SYMBOLIC TECHNIQUES FOR QUANTITATIVE REASONING

Brief Report

Temur Kutsia
RISC, Johannes Kepler University Linz

JOHANNES KEPLER
UNIVERSITY LINZ

Overview

Quantitative equational theories: equality is replaced by its quantitative approximation.

Quantitative reasoning techniques:

- solving: matching, unification
- computing: rewriting, narrowing
- proving: completion, resolution

Quantitative equational theories

Equalities are relaxed to its quantitative approximation.

A very abstract notion of proximity between two terms, expressed using an element of a quantale.

The approach to quantitative equational theories introduced in

> F. Gavazzo, C. Di Florio: Elements of Quantitative Rewriting. Proc. ACM Program. Lang. 7(POPL), 1832–1863 (2023).

Quantales

A quantale $\Omega = (\Omega, \precsim, \otimes, \kappa)$: an algebraic structure, where

- $(\Omega, \kappa, \otimes)$ is a monoid,
- $(\Omega, \precsim)$ is a complete lattice (with join $\vee$ and meet $\wedge$),
- the following distributivity laws are satisfied:
  $\delta \otimes \left(\bigvee_{i \in I} \varepsilon_i\right) = \bigvee_{i \in I}(\delta \otimes \varepsilon_i)$ and
  $\left(\bigvee_{i \in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i \in I}(\varepsilon_i \otimes \delta)$.

Terminology:

- Ω: carrier set
- $\precsim$: order
- κ: unit
- $\otimes$: tensor

Quantale examples

Correspondence between quantales $\mathbb{\Omega}$ (generic), $\mathbb{2}$ (Boolean), $\mathbb{I}$ (fuzzy), $\mathbb{L}$ (Lawvere), $\mathbb{L}^{\max}$ (strong Lawvere).

	$\mathbb{\Omega}$	$\mathbb{2}$	$\mathbb{I}$	$\mathbb{L}$	$\mathbb{L}^{\max}$
Carrier	Ω	$\{0,1\}$	$[0,1]$	$[0,\infty]$	$[0,\infty]$
Order	$\precsim$	$\leqslant$	$\leqslant$	$\geqslant$	$\geqslant$
Unit	κ	1	1	0	0
Tensor	$\otimes$	$\wedge$	left-cont. T-norm	$+$	$\max$
Join	$\vee$	$\sup$	$\sup$	$\inf$	$\inf$
Meet	$\wedge$	$\inf$	$\inf$	$\sup$	$\sup$

Quantales: terminology

$\top$, $\perp$: the top and bottom elements of a quantale.

- integral quantale: $\kappa = \top$
- commutative quantale: $\otimes$ is commutative
- nontrivial quantale: $\kappa \neq \perp$
- cointegral quantale: $\varepsilon \otimes \delta = \perp$ implies $\varepsilon = \perp$ or $\delta = \perp$
- idempotent element: $\varepsilon \in \Omega$ such that $\varepsilon \otimes \varepsilon = \varepsilon$
- idempotent quantale: every element is idempotent

Quantales: terminology

Lawvereian quantales: integral, commutative, nontrivial, and cointegral quantales.

Quantales: terminology

Lawvereian quantales: integral, commutative, nontrivial, and cointegral quantales.

$\mathbb{2}$, $\mathbb{L}$, and $\mathbb{L}^{\max}$ are Lawvereian quantales.

$\mathbb{I}$ is Lawvereian for $\min$ and product t-norms, but not for Łukasiewicz t-norm.

Quantales: terminology

Lawvereian quantales: integral, commutative, nontrivial, and cointegral quantales.

$\mathbb{2}$, $\mathbb{L}$, and $\mathbb{L}^{\max}$ are Lawvereian quantales.

$\mathbb{I}$ is Lawvereian for $\min$ and product t-norms, but not for Łukasiewicz t-norm.

$\mathbb{2}$ and $\mathbb{L}^{\max}$ are idempotent quantales.

