Solving Quantitative Equations

Formal Methods and Automated Reasoning Seminar

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FWF Österreichischer Wissenschaftsfonds

RISC

(Quantitative) Equational Theories

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  Equations $s \approx t$ between terms $s, t \in T(\mathcal{F}, X)$.

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  Similarity/proximity rather than strict equality!

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- Quantitative setting:
  Similarity/proximity rather than strict equality!
  $\rightsquigarrow$ Equip equations $s \approx t$ with some element ε that measures the "degree to which they hold true".

Quantitative Equational Theories

Denote such equations by $\varepsilon \Vdash s \approx t$.

Example

- Fuzzy reasoning:
  $\varepsilon \in [0, 1]$ ($\sim$probability that s and t are equal).

  "Transitivity": $\varepsilon \Vdash t \approx s, \delta \Vdash s \approx r \Longrightarrow \min(\varepsilon, \delta) \Vdash t \approx r$
  "Weakening": $\varepsilon \geqslant \delta$, $\varepsilon \Vdash t \approx s \Longrightarrow \delta \Vdash t \approx s$.

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Two main requirements for the degrees:
It should be possible to combine and compare them.

Quantales

Definition (Quantale)

Quantale: $\mathbb{\Omega} = (\Omega, \precsim, \otimes, \kappa)$ such that

- $(\Omega, \precsim)$ is a complete lattice (poset where every subset has a supremum and infimum, denoted $\vee$ and $\wedge$)
- $(\Omega, \otimes, \kappa)$ is a monoid

satisfying the following distributivity laws:

$$\delta \otimes \left(\bigvee_{i \in I} \varepsilon_i \right) = \bigvee_{i \in I} (\delta \otimes \varepsilon_i), \qquad \left(\bigvee_{i \in I} \varepsilon_i \right) \otimes \delta = \bigvee_{i \in I} (\varepsilon_i \otimes \delta).$$

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Example

- $\mathbb{I} = ([0, 1], \leqslant, \min, 1)$ “fuzzy quantale”
- $\mathbb{L} = ([0, \infty], \geqslant, +, 0)$ “Lawvere quantale”
- $\mathbb{2} = (\{0, 1\}, \leqslant, \cdot, 1)$ “Boolean quantale”

Assume that we are working with Lawvereian quantales.

Definition

A quantale $\Omega = (\Omega, \precsim, \otimes, \kappa)$ is called *Lawvereian* if

- $\otimes$ is commutative
- Ω is *integral*: $\kappa = \top$
- Ω is *co-integral*: if $\varepsilon \otimes \delta = \bot$, then either $\varepsilon = \bot$ or $\delta = \bot$ (where $\bot$ is the bottom element)
- Ω is *non-trivial*: $\kappa \neq \bot$

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Definition

A quantale $\mho = (\Omega, \precsim, \otimes, \kappa)$ is called *Lawvereian* if

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- $\mho$ is *non-trivial*: $\kappa \neq \bot$

Remark

For $\varepsilon, \delta \in \Omega$, there exists an element $\varepsilon \multimap \delta$ (called *adjoint*), which has the following property:

$$\varepsilon \otimes \eta \precsim \delta \iff \eta \precsim \varepsilon \multimap \delta.$$

It can be computed as $\varepsilon \multimap \delta = \bigvee \{\eta \mid \varepsilon \otimes \eta \precsim \delta\}$.

Inference rules for quantitative equational logic (Gavazzo and Di Florio 2023)

$$\text{(Ax)}\ \frac{\varepsilon \Vdash t \approx_E s}{\varepsilon \Vdash t =_E s} \qquad \text{(Refl)}\ \frac{}{\kappa \Vdash t =_E t} \qquad \text{(Trans)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \Vdash s =_E r}{\varepsilon \otimes \delta \Vdash t =_E r}$$

$$\text{(NExp)}\ \frac{\varepsilon_1 \Vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_E s_n}{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \Vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)} \qquad \text{(Sym)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash s =_E t}$$

$$\text{(Ord)}\ \frac{\varepsilon \Vdash t =_E s \quad \delta \precsim \varepsilon}{\delta \Vdash t =_E s} \qquad \text{(Subst)}\ \frac{\varepsilon \Vdash t =_E s}{\varepsilon \Vdash t\sigma =_E s\sigma}$$

$$\text{(Arch)}\ \frac{\forall \delta \ll \varepsilon.\, \delta \Vdash t =_E s}{\varepsilon \Vdash t =_E s} \qquad \text{(Join)}\ \frac{\varepsilon_1 \Vdash t =_E s \quad \cdots \quad \varepsilon_n \Vdash t =_E s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_E s}$$

Remark

The examples mentioned before are recovered as special cases of quantitative equational theories by fixing the quantale Ω.

