalization problem

Given: An equational theory E and two terms t_1 and t_2.

Find: A term r such that for some substitutions σ_1 and σ_2,

$$r\sigma_1 =_E t_1 \quad \text{and} \quad r\sigma_2 =_E t_2.$$

r is more general than t_1 and t_2 (notation: $r \preceq_E t_i$, $i = 1, 2$).

σ_1 and σ_2: witness substitutions.

r: a common generalization of t_1 and t_2.

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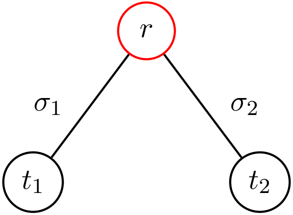
σ_1 and σ_2: witness substitutions.

r: a common generalization of t_1 and t_2.

r is a least general generalization (lgg) of t_1 and t_2 if there is no r' such that

- r' is a generalization of t_1 and t_2, and
- r' is strictly less general than r.

Lest General Generalizations



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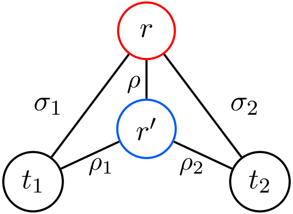

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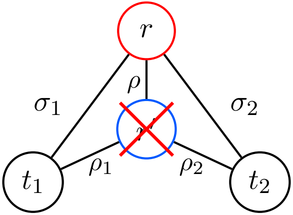

Examples

$E = \emptyset :$ syntactic equality

$t_1 = f(a, g(a))$

$t_2 = f(b, g(b))$

$r = f(x, g(x)) :$ a single lgg

$$r\{x \mapsto a\} =_E t_1$$

$$r\{x \mapsto b\} =_E t_2$$

Examples

$E = \{f(x,y) = f(y,x)\}$: f is commutative

$t_1 = g(a, f(a,b))$

$t_2 = g(b, f(b,a))$

$r_1 = g(x, f(x,y))$: an lgg

$$r_1\{x \mapsto a, y \mapsto b\} =_E t_1$$

$$r_1\{x \mapsto b, y \mapsto a\} =_E t_2$$

Examples

$$E = \{f(x,y) = f(y,x)\}: \ f \text{ is commutative}$$

$$t_1 = g(a, f(a,b))$$

$$t_2 = g(b, f(b,a))$$

$$r_1 = g(x, f(x,y)): \ \text{an lgg}$$

$$r_1\{x \mapsto a, y \mapsto b\} =_E t_1$$

$$r_1\{x \mapsto b, y \mapsto a\} =_E t_2$$

$$r_2 = g(z, f(a,b)): \ \text{another lgg}$$

$$r_2\{z \mapsto a\} =_E t_1$$

$$r_1\{z \mapsto b\} =_E t_2$$

Examples

$E = \{f(f(x,y),z) = f(x,f(y,z))\}$: f is associative

$t_1 = f(a,b,b,a)$

$t_2 = f(a,b,a)$

$r_1 = f(a,b,x)$: an lgg

$$r_1\{x \mapsto f(b,a)\} =_E t_1$$

$$r_1\{x \mapsto a\} =_E t_2$$

Examples

$E = \{f(f(x, y), z) = f(x, f(y, z))\}$: f is associative

$t_1 = f(a, b, b, a)$

$t_2 = f(a, b, a)$

$r_1 = f(a, b, x)$: an lgg

$r_1\{x \mapsto f(b, a)\} =_E t_1$

$r_1\{x \mapsto a\} =_E t_2$

$r_2 = f(y, b, a)$: another lgg

$r_2\{y \mapsto f(a, b)\} =_E t_1$

$r_1\{y \mapsto a\} =_E t_2$

Examples

$$E = \{f(f(x,y),z) = f(x,f(y,z)), f(x,e) = x, f(e,x) = x\}:$$

f is associative, e its unit element

$$t_1 = f(a,b,b,a)$$

$$t_2 = f(a,b,a)$$

$r_1 = f(a,b,x,a):$ one lgg

$$r_1\{x \mapsto b\} =_E t_1$$

$$r_1\{x \mapsto e\} =_E t_2$$

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$r_2 = f(a,y,b,a):$ another lgg