$\mathbb{I}$ is idempotent for the $\min$ t-norm, but not for the product and Łukasiewicz t-norms.

$\mathbb{L}$ is not an idempotent quantale.

Quantales: terminology

Lawvereian quantales: integral, commutative, nontrivial, and cointegral quantales.

$\mathbb{2}$, $\mathbb{L}$, and $\mathbb{L}^{\max}$ are Lawvereian quantales.

$\mathbb{I}$ is Lawvereian for $\min$ and product t-norms, but not for Łukasiewicz t-norm.

$\mathbb{2}$ and $\mathbb{L}^{\max}$ are idempotent quantales.

$\mathbb{I}$ is idempotent for the $\min$ t-norm, but not for the product and Łukasiewicz t-norms.

$\mathbb{L}$ is not an idempotent quantale.

Only Lawvereian quantales are considered.

Quantitative equational theory

Given:

- a quantale $\mathbb{\Omega} = (\Omega, \precsim, \kappa, \otimes)$,
- a set of terms $\mathcal{T}$,
- a set of triples $E \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ ($\mathbb{\Omega}$-equalities).

Notation: $\varepsilon \Vdash t \approx_E s$ for $(t, \varepsilon, s) \in E$.

Intuition: E is a set of axioms that induces an equational theory.

Quantitative equational theory

Given:

- a quantale $\mathbb{\Omega} = (\Omega, \precsim, \kappa, \otimes)$,
- a set of terms $\mathcal{T}$,
- a set of triples $E \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ ($\mathbb{\Omega}$-equalities).

Notation: $\varepsilon \Vdash t \approx_E s$ for $(t, \varepsilon, s) \in E$.

Intuition: E is a set of axioms that induces an equational theory.

Quantitative equational theory induced by E (wrt $\mathbb{\Omega}$): a ternary relation $=_E \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ defined by the rules on the next slide.

Informally, $\varepsilon \Vdash t =_E s$ is read as

- "t and s are at most ε-apart modulo E" or
- "t and s are equal modulo E with degree ε".

Quantitative equational theory: rules

The rules define a non-expansive quantitative equational theory.

$$\text{(Ax)}\ \frac{\varepsilon \Vdash t \approx_E s}{\varepsilon \Vdash t =_E s}$$

$$\text{(Refl)}\ \frac{}{\kappa \Vdash t =_E t} \qquad \text{(Sym)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash s =_E t} \qquad \text{(Trans)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \Vdash s =_E r}{\varepsilon \otimes \delta \Vdash t =_E r}$$

$$\text{(NExp)}\ \frac{\varepsilon_1 \Vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_E s_n}{\bigotimes_{i=1}^n \varepsilon_i \Vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)} \qquad \text{(Subst)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash t\sigma =_E s\sigma}$$

$$\text{(Ord)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \precsim \varepsilon}{\delta \Vdash t =_E s} \qquad \text{(Join)}\ \frac{\varepsilon_1 \Vdash t =_E s \quad \cdots \quad \varepsilon_n \Vdash t =_E s}{\bigvee_{i=1}^n \varepsilon_i \Vdash t =_E s}$$

$$\text{(Arch)}\ \frac{\forall \delta \ll \varepsilon.\, \delta \Vdash t =_E s}{\varepsilon \Vdash t =_E s}$$

SOLVING

Quantitative unification and matching

Given: A quantale $\mathbb{\Omega}$, $\varepsilon \in \Omega$ (the threshold) with $\varepsilon \neq \bot$,
a set of $\mathbb{\Omega}$-equalities E, and two terms t and s.

Find: A substitution σ such that

- $\varepsilon \Vdash t\sigma =_E s\sigma$ (unification problem),
- $\varepsilon \Vdash t\sigma =_E s$ (matching problem).

Quantitative unification and matching

Solvability of unification and matching problems depends on E.