- $\Omega = 2 = (\{0,1\}, \leqslant, \cdot, 1)$: Classical equational reasoning (read $1 \Vdash s \approx_E t$ as $s \approx_E t$)
- $\Omega = \mathbb{I} = ([0,1], \leqslant, \min, 1)$: Fuzzy reasoning
- $\Omega = \mathbb{L} = ([0,\infty], \geqslant, +, 0)$: quantitative algebraic theories in the sense of Mardare, Panangaden, and Plotkin 2016 (with slightly modified (NExp) rule)

Unification Problems

Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of equations.

(Classical) Unification problem: $s \overset{?}{\approx}_E t$

Find a substitution σ such that $s\sigma \approx_E t\sigma$.

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Example

$E = \{f(x, y) \approx f(y, x)\}$.

The problem

$$f(g(x), f(b, a)) \approx^?_E f(f(x, b), y)$$

has solution

$$\sigma = \{x \mapsto a, y \mapsto g(a)\}.$$

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Let $s, t \in T(\mathcal{F}, X)$ be terms, E a set of Ω-equations, $\varepsilon \in \Omega$.

Quantitative unification problem: $s \approx^{?}_{E,\varepsilon} t$

Find a substitution σ such that $\varepsilon \Vdash s\sigma \approx_E t\sigma$.

Useful concepts in the classical case:

Definition (Instantiation preorder)

For substitutions σ and τ and a theory E, set
$\sigma \lesssim_E \tau :\iff \exists\varphi$ such that $x\sigma\varphi \approx_E x\tau$ for every $x \in X$.

If σ solves $s \approx_E^? t$ and $\sigma \lesssim \tau$, then τ solves $s \approx_E^? t$.

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Definition (mcsu)

Minimal complete set of unifiers (mcsu) of s and t modulo E:
A set U of substitutions such that

- Each $\sigma \in U$ is a solution of $s \approx_E^? t$,
- If ρ is a solution of $s \approx_E^? t$, then $\sigma \lesssim \rho$ holds for some $\sigma \in U$.
- If $\sigma, \tau \in U$ satisfy $\sigma \lessapprox \tau$, then $\sigma = \tau$.

If U is a mcsu of $s \approx_E^? t$, then the set of all solutions of $s \approx_E^? t$ is given by $\{\sigma\tau \mid \sigma \in U, \tau \text{ some substitution}\}$.

Quantitative analogues

"Quantitative instantiation" (for fixed $\varepsilon \in \Omega$):
$\sigma \lesssim_{E,\varepsilon} \tau :\iff \exists\varphi$ such that $\varepsilon \Vdash x\sigma\varphi \approx_E x\tau$ holds for every variable x.

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Problem: $\lesssim_{E,\varepsilon}$ is usually not transitive!
We only know that $\rho \lesssim_{E,\delta} \sigma$, $\sigma \lesssim_{E,\varepsilon} \tau$ implies $\rho \lesssim_{E,\varepsilon+\delta} \tau$.
If $\iota \in \Omega$ is idempotent ($\iota \otimes \iota = \iota$), then $\lesssim_{E,\iota}$ is a preorder.

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Definition

If $\iota \in \Omega$ is idempotent and $\iota \otimes \varepsilon = \varepsilon$, then a set $\mathcal{C}$ of (E, ε)-unifiers of s and t is said to be *ι-complete* if it satisfies the following:

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Remark

- In an arbitrary quantale, we can always choose $\iota = \kappa$.
- If Ω is idempotent, we can even choose $\iota = \varepsilon$.

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Minimality: $\mathcal{C}$ is a *minimal ι-complete set of unifiers* if moreover the following holds:

$$\text{If } \sigma, \tau \in \mathcal{C} \text{ and } \sigma \lesssim_{E,\iota} \tau, \text{ then } \sigma = \tau.$$

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Goal:

- General case: compute a κ-mcsu.
- Idempotent case: compute an ε-mcsu.

Assumptions

Assume that E is a finite set of quantitative equations of the form

$$\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n),$$

where f, g are n-ary function symbols and $x_1, \ldots, x_n$ are distinct variable symbols.