$r_2\{y \mapsto b\} =_E t_1$

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Examples

$$E = \{f(f(a,a),a) = f(a,a),\ f(f(b,b),b) = f(b,b)\}$$

$$t_1 = f(a,a)$$

$$t_2 = f(b,b)$$

$$r_1 = f(x_1, x_1),$$

$$r_2 = f(f(x_2, x_2), x_2),$$

$$r_3 = f(f(f(x_3, x_3), x_3), x_3),$$

$$\dots$$

infinitely many lggs

$r_i\{x_i \mapsto a\} =_E t_1$ for any $i \geq 1$

$r_i\{x_i \mapsto b\} =_E t_2$ for any $i \geq 1$

Examples

$$E = \{f(a) = a, f(b) = b\}$$

$$t_1 = a$$

$$t_2 = b$$

A chain of strictly decreasing generalizations

$r_1 = x_1 \prec_E r_2 = f(x_2) \prec_E r_3 = f(f(x_3)) \cdots$

$r_i\{x_i \mapsto a\} =_E t_1$ for any $i \geq 1$

$r_i\{x_i \mapsto b\} =_E t_2$ for any $i \geq 1$

Minimal Complete Set of Generalizations

Given an equational theory E, a set of terms $\mathcal{G}$ is called a complete set of E-generalizations of the given terms t_1 and t_2, if the following properties are satisfied:

- **Soundness:** each $r \in \mathcal{G}$ is an E-generalization of t_1 and t_2.
- **Completeness:** for each E-generalization r' of t_1 and t_2 there exists $r \in \mathcal{G}$ such that $r' \preceq_E r$.

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The set $\mathcal{G}$ is called a minimal complete set of E-generalizations of t_1 and t_2, denoted by $\mathsf{mcsg}_E(t_1, t_2)$, if, in addition, the following holds:

- **Minimality:** no distinct elements of $\mathcal{G}$ are $\preceq_E$-comparable.

Generalization Algorithm in the Free Theory

A rule-base algorithm working on configurations $P; S; \sigma$, where

- P is a set of unsolved problems,
- S is a set of solved problems, and
- σ is a substitution representing the generalization computed so far.

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The process stops in some final configuration $\emptyset; S; \sigma$ such that

- $x\sigma$ is an lgg of t and s,
- S gives the corresponding witness substitutions.

Generalization Algorithm in the Free Theory

DEC: **Decomposition**

$$\{x : f(t_1, \ldots, t_n) \triangleq f(s_1, \ldots, s_n)\} \uplus P; S; \sigma \Longrightarrow$$
$$P \cup \{y_1 : t_1 \triangleq s_1, \ldots, y_n : t_n \triangleq s_n\}; S; \sigma\{x \mapsto f(y_1, \ldots, y_n)\},$$

where $n \geq 0$ and $y_1, \ldots, y_n$ are fresh.

SOL: **Solve**

$$\{x : t \triangleq s\} \uplus P; S; \sigma \Longrightarrow P; S \cup \{x : t \triangleq s\}; \sigma,$$

if $\mathsf{root}(t) \neq \mathsf{root}(s)$.

MER: **Merge**

$$\emptyset; \{x : t \triangleq s,\ x' : t \triangleq s\} \uplus S; \sigma \Longrightarrow \emptyset; S \cup \{x : t \triangleq s\}; \sigma\{x' \mapsto x\}.$$

Generalization Algorithm in the Free Theory

Example

Syntactic generalization of $f(a, g(a))$ and $f(b, g(b))$:

