Quantitative unification over "simply permutative" theories

Georg Ehling

Formal Methods & Automated Reasoning Seminar

November 4th, 2024

JOHANNES KEPLER UNIVERSITÄT LINZ

FWF Österreichischer Wissenschaftsfonds

RISC

(Quantitative) Equational Theories

Fix a signature $\mathcal{F}$ and a set of variables X.

- "Classical" setting:
  Equations $s = t$ between terms $s, t \in T(\mathcal{F}, X)$.

(Quantitative) Equational Theories

Fix a signature $\mathcal{F}$ and a set of variables X.

- "Classical" setting:
  Equations $s = t$ between terms $s, t \in T(\mathcal{F}, X)$.
  - Equations can be true or false (modulo a given theory E): either $s =_E t$, or $s \neq_E t$.
  - $=_E$ is reflexive, transitive, symmetric, stable under substitutions and compatible with $\mathcal{F}$-operations

(Quantitative) Equational Theories

Fix a signature $\mathcal{F}$ and a set of variables X.

- "Classical" setting:
  Equations $s = t$ between terms $s, t \in T(\mathcal{F}, X)$.
  - Equations can be true or false (modulo a given theory E): either $s =_E t$, or $s \neq_E t$.
  - $=_E$ is reflexive, transitive, symmetric, stable under substitutions and compatible with $\mathcal{F}$-operations
- Quantitative setting:
  Similarity/proximity rather than strict equality!

(Quantitative) Equational Theories

Fix a signature $\mathcal{F}$ and a set of variables X.

- "Classical" setting:
  Equations $s = t$ between terms $s, t \in T(\mathcal{F}, X)$.
  - Equations can be true or false (modulo a given theory E): either $s =_E t$, or $s \neq_E t$.
  - $=_E$ is reflexive, transitive, symmetric, stable under substitutions and compatible with $\mathcal{F}$-operations
- Quantitative setting:
  Similarity/proximity rather than strict equality!
  $\rightsquigarrow$ Equip equations $s = t$ with some element ε that measures the "degree to which they hold true".

Quantales

General notion of proximity requires a general notion of distances.

Definition (Quantale)

Quantale: $\mho = (\Omega, \precsim, \otimes, \kappa)$ such that

- $(\Omega, \precsim)$ is a complete lattice (poset where every subset has a supremum and infimum, denoted $\vee$ and $\wedge$)
- $(\Omega, \otimes, \kappa)$ is a monoid

satisfying the following distributivity laws:

$$\delta \otimes \left(\bigvee_{i\in I} \varepsilon_i\right) = \bigvee_{i\in I}(\delta \otimes \varepsilon_i), \qquad \left(\bigvee_{i\in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i\in I}(\varepsilon_i \otimes \delta).$$

Quantales

General notion of proximity requires a general notion of distances.

Definition (Quantale)

Quantale: $\mathbb{Q} = (\Omega, \precsim, \otimes, \kappa)$ such that

- $(\Omega, \precsim)$ is a complete lattice (poset where every subset has a supremum and infimum, denoted $\vee$ and $\wedge$)
- $(\Omega, \otimes, \kappa)$ is a monoid

satisfying the following distributivity laws:

$$\delta \otimes \left(\bigvee_{i \in I} \varepsilon_i\right) = \bigvee_{i \in I} (\delta \otimes \varepsilon_i), \qquad \left(\bigvee_{i \in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i \in I} (\varepsilon_i \otimes \delta).$$

Example

- $\mathbb{I} = ([0,1], \leqslant, \min, 1)$ "Fuzzy quantale"
- $\mathbb{L} = ([0,\infty], \geqslant, +, 0)$ "Lawvere quantale"
- $\mathbb{2} = (\{0,1\}, \leqslant, \cdot, 1)$ "Boolean quantale"

Assume that we are working with Lawverean quantales.