We started from the simplest case: E is a set of quantitative equations of the form

$$\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n), \quad f \neq g, \quad n \geq 0.$$

Quantitative unification: examples

Example

Take

$\Omega = \mathbb{L}$ (Lawvere quantale),

$E = \{1 \Vdash a \approx b, 1 \Vdash b \approx c, 1 \Vdash c \approx d\}$,

and the unification problem

$1 \Vdash f(x, b) =^?_E f(c, x)$.

Solutions: $\sigma = \{x \mapsto b\}$, $\vartheta = \{x \mapsto c\}$.

- $f(x, b)\sigma = f(b, b)$, $f(c, x)\sigma = f(c, b)$ and $1 \Vdash f(b, b) =_E f(c, b)$,
- $f(x, b)\vartheta = f(c, b)$, $f(c, x)\vartheta = f(c, c)$ and $1 \Vdash f(c, b) =_E f(c, c)$.

Neither $\{x \mapsto a\}$ nor $\{x \mapsto d\}$ is an $(E, 1)$-unifier of $f(x, b)$ and $f(c, x)$ (but they are their $(E, 3)$-unifiers).

Quantitative unification: examples

Example

Take

$$\Omega = \mathbb{L} \quad \text{(Lawvere quantale)},$$

$$E = \{1 \Vdash a \approx b,\ 1 \Vdash b \approx c,\ 1 \Vdash f(x_1, x_2) \approx g(x_1, x_2)\},$$

and the unification problem

$$5 \Vdash f(y, g(x, x)) =^?_E g(f(c, a), y).$$

It has multiple “incomparable” solutions, among them

$$\sigma = \{y \mapsto f(b, a), x \mapsto a\} \quad \text{(with degree } 4\text{)}$$

$$f(y, g(x, x))\sigma = f(f(b, a), g(a, a))$$

$$g(f(c, a), y)\sigma = g(f(c, a), f(b, a))$$

$$4 \Vdash f(f(b, a), g(a, a)) =_E g(f(c, a), f(b, a)).$$

Quantitative unification: examples

Example

Take $\Omega = \mathbb{L}$, $E = \{1 \Vdash f(x) \approx g(x)\}$ and the unification problem

$$2 \Vdash x =_E^? f(y).$$

One of its solutions is given by a pair $(\sigma, \{\mathsf{eq}\})$, where

$\sigma = \{x \mapsto g(z)\},$ where z is fresh,

$\mathsf{eq} = \gamma \Vdash z =_E y$ with the constraint $\gamma \leqslant 1$.

Then

$$x\sigma = g(z), \qquad f(y)\sigma = f(y)$$

and $\sigma\vartheta$ for any solution ϑ of eq is a unifier of the original problem.

$\sigma\vartheta = \{x \mapsto g(a), y \mapsto a, z \mapsto a\},$ degree 1,

$\sigma\vartheta = \{x \mapsto g(f(a)), y \mapsto g(a), z \mapsto f(a)\},$ degree 2.

Quantitative unification and matching

Ongoing and future work:

- Increase expressiveness without violating "good" computational behavior, e.g., by considering shallow regular theories such as $E = \{1 \Vdash a \approx b,\ 2 \Vdash f(x, y, h(a)) \approx g(y, y, c, x)\}$.
- Efficient special cases: idempotent quantales, linear solutions, $\ldots$.
- Specialized algorithms for matching.
- Generalizing results from non-expansive to graded equational theories.

COMPUTING

Quantitative abstract rewriting

Given a Lawverean quantale Ω, a quantitative abstract rewriting system (wrt Ω) is a pair (A, R), where R is a function $A \times A \longrightarrow \Omega$.

Terminology:

- For $a, b \in A$, a rewrites to b if $R(a, b) \neq \bot$.
- $R(a, b)$: the distance, the degree, the cost, or the resource of reduction.

The notions of Diamond property, confluence, Church-Rosser, local confluence, termination, etc. are extended from standard to quantitative rewriting systems.