$$\{x : f(a, g(a)) \triangleq f(b, g(b))\}; \emptyset; Id \Longrightarrow_{\mathsf{Dec}}$$
$$\{y : a \triangleq b, z : g(a) \triangleq g(b)\}; \emptyset; \{x \mapsto f(y, z)\} \Longrightarrow_{\mathsf{Sol}}$$
$$\{z : g(a) \triangleq g(b)\}; \{y : a \triangleq b\}; \{x \mapsto f(y, z)\} \Longrightarrow_{\mathsf{Dec}}$$
$$\{u : a \triangleq b\}; \{y : a \triangleq b\}; \{x \mapsto f(y, g(u)), z \mapsto g(u)\} \Longrightarrow_{\mathsf{Sol}}$$
$$\emptyset; \{y : a \triangleq b,\ u : a \triangleq b\}; \{x \mapsto f(y, g(u)),\ z \mapsto g(u)\} \Longrightarrow_{\mathsf{Mer}}$$
$$\emptyset; \{y : a \triangleq b\}; \{x \mapsto f(y, g(y)), z \mapsto g(y), u \mapsto y\}.$$

The lgg: the instance of x under the computed subst.: $f(y, g(y))$.

The witness substitutions: $\{y \mapsto a\}$ and $\{y \mapsto b\}$.

Generalization Type

The type of the E-generalization problem between t_1 and t_2 is

- **unitary** $(\mathbf{1})$: if $\mathsf{mcsg}_E(t_1, t_2)$ is a singleton,
- **finitary** (ω): if $\mathsf{mcsg}_E(t_1, t_2)$ is finite (not a singleton),
- **infinitary** (∞): if $\mathsf{mcsg}_E(t_1, t_2)$ is infinite,
- **nullary** $(\mathbf{0})$: if $\mathsf{mcsg}_E(t_1, t_2)$ does not exist
  (i.e., minimality and completeness contradict each other).

The generalization type of an equational theory E:

- the $<$-maximal type of E-generalization problems, where
  $\mathbf{1} < \omega < \infty < \mathbf{0}$.

Generalization Type: Examples

- **Unitary**:

  $E = \emptyset$.

- **Finitary**:

  commutativity, associativity, associativity with unit.

- **Infinitary**:

  $E = \{f(f(a,a),a) = f(a,a),\ f(f(b,b),b) = f(b,b)\}$

- **Nullary**:

  $E = \{f(a) = a,\ f(b) = b\}$.

Combination Problem

Is it possible to derive a generalization algorithm for a union of equational theories from the existing generalization algorithms for the component theories?

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Is it possible to derive a generalization algorithm for a union of equational theories from the existing generalization algorithms for the component theories?

Given (complete) algorithms for theory E_1 and for theory E_2, can we obtain a (complete) algorithm for $E_1 \cup E_2$?

Modularity.

Combination Problem

For theories sharing function symbols, the combination problem is very challenging.

Generalization type might change.

Each of $E_1 = \{f(a) = a\}$ and $E_2 = \{f(b) = b\}$ are theories with the unitary generalization type, but $E_1 \cup E_2$ is nullary.

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We focus on signature-disjoint union (no shared symbols in the axioms).

Combination Problem

Even signature-disjoint union can be problematic.

Each of

- $E_1 = \{f(x,0) = x,\ f(0,x) = x\}$ and
- $E_2 = \{g(x,1) = x,\ g(1,x) = x\}$

are theories with the finitary generalization type, but $E_1 \cup E_2$ is nullary.

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Even signature-disjoint union can be problematic.

Each of

- $E_1 = \{f(x, 0) = x,\ f(0, x) = x\}$ and
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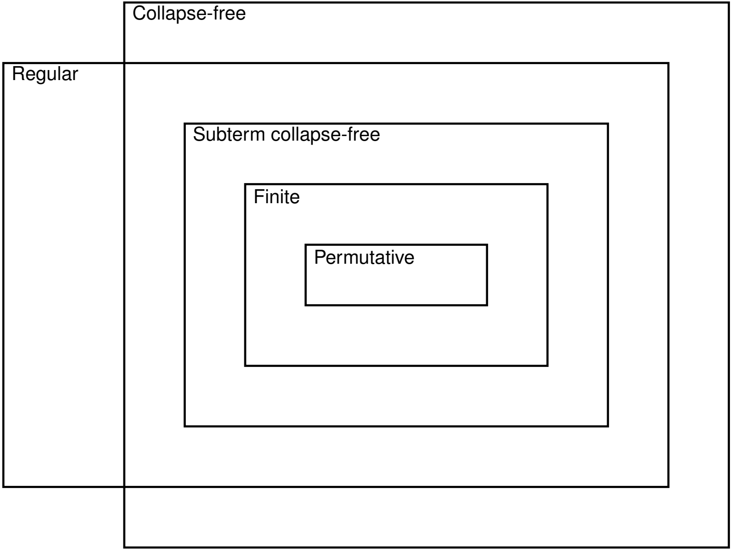
are theories with the finitary generalization type, but $E_1 \cup E_2$ is nullary.