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Main idea for unification:
Decomposition:
Instead of solving $f(t_1, \ldots, t_n) \approx^?_\varepsilon g(s_1, \ldots, s_n)$, solve

$$t_1 \approx^?_{\alpha_1} s_1, \ldots, t_n \approx^?_{\alpha_n} s_n,$$

where $\alpha_1, \ldots, \alpha_n \in \Omega$ are yet to be determined and should satisfy $\mathfrak{d}_E(f, g) \multimap \varepsilon \precsim \alpha_1 \otimes \cdots \otimes \alpha_n$.
Here,

$$\mathfrak{d}_E(f, g) := \bigvee\{\varepsilon : \varepsilon \Vdash f(x_1, \ldots, x_n) \approx_E g(x_1, \ldots, x_n)\}.$$

The calculus

Operate on **configurations**: Quadruples $P; C; \delta; \sigma$, where

- P is a set of quantitative unification problems, indexed by metavariables $\alpha_1, \ldots, \alpha_n$ (the remainder of the problem)
- C is a constraint of the form $\alpha_1 \otimes \cdots \otimes \alpha_n \precsim \lambda$, where $\lambda \in \Omega$
- $\delta \in \Omega$ (the current approximation degree)
- σ is a substitution (the solution computed so far)

Rules

Tri : **Trivial**

$$\{t =^?_\alpha t\} \uplus P; \zeta \precsim \alpha \otimes \Delta; \delta; \sigma \Longrightarrow P; \zeta \precsim \Delta; \delta; \sigma.$$

Dec : **Decompose**

$$\{f(t_1, \ldots, t_n) =^?_\alpha g(s_1, \ldots, s_n)\} \uplus P; \zeta \precsim \alpha \otimes \Delta; \delta; \sigma \Longrightarrow$$
$$\{t_1 =^?_{\beta_1} s_1, \ldots, t_n =^?_{\beta_n} s_n\} \cup P;$$
$$\mathfrak{d}_E(f, g) \multimap \zeta \precsim \beta_1 \otimes \cdots \otimes \beta_n \otimes \Delta; \delta \otimes \mathfrak{d}_E(f, g); \sigma,$$

where $\beta_1, \ldots, \beta_n$ are new metavariables and $\zeta \precsim \mathfrak{d}_E(f, g)$.

Cla : **Clash**

$$\{f(t_1, \ldots, t_n) =^?_\alpha g(s_1, \ldots, s_m)\} \uplus P; \zeta \precsim \alpha \otimes \Delta; \delta; \sigma \Longrightarrow \mathbf{F},$$

if $\zeta \not\precsim \mathfrak{d}_E(f, g)$.

Rules (cont.)

$L-Sub$: **Substitute (lazy)**

$$\{x =^?_\alpha f(s_1,\dots,s_n)\} \uplus P; \zeta \precsim \alpha \otimes \Delta; \delta; \sigma \Longrightarrow$$
$$\{x_1 =^?_{\beta_1} s_1,\dots,x_n =^?_{\beta_n} s_n\} \cup P\rho;$$
$$\mathfrak{d}_E(f,g) \multimap \zeta \precsim \beta_1 \otimes \cdots \otimes \beta_n \otimes \Delta; \delta \otimes \mathfrak{d}_E(f,g); \sigma\rho,$$

where x does not appear in an occurrence cycle in $\{x =^?_\alpha f(s_1,\dots,s_n)\} \cup P$, and $\rho = \{x \mapsto g(x_1,\dots,x_n)\}$ with $x_1,\dots,x_n$ being fresh variables and $\zeta \precsim \mathfrak{d}_E(f,g)$.

CCh : **Cycle check**

$$\{x =^?_\alpha t\} \uplus P; C; \delta; \sigma \Longrightarrow \mathbf{F},$$

if x appears in an occurrence cycle in $\{x =^?_\alpha t\} \uplus P$.

Ori : **Orient**

$$\{t =^?_\alpha x\} \uplus P; C; \delta; \sigma \Longrightarrow P \cup \{x =^?_\alpha t\}; C; \delta; \sigma, \text{ where } t \notin \mathcal{V}.$$

Unification algorithm QUNIFY

Input: E and $t \approx^?_\varepsilon s$.

- Initial configuration: $\{t \approx^?_\alpha s\}; \varepsilon \precsim \alpha; \kappa; \mathrm{Id}$
- Apply rules as long as possible.
- Obtain **F** or a terminal configuration $P_t; C; \delta; \sigma$, where P_t contains only equations between variables.