Definition

A quantale $\Omega = (\Omega, \precsim, \otimes, \kappa)$ is called *Lawverean* if

- $\otimes$ is commutative
- Ω is *integral*: $\kappa = \top$ (where $\top$ is the top element)
- Ω is *co-integral*: if $\varepsilon \otimes \delta = \bot$, then either $\varepsilon = \bot$ or $\delta = \bot$ (where $\bot$ is the bottom element)
- Ω is *non-trivial*: $\kappa \neq \bot$

Inference rules for quantitative equational logic (Gavazzo and Di Florio 2023)

$$\text{(Ax)}\ \frac{\varepsilon \Vdash t \approx s \in E}{\varepsilon \Vdash t =_E s} \qquad \text{(Refl)}\ \frac{}{\varepsilon \Vdash t =_E t} \qquad \text{(Sym)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash s =_E t}$$

$$\text{(Trans)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \Vdash s =_E r}{\varepsilon \otimes \delta \Vdash t =_E r}$$

$$\text{(NExp)}\ \frac{\varepsilon_1 \Vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_E s_n}{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \Vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)}$$

$$\text{(Subst)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash t\sigma =_E s\sigma} \qquad \text{(Join)}\ \frac{\varepsilon_1 \Vdash t =_E s \quad \cdots \quad \varepsilon_n \Vdash t =_E s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_E s}$$

$$\text{(Ord)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \precsim \varepsilon}{\delta \Vdash t =_E s} \qquad \text{(Arch)}\ \frac{\forall \delta \ll \varepsilon.\, \delta \Vdash t =_E s}{\varepsilon \Vdash t =_E s}$$

Example

- $\Omega = \mathbb{2} = (\{0,1\}, \leqslant, \cdot, 1)$:
  Classical equational reasoning (read $1 \Vdash s =_E t$ as $s =_E t$)
- $\Omega = \mathbb{I} = ([0,1], \leqslant, \min, 1)$:
  Fuzzy reasoning
- $\Omega = \mathbb{L} = ([0,\infty], \geqslant, +, 0)$:
  quantitative algebraic theories in the sense of Mardare, Panangaden, and Plotkin 2016 (with slightly modified (NExp) rule)

Unification Problems

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of equations.

(Classical) Unification problem: $s =^?_E t$

Find a substitution σ such that $s\sigma =_E t\sigma$.

Unification Problems

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of equations.

(Classical) Unification problem: $s =^?_E t$

Find a substitution σ such that $s\sigma =_E t\sigma$.

Example

$E = \{f(x, y) \approx f(y, x)\}$.

The problem

$$f(g(x), f(b, a)) =^?_E f(f(x, b), y)$$

has the solution $\sigma = \{x \mapsto a, y \mapsto g(a)\}$.

Unification Problems

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of equations.

(Classical) Unification problem: $s =^?_E t$

Find a substitution σ such that $s\sigma =_E t\sigma$.

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of Ω-equations, $\varepsilon \in \Omega$.

Quantitative unification problem: $s =^?_{E,\varepsilon} t$

Find a substitution σ such that $\varepsilon \Vdash s\sigma =_E t\sigma$.

Unification Problems

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of equations.

(Classical) Unification problem: $s =^?_E t$

Find a substitution σ such that $s\sigma =_E t\sigma$.

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of $\mho$-equations, $\varepsilon \in \Omega$.

Quantitative unification problem: $s =^?_{E,\varepsilon} t$

Find a substitution σ such that $\varepsilon \Vdash s\sigma =_E t\sigma$.

Example

$\mho = \mathbb{L}$, $E = \{1 \Vdash a \approx b, 1 \Vdash b \approx c, 1 \Vdash c \approx d\}$.

The problem

$$f(x, x) =^?_{E,2} f(a, c)$$

has solutions $\{x \mapsto a\}, \{x \mapsto b\}, \{x \mapsto c\}$.

Assumption

- Consider a special class of quantitative theories:

  Let E_π be a finite set of quantitative equations of the form

$$\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_{\pi(1)} \ldots x_{\pi(n)}),$$

  where $x_1, \ldots, x_n$ are distinct variable symbols and $\pi \in \mathfrak{S}_n$ is a permutation.