$\longrightarrow$ Signature-disjoint union for a restricted class of equational theories.

Classes of Equational Theories



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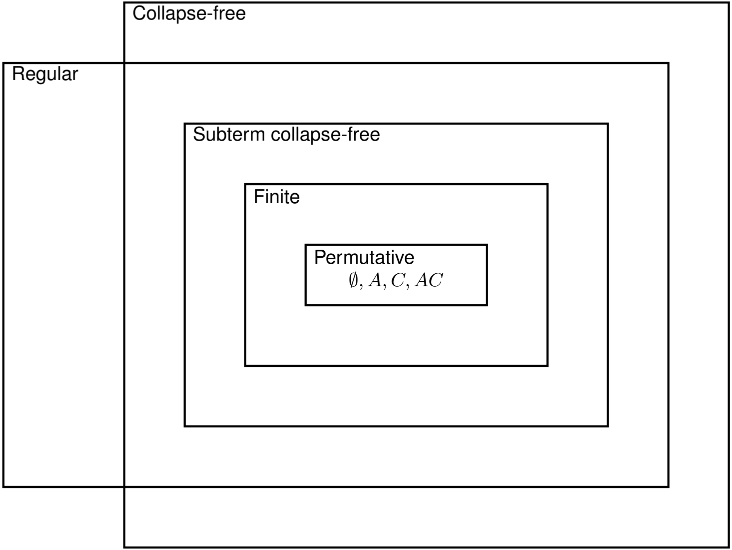

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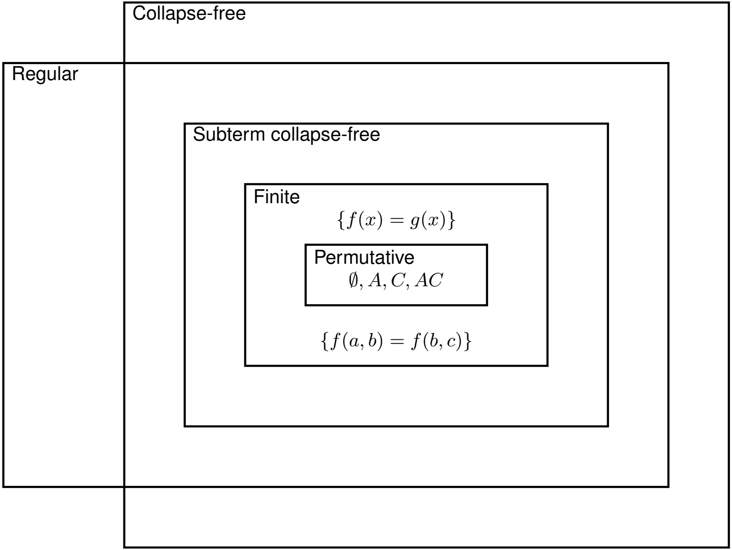

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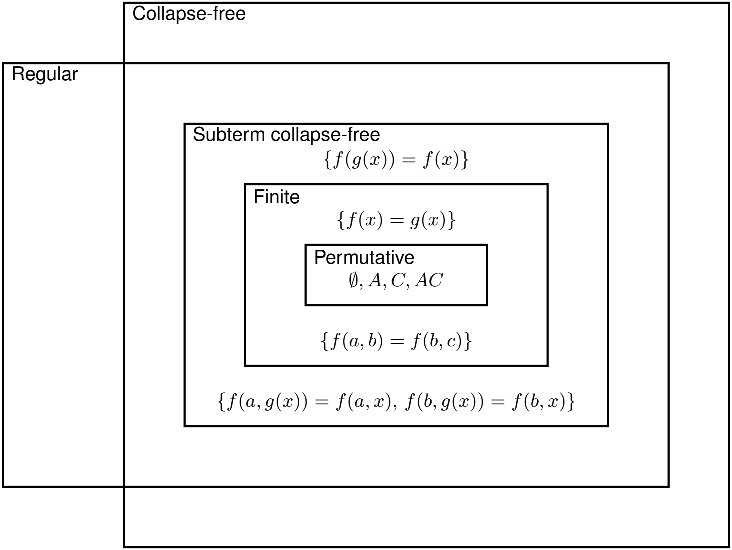