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Theorem (Termination)

Any run of QUNIFY *terminates.*

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Theorem (Termination)

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Theorem (Soundness and Completeness)

Soundness: *If QUNIFY yields a terminal configuration, then any "solution" of this configuration is an (E, ε)-unifier of t and s.*

Completeness: *If σ is an (E, ε)-unifier of t and s, then there exists a run of QUNIFY that yields a terminal configuration for which σ is a "solution"*

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Definition (Solution of a configuration)

A substitution τ is a *solution* of the configuration $P; \zeta \precsim \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n; \delta; \sigma$ if there exists a function μ mapping metavariables to elements of Ω such that

1. $\zeta \precsim \mu(\alpha_1) \otimes \mu(\alpha_2) \otimes \cdots \otimes \mu(\alpha_n)$ is valid,
2. $\mu(\beta) \Vdash s\tau =_E t\tau$ holds for every equation $s =^?_\beta t$ in P.
3. $x\tau = x\sigma\tau$ (syntactic equality) holds for every variable $x \in \mathrm{dom}(\sigma)$.

Example derivation

Example

$\Omega = \mathbb{L}$, $E = \{1 \Vdash f(x,y) \approx g(x,y)\}$.
Unification problem: $g(a,x) =^?_3 f(y, g(b,z))$.

$$\{g(a,x) =^?_\alpha f(y, g(b,z))\};\ 3 \geqslant \alpha;\ 0;\ \mathrm{Id}$$

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$$\Longrightarrow_{Dec} \{a =^?_{\beta_1} y,\ x =^?_{\beta_2} g(b,z)\};\ 2 \geqslant \beta_1 + \beta_2;\ 1;\ \mathrm{Id}$$

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&\Longrightarrow^{x \mapsto f(x_1,x_2)}_{L-Sub} \{x_1 =^?_{\gamma_1} b,\ x_2 =^?_{\gamma_2} z\};\ 1 \geqslant \gamma_1 + \gamma_2;\ 2;\\
&\qquad\qquad \{y \mapsto a, x \mapsto f(x_1,x_2)\}
\end{aligned}
$$

Example derivation

Example

$\mho = \mathbb{L}$, $E = \{1 \Vdash f(x,y) \approx g(x,y)\}$.
Unification problem: $g(a,x) =^?_3 f(y, g(b,z))$.

$$
\begin{aligned}
& \{g(a,x) =^?_\alpha f(y,g(b,z))\};\ 3 \geqslant \alpha;\ 0;\ \mathrm{Id} \\
& \Longrightarrow_{Dec} \{a =^?_{\beta_1} y,\ x =^?_{\beta_2} g(b,z)\};\ 2 \geqslant \beta_1 + \beta_2;\ 1;\ \mathrm{Id} \\
& \Longrightarrow_{Ori} \{y =^?_{\beta_1} a,\ x =^?_{\beta_2} g(b,z)\};\ 2 \geqslant \beta_1 + \beta_2;\ 1;\ \mathrm{Id} \\
& \Longrightarrow^{y \mapsto a}_{L-Sub} \{x =^?_{\beta_2} g(b,z)\};\ 2 \geqslant \beta_2;\ 1;\ \{y \mapsto a\} \\
& \Longrightarrow^{x \mapsto f(x_1,x_2)}_{L-Sub} \{x_1 =^?_{\gamma_1} b,\ x_2 =^?_{\gamma_2} z\};\ 1 \geqslant \gamma_1 + \gamma_2;\ 2; \\
& \qquad\qquad \{y \mapsto a, x \mapsto f(x_1,x_2)\} \\
& \Longrightarrow^{x_1 \mapsto b}_{L-Sub} \{x_2 =^?_{\gamma_2} z\};\ 1 \geqslant \gamma_2;\ 2;\ \{y \mapsto a, x \mapsto f(b,x_2), x_1 \mapsto b\}
\end{aligned}
$$

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Reached a terminal configuration:

$$\{x_2 =^?_{\gamma_2} z\};\ 1 \geqslant \gamma_2;\ 2;\ \{y \mapsto a, x \mapsto f(b, x_2), x_1 \mapsto b\}$$

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Any substitution that solves this configuration is a solution.
Unifiers that can be obtained in this way include:

- $\{y \mapsto a, x \mapsto f(b,u), z \mapsto u\}$, where u is a fresh variable
- $\{y \mapsto a, x \mapsto f(b, f(a,a)), z \mapsto g(a,a)\}$

Conclusion/Outlook

So far:

- Established quantitative analogues of central notions of equational unification (instantiation relation, mcsu)
- Solved quantitative unification over a general quantale for a very specific class of theories
- Stronger results in the case of idempotent quantales

Future research directions:

- Extend this approach to more general classes of theories
- Quantitative anti-unification

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