  (Previously: only $\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n)$ was allowed)

Assumption

- Consider a special class of quantitative theories:

  Let E_π be a finite set of quantitative equations of the form

$$\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_{\pi(1)} \ldots x_{\pi(n)}),$$

  where $x_1, \ldots, x_n$ are distinct variable symbols and $\pi \in \mathfrak{S}_n$ is a permutation.

  (Previously: only $\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n)$ was allowed)

Example

Near-commutativity: $E_\pi = \{\varepsilon \Vdash f(x, y) \approx f(y, x)\}$.

Assumption

- Consider a special class of quantitative theories:

  Let E_π be a finite set of quantitative equations of the form

  $$\varepsilon \Vdash f(x_1, \dots, x_n) \approx g(x_{\pi(1)} \dots x_{\pi(n)}),$$

  where $x_1, \dots, x_n$ are distinct variable symbols and $\pi \in \mathfrak{S}_n$ is a permutation.

  (Previously: only $\varepsilon \Vdash f(x_1, \dots, x_n) \approx g(x_1, \dots, x_n)$ was allowed)

Example

Near-commutativity: $E_\pi = \{\varepsilon \Vdash f(x, y) \approx f(y, x)\}$.

- *Degrees:*

$$\mathfrak{d}_E(f, g, \pi) := \bigvee \{\varepsilon \in \Omega : \varepsilon \Vdash f(x_1, \dots, x_n) =_E g(x_{\pi(1)}, \dots, x_{\pi(n)})\}$$

The calculus

- **Configurations:** Triples $P; \delta; \sigma$, where
  - P is a set of unification problems (the remainder of the problem)
  - $\delta \in \Omega$ (the current approximation degree)
  - σ is a substitution (the solution computed so far)
- **Initial configuration** for $s =^?_{E,\varepsilon} t$:
  $\{s =^? t\}; \kappa; \mathrm{Id}$
- **Unification algorithm** $\textsc{Qunify-}\pi$:
  Construct initial configuration and apply unification rules as long as possible. Return terminal configuration(s).

Unification rules

Tri_{E_π}: **Trivial**

$$\{t =^? t\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad P; \delta; \sigma$$

Dec_{E_π}: **Decompose**

$$\{f(t_1, \ldots, t_n) =^? g(s_1, \ldots, s_n)\} \uplus P;\ \delta;\ \sigma \quad \Longrightarrow$$
$$\{t_{\pi(1)} =^? s_1, \ldots, t_{\pi(n)} =^? s_n\} \cup P;\ \delta \otimes \mathfrak{d}_{E_\pi}(f, g, \pi);\ \sigma,$$

where f and g are n-ary function symbols, $\pi \in \mathfrak{S}_n$ and $\delta \otimes \mathfrak{d}_{E_\pi}(f, g, \pi) \succsim \varepsilon$.

Cla_{E_π}: **Clash**

$$\{f(t_1, \ldots, t_n) =^? g(s_1, \ldots, s_m)\} \uplus P;\ \delta;\ \sigma \quad \Longrightarrow \quad \mathbf{F},$$

if $\delta \otimes \mathfrak{d}_{E_\pi}(f, g, \pi) \not\succsim \varepsilon$ for all $\pi \in \mathfrak{S}_n$.

Unification rules (cont.)

L-Sub$_{E_\pi}$: **Substitute (lazy)**

$$\{x =^? f(t_1, \dots, t_n)\} \uplus P;\ \delta;\ \sigma \quad \Longrightarrow$$
$$\{x_{\pi(1)} =^? t_1, \dots, x_{\pi(n)} =^? t_n\} \cup P\rho;\ \delta \otimes \mathfrak{d}_{E_\pi}(f, g, \pi);\ \sigma\rho,$$

where x does not appear in an occurrence cycle in $\{x =^? f(t_1, \dots, t_n)\} \cup P$, and $\rho = \{x \mapsto g(x_1, \dots, x_n)\}$ with $x_1, \dots, x_n$ being fresh variables and $\delta \otimes \mathfrak{d}_{E_\pi}(f, g, \pi) \gtrsim \varepsilon$.