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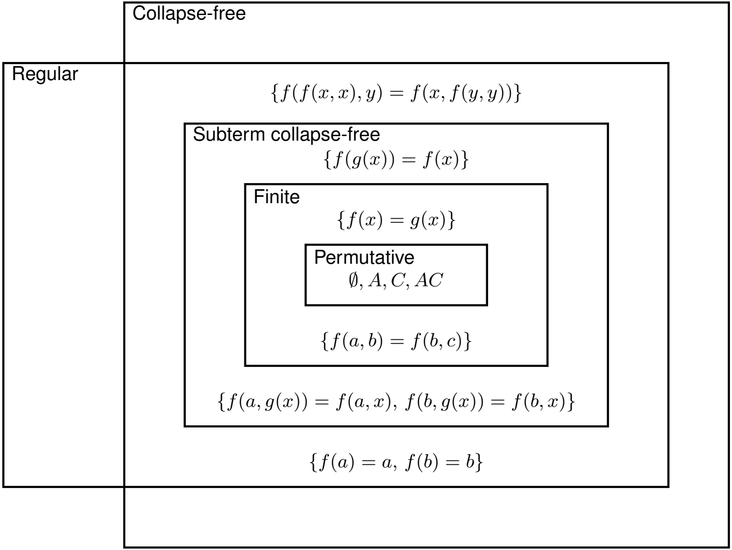

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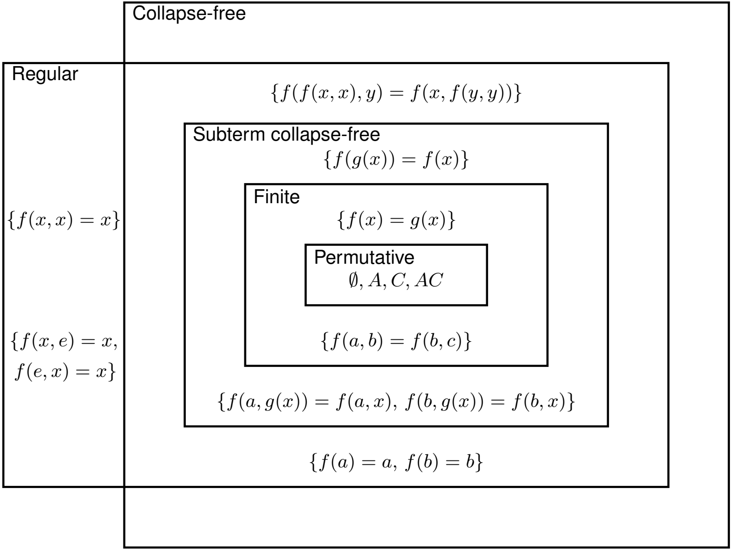

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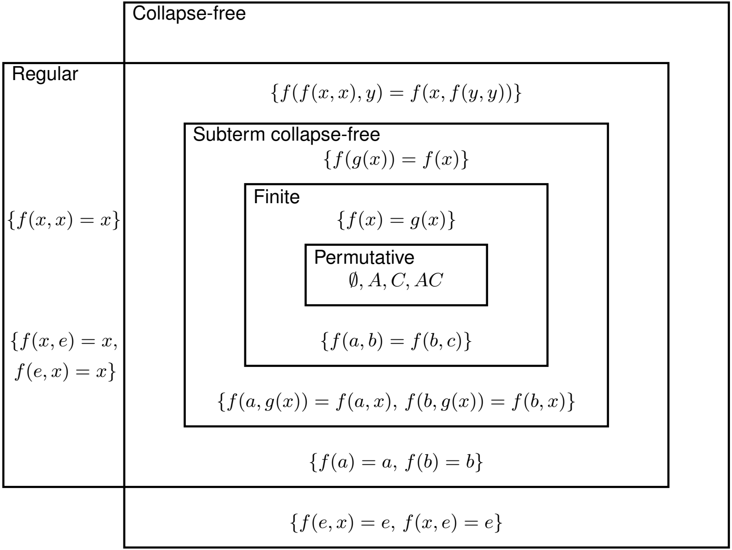