Quantitative abstract rewriting

Diamond Property

$$\begin{tikzcd}
& c \arrow[dl, "R"'] \arrow[dr, "R"] & \\
a \arrow[dr, dotted, "R"'] & & b \arrow[dl, dotted, "R"] \\
& d &
\end{tikzcd}$$

$$\bigvee_{c} R(c, a) \otimes R(c, b) \leq \bigvee_{d} R(a, d) \otimes R(b, d)$$

Quantitative counterparts of the standard properties of abstract rewriting hold.

Quantitative term rewriting

Non-expansive quantitative TRSs: reducing terms inside contexts non-expansively propagates distances.

If t reduces to s with distance ε, then $C[t]$ reduces to $C[s]$ with distance ε too.

Nonlinear rules break non-expansiveness: distance amplification.

Quantitative term rewriting

Given:

- a quantale $\mathbb{\Omega} = (\Omega, \precsim, \kappa, \otimes)$,
- a set of terms $\mathcal{T}$,
- a set of triples $\mathcal{R} \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ ($\mathbb{\Omega}$-rewrite rules).

Notation: $\varepsilon \Vdash t \mapsto_{\mathcal{R}} s$ for $(t, \varepsilon, s) \in \mathcal{R}$.

Intuition: $\mathcal{R}$ is a set of rewrite rules that induces a rewrite relation.

A (nonexpansive) rewrite relation induced by $\mathcal{R}$ (wrt $\mathbb{\Omega}$): a ternary relation $\to_{\mathcal{R}} \subseteq \mathcal{T} \times \Omega \times \mathcal{T}$ defined by the rules on the next slide.

Non-expansive rewrite relation

$$\frac{\varepsilon \Vdash t \mapsto_{\mathcal{R}} s}{\varepsilon \Vdash C[t\sigma] \to_{\mathcal{R}} C[s\sigma]}$$

$$\frac{\varepsilon \Vdash t \to_{\mathcal{R}} s \quad \delta \precsim \varepsilon}{\delta \Vdash t \to_{\mathcal{R}} s}$$

$$\frac{\varepsilon_1 \Vdash t \to_{\mathcal{R}} s \quad \cdots \quad \varepsilon_n \Vdash t \to_{\mathcal{R}} s}{\bigvee_{i=1}^{n} \varepsilon_i \Vdash t \to_{\mathcal{R}} s}$$

$$\text{(Arch)}\ \frac{\forall \delta \ll \varepsilon.\, \delta \Vdash t \to_{\mathcal{R}} s}{\varepsilon \Vdash t \to_{\mathcal{R}} s}$$

Quantitative rewriting

Future work:

- Non-expansive quantitative rewriting modulo equational theories:

$$\frac{\varepsilon \Vdash t \mapsto_{\mathcal{R}} s \qquad \delta \Vdash t\sigma =_E r}{\varepsilon \otimes \delta \Vdash C[r] \to_{\mathcal{R},E} C[s\sigma]}$$

- Non-expansive quantitative narrowing:

$$\frac{\varepsilon \Vdash t \mapsto_{\mathcal{R}} s \qquad t\sigma = r\sigma}{\varepsilon \Vdash C[r] \rightsquigarrow_{\mathcal{R}} C\sigma[s\sigma]}$$

- Extending these non-expansive techniques to their graded counterparts.
- Quantitative version(s) of rewriting logic.

PROVING

Quantitative completion

Unfailing completion is a popular equational proving method.

Its quantitative counterpart can play a similar role for proving quantitative equalities.

First steps towards quantitative completion: for special shallow regular theories.

Future work: unfailing completion procedure for non-expansive and graded variants of quantitative rewrite systems.

Quantitative resolution

Has been studied in the context of logic programming with fuzzy similarity and proximity relations (Sessa, Julian Iranzo et al).

Can be extended to arbitrary quantales.

Semantics of such extended logic programming languages has to be investigated.

Related work: Weber et al. NLProlog: Reasoning with Weak Unification for Question Answering in Natural Language. ACL 2019.