CCh$_{E_\pi}$: **Cycle check**

$$P;\ \delta;\ \sigma \Longrightarrow \mathbf{F},$$

if P contains an occurrence cycle.

Ori$_{E_\pi}$: **Orient**

$$\{t =^? x\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad \{x =^? t\} \cup P; \delta; \sigma,$$

where x is a variable and t is a non-variable term.

Example

Example

$\Omega = \mathbb{L},\ E_\pi = \{1 \Vdash f(x, y) \approx f(y, x)\}$.

Solve

$$f(f(z, a), x) =^?_{E_\pi, 2} f(c, f(y, b)).$$

We have $\mathfrak{d}_{E_\pi}(f, f, \mathrm{Id}) = 0$ and $\mathfrak{d}_{E_\pi}(f, f, (1\ 2)) = 1$.

$$
\begin{array}{lr}
\text{Initial config.:} & \{f(f(z, a), x) =^? f(c, f(y, b))\}; 0; \mathrm{Id} \\
\Longrightarrow \ \mathsf{Dec}_{(1\,2)} & \{x =^? c,\ f(z, a) =^? f(y, b)\}; 1; \mathrm{Id} \\
\Longrightarrow \ \mathsf{L\text{-}Sub}_{x \mapsto c} & \{f(z, a) =^? f(y, b)\}; 1; \{x \mapsto c\} \\
\Longrightarrow \ \mathsf{Dec}_{(1\,2)} & \{a =^? y,\ z =^? b\}; 2; \{x \mapsto c\} \\
\Longrightarrow \ \mathsf{L\text{-}Sub}_{z \mapsto b} & \{a =^? y\}; 2; \{x \mapsto c, z \mapsto b\} \\
\Longrightarrow \ \mathsf{Ori} & \{y =^? a\}; 2; \{x \mapsto c, z \mapsto b\} \\
\Longrightarrow \ \mathsf{L\text{-}Sub} & \emptyset; 2; \{x \mapsto c, z \mapsto b, y \mapsto a\}
\end{array}
$$

$\rightsquigarrow$ obtain the solution $\{x \mapsto c, y \mapsto a, z \mapsto b\}$

Why lazy substitution?

Q: Why cannot we use a simpler eager substitution rule instead of L-Sub?

E-Sub$_{E_\pi}$: **Substitute (eager)**

$$\{x =^? t\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad P\{x \mapsto t\}; \delta; \sigma\{x \mapsto t\},$$

if $P \cup \{x =^? t\}$ does not contain occurrence cycles

Why lazy substitution?

Q: Why cannot we use a simpler eager substitution rule instead of L-Sub?

E-Sub$_{E_\pi}$: **Substitute (eager)**

$$\{x =^? t\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad P\{x \mapsto t\}; \delta; \sigma\{x \mapsto t\},$$

if $P \cup \{x =^? t\}$ does not contain occurrence cycles

Example

$\Omega = \mathbb{L}, \; E = \{1 \Vdash a \approx d, \; 1 \Vdash b \approx d, \; 1 \Vdash c \approx d\}$.

Solve

$$f(x, x, x) =^?_{E,3} f(a, b, c).$$

Why lazy substitution?

Q: Why cannot we use a simpler eager substitution rule instead of L-Sub?

E-Sub$_{E_\pi}$: **Substitute (eager)**

$$\{x =^? t\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad P\{x \mapsto t\}; \delta; \sigma\{x \mapsto t\},$$

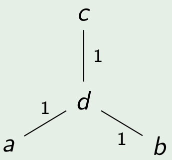
if $P \cup \{x =^? t\}$ does not contain occurrence cycles

Example

$\mho = \mathbb{L}$, $E = \{1 \Vdash a \approx d,\ 1 \Vdash b \approx d,\ 1 \Vdash c \approx d\}$.
Solve

$$f(x,x,x) =^?_{E,3} f(a,b,c).$$

The only solution $\{x \mapsto d\}$ cannot be found by E-Sub.