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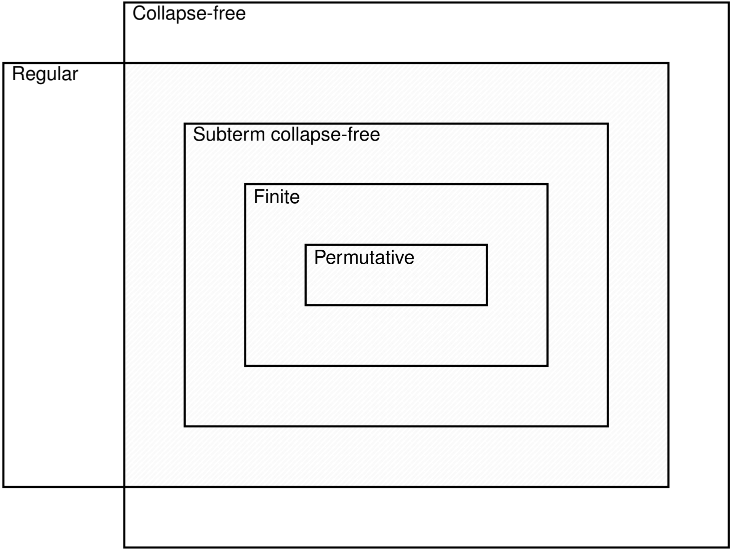

Regular Collapse-Free Theories

We designed a combination method for generalization in the union of signature-disjoint regular collapse-free theories, where

- free (uninterpreted) constants are allowed in generalization problems together with symbols from the component signatures,
- for each component theory (enriched with free constants), we have a procedure that computes a minimal complete set of generalizations together with their witness substitutions,
- equality is decidable in the combined theory.

Combination Algorithms: Two Versions

Black-box combination:

- does not require the knowledge of how the component algorithms work,
- only an input-output interface with the them is needed,
- more generic, but less transparent.

Combination Algorithms: Two Versions

White-box combination:

- both component algorithms work on the same data-structure: $P; S; \sigma$, where P is a set of unsolved problems, S is a set of solved ones, σ is a substitution representing the generalization computed so far.
- PSS: problem, solved set, substitution,
- less generic than black-box, but more transparent and provides more flexibility,
- many existing algorithms for particular theories are PSS algorithms.

Alien Subterm Abstraction

A technique used in various combination algorithms.

Idea:

- If the root of a term t is a symbol from the signature Σ_1 (resp. Σ_2), replace all maximal subterms of t rooted in Σ_2 (resp. Σ_1) by fresh free constants.
- Equal subterms are replaces by the same constants.
- Hence, if the original term contains symbols from Σ_1, Σ_2, and C (free constants), and is rooted in Σ_1, then the abstracted term contains symbols from Σ_1 and C.

Alien Subterm Abstraction

Idea by example.

Assume

- $\Sigma_1 = \{f, h\}$,
- $\Sigma_2 = \{g, a, b\}$ with $g(a) =_{E_2} b$,
- c is a free constant.

Then

- Blue subterms of $f(g(a), b, h(g(b)), c)$ are alien in it.
- Abstraction of $f(g(a), b, h(g(b)), c)$ is $f(c_1, c_1, h(c_2), c)$, where c_1 and c_2 are new free constants.