Why lazy substitution?

Q: Why cannot we use a simpler eager substitution rule instead of L-Sub?

E-Sub$_{E_\pi}$: **Substitute (eager)**

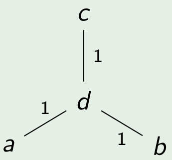
$$\{x =^? t\} \uplus P; \delta; \sigma \quad \Longrightarrow \quad P\{x \mapsto t\}; \delta; \sigma\{x \mapsto t\},$$

if $P \cup \{x =^? t\}$ does not contain occurrence cycles

Example

$\Omega = \mathbb{L}$, $E = \{1 \Vdash a \approx d,\ 1 \Vdash b \approx d,\ 1 \Vdash c \approx d\}$.
Solve

$$f(x, x, x) =^?_{E,3} f(a, b, c).$$



The only solution $\{x \mapsto d\}$ cannot be found by E-Sub.

Eager substitution ignores the quantitative aspect!

Results

Theorem (Soundness and Completeness)

Soundness: *If* QUNIFY-π *yields a terminal configuration, then any "solution" of this configuration is an* (E, ε)*-unifier of* t *and* s.

Completeness: *If* σ *is an* (E, ε)*-unifier of* t *and* s*, then there exists a run of* QUNIFY-π *that yields a terminal configuration for which* σ *is a "solution"*

Results

Theorem (Soundness and Completeness)

Soundness: *If* QUNIFY-π *yields a terminal configuration, then any "solution" of this configuration is an* (E, ε)*-unifier of* t *and* s.

Completeness: *If* σ *is an* (E, ε)*-unifier of* t *and* s*, then there exists a run of* QUNIFY-π *that yields a terminal configuration for which* σ *is a "solution"*

Theorem (Termination)

Any run of QUNIFY-π *terminates, provided that* L-Sub *is not used as long as* Dec *or* Cla *can be applied.*

Termination

Termination is not straight-forward:

Example

Consider the configuration

$$\{x =^? f(a, y),\ \ y =^? f(g(z), b), z =^? b\}; \delta; \mathrm{Id}$$

Apply L-Sub, via $\{y \mapsto f(y_1, y_2)\}$:

$$\Longrightarrow \{x =^? f(a, f(y_1, y_2)), y_1 =^? g(z), y_2 =^? b, z =^? b\};$$
$$\delta; \{y \mapsto f(y_1, y_2)\}$$

Termination

Termination is not straight-forward:

Example

Consider the configuration

$$\{x =^? f(a, y),\ y =^? f(g(z), b), z =^? b\}; \delta; \mathrm{Id}$$

Apply L-Sub, via $\{y \mapsto f(y_1, y_2)\}$:

$$\Longrightarrow \{x =^? f(a, f(y_1, y_2)), y_1 =^? g(z), y_2 =^? b, z =^? b\};$$
$$\delta; \{y \mapsto f(y_1, y_2)\}$$

L-Sub increases the total size of the problem as well as the number of variables!

Termination

Termination is not straight-forward:

Example

Consider the configuration

$$\{x =^? f(a, y),\ \ y =^? f(g(z), b), z =^? b\}; \delta; \mathrm{Id}$$

Apply L-Sub, via $\{y \mapsto f(y_1, y_2)\}$:

$$\Longrightarrow \{x =^? f(a, f(y_1, y_2)), y_1 =^? g(z), y_2 =^? b, z =^? b\};$$
$$\delta; \{y \mapsto f(y_1, y_2)\}$$

L-Sub increases the total size of the problem as well as the number of variables!

$\rightsquigarrow$ We also need to measure the dependencies between variables!

Termination (cont.)

Dependency graph of a configuration $P; \delta; \sigma$:

- Nodes: $\mathrm{var}(P) \cup \{G\}$
- Edges:
  - $x \to_d y$ whenever $x =^? t[y]_p \in P$, where $d = |p|$;
  - $x \to_d G$ whenever $x =^? t[c]_p \in P$, where c is a constant and $d = |p| + 1$.

Termination (cont.)

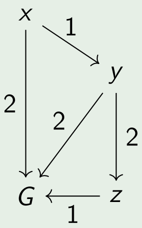
Dependency graph of a configuration $P; \delta; \sigma$:

- Nodes: $\mathrm{var}(P) \cup \{G\}$
- Edges:
  - $x \to_d y$ whenever $x =^? t[y]_p \in P$, where $d = |p|$;
  - $x \to_d G$ whenever $x =^? t[c]_p \in P$, where c is a constant and $d = |p| + 1$.

Example

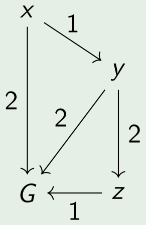
Dependency graph corresponding to the configuration

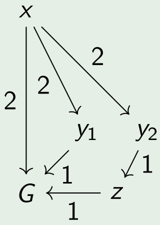
$$\{x =^? f(a, y),\ y =^? f(g(z), b), z =^? b\}; \delta; \mathrm{Id} :$$



Example

$$\{x =^? f(a,y),\ y =^? f(g(z),b), z =^? b\}; \delta; \mathrm{Id} \quad \Longrightarrow$$
$$\{x =^? f(a, f(y_1,y_2)), y_1 =^? g(z), y_2 =^? b, z =^? b\}; \delta; \{y \mapsto f(y_1,y_2)\}$$





For each configuration, consider now the multiset of the maximal lengths of walks in the dependency graph starting from each variable:

$$\{4,3,1\} > \{4,1,2,1\}.$$

Computing degrees

Input : A simply permutative theory E_π
Output: The values of $\mathfrak{d}_{E_\pi}(f, g, \pi)$ for any f, g of arity n and $\pi \in \mathfrak{S}_n$.
Initialization:

- $\mathfrak{d}_0(f, f, \mathrm{Id}) \leftarrow \kappa$
- $\mathfrak{d}_0(f, g, \pi) \leftarrow \bigvee\{\varepsilon \mid \varepsilon \Vdash f(x_1, \dots, x_n) \dot{\approx} g(x_{\pi(1)}, \dots, x_{\pi(n)}) \in E_\pi\}$
- $n \leftarrow 0$

while *true* **do**
  for f, g *of arity* n, $\pi \in \mathfrak{S}_n$ **do**
    $$\mathfrak{d}_{N+1}(f, g, \pi) \leftarrow \mathfrak{d}_N(f, g, \pi) \vee \bigvee_{\substack{h \in \mathcal{F}, \\ \rho \circ \sigma = \pi}} \mathfrak{d}_N(f, h, \rho) \otimes \mathfrak{d}_N(h, g, \sigma)$$
  end
  if $\mathfrak{d}_{N+1} \neq \mathfrak{d}_N$ **then**
    $N \leftarrow N + 1$
  else
    return $\mathfrak{d}_N$;
  end
end

Conclusion/Outlook

So far:

- Solved quantitative unification over a general quantale for a specific class of shallow theories

Future research directions:

- Quantitative unification over more general classes of theories: Some approaches for special classes of syntactic theories might allow for an adaptation to the quantitative setting
  - Hubert Comon, Marianne Haberstrau, and Jean-Pierre Jouannaud (1994). "Syntacticness, Cycle-Syntacticness, and Shallow Theories". In: *Inf. Comput.* 111.1, pp. 154–191
  - Christopher Lynch and Barbara Morawska (2002). "Basic Syntactic Mutation". In: *Automated Deduction—CADE-18*. Berlin, Germany: Springer, pp. 471–485
- Quantitative matching and anti-unification