Black-Box Combination Algorithm: Rules

E_i-GEN: **Generalization in the E_i-theory**

$$\{x : t \triangleq s\} \uplus A; S; \sigma \Longrightarrow$$
$$(P \setminus S_i) \cup A; S_i \cup S; \sigma\{x \mapsto \pi^{-1}(r)\},$$

where

- $\mathsf{root}(t) \in \Sigma_i \cup C$ and $\mathsf{root}(s) \in \Sigma_i \cup C$,
- π is the alien subterm abstraction mapping,
- $r \in \mathsf{mcsg}_{E_i}(\pi(t), \pi(s))$, and ϕ, ψ are its witnesses,
- $P = \{y : \pi^{-1}(y\phi) \triangleq \pi^{-1}(y\psi) \mid y \in \mathsf{var}(r)\}$,
- $S_i = \{y : t' \triangleq s' \mid (y : t' \triangleq s') \in P,$
  $\mathsf{root}(t') \in \Sigma_i \cup C,\ \mathsf{root}(s') \in \Sigma_i \cup C\}$.

Black-Box Combination Algorithm

$E_{1,2}$-SOL: **Solving in the combined theory**

$$\{x : t \triangleq s\} \uplus A; S; \sigma \Longrightarrow A; S \cup \{x : t \triangleq s\}; \sigma,$$

if $\mathrm{root}(t) \in \Sigma_i$ and $\mathrm{root}(s) \in \Sigma_{3-i}$, $i = 1, 2$.

$E_{1,2}$-MER: **Merging in the combined theory**

$$\emptyset; \{x : t \triangleq s,\ x' : t' \triangleq s'\} \uplus S; \sigma \Longrightarrow$$
$$\emptyset; S \cup \{x : t \triangleq s\}; \sigma\{x' \mapsto x\},$$

if $t =_{E_1 \cup E_2} t'$ and $s =_{E_1 \cup E_2} s'$.

White-Box Combination Algorithm: Rules

E_i-STEP: **Step in the E_i-theory**

$$\{x : t \triangleq s\} \uplus P;\, S;\, \sigma \Longrightarrow$$
$$P \cup \pi^{-1}(P_i);\, S \cup \pi^{-1}(S_i);\, \sigma(\pi^{-1}(\sigma_i)),$$

where

- $\mathsf{root}(t) \in \Sigma_i \cup C$ and $\mathsf{root}(s) \in \Sigma_i \cup C$,
- π is the alien subterm abstraction mapping, and
- $\{x : \pi(t) \triangleq \pi(s)\}; \emptyset; Id \Longrightarrow_{\mathfrak{G}_i} P_i; S_i; \sigma_i$, where $\Longrightarrow_{\mathfrak{G}_i}$ is a step performed by the $\mathfrak{G}_i$ algorithm for theory E_i.

White-Box Combination Algorithm

$E_{1,2}$-SOL: **Solving in the combined theory**

$$\{x : t \triangleq s\} \uplus A; S; \sigma \Longrightarrow A; S \cup \{x : t \triangleq s\}; \sigma,$$

if $\mathrm{root}(t) \in \Sigma_i$ and $\mathrm{root}(s) \in \Sigma_{3-i}$, $i = 1, 2$.

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$$\emptyset; \{x : t \triangleq s,\ x' : t' \triangleq s'\} \uplus S; \sigma \Longrightarrow$$
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if $t =_{E_1 \cup E_2} t'$ and $s =_{E_1 \cup E_2} s'$.

Properties

We designed a combination method for generalization in the union of signature-disjoint regular collapse-free theories, where

- free (uninterpreted) constants are allowed in generalization problems together with symbols from the component signatures,
- for each component theory (enriched with free constants), we have a procedure that computes a minimal complete set of generalizations together with their witness substitutions,
- equality is decidable in the combined theory.

Both black-box and white-box combined algorithms are complete.

If at least one of the component theories is not finitary, the white-box combination is preferred over the black-box, since the latter may generate an infinitely branching derivation tree.

Next Steps

Relaxing the disjointness restriction: shared constructors?

Replacing substitutions by a certain kind of tree grammars (finitary representation of an infinite set of generalizations) $\longrightarrow$ terminating algorithms for some non-finitary problems?

Relaxing regularity and collapse-freeness restrictions.