# Specifying and Verifying Programs (Part 1)

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# Specifying and Verifying Programs

We will discuss three (closely interrelated) calculi.

- Hoare Calculus: $\{P\}\ c\ \{Q\}$
  - If command $c$ is executed in a pre-state with property $P$ and terminates, it yields a post-state with property $Q$.

    $\{x = a \land y = b\}x := x + y\{x = a + y \land y = b\}$
- Predicate Transformers: $\text{wp}(c, Q) = P$
  - If the execution of command $c$ shall yield a post-state with property $Q$, it must be executed in a pre-state with property $P$.

    $\text{wp}(x := x + y, x = a + y \land y = b) = (x + y = a + y \land y = b)$
- State Relations: $c : [P \Rightarrow Q]^{x,\ldots}$
  - The post-state generated by the execution of command $c$ is related to the pre-state by $P \Rightarrow Q$ (where only variables $x, \ldots$ have changed).

    $x = x + y : [\text{var } x = \text{old } x + \text{old } y]^x$

1. The Hoare Calculus
2. Checking Verification Conditions
3. Predicate Transformers
4. Termination
5. Abortion
6. Generating Verification Conditions
7. Proving Verification Conditions
8. Procedures

# The Hoare Calculus

First and best-known calculus for program reasoning (C.A.R. Hoare).

- "Hoare triple": $\{P\}\ c\ \{Q\}$
  - Logical propositions $P$ and $Q$, program command $c$.
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.
- Partial correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If $c$ is executed in a state in which $P$ holds, then it terminates in a state in which $Q$ holds unless it aborts or runs forever."

  - Program does not produce wrong result.
  - But program also need not produce any result.
    - Abortion and non-termination are not (yet) ruled out.
- Total correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If $c$ is executed in a state in which $P$ holds, then it terminates in a state in which $Q$ holds."

  - Program produces the correct result.

We will use the partial correctness interpretation for the moment.

# The Rules of the Hoare Calculus

Hoare calculus rules are inference rules with Hoare triples as proof goals.

$$\frac{\{P_1\}\ c_1\ \{Q_1\}\ \ \ldots\ \ \{P_n\}\ c_n\ \{Q_n\}\quad VC_1, \ldots, VC_m}{\{P\}\ c\ \{Q\}}$$

- Application of a rule to a triple $\{P\}\ c\ \{Q\}$ to be verified yields
  - other triples $\{P_1\}\ c_1\ \{Q_1\} \ldots \{P_n\}\ c_n\ \{Q_n\}$ to be verified, and
  - formulas $VC_1, \ldots, VC_m$ (the verification conditions) to be proved.
- Given a Hoare triple $\{P\}c\{Q\}$ as the root of the verification tree:
  - The rules are repeatedly applied until the leaves of the tree do not contain any more Hoare triples.
  - If all verification conditions in the tree can be proved, the root of the tree represents a valid Hoare triple.

The Hoare calculus generates verification conditions such that the validity of the conditions implies the validity of the original Hoare triple.

# Weakening and Strengthening

$$\frac{P \Rightarrow P' \quad \{P'\}\ c\ \{Q'\} \quad Q' \Rightarrow Q}{\{P\}\ c\ \{Q\}}$$

- Logical derivation: $\frac{A_1 \ A_2}{B}$
  - Forward: If we have shown $A_1$ and $A_2$, then we have also shown $B$.
  - Backward: To show $B$, it suffices to show $A_1$ and $A_2$.
- Interpretation of above sentence:
  - To show that, if $P$ holds, then $Q$ holds after executing $c$, it suffices to show this for a $P'$ weaker than $P$ and a $Q'$ stronger than $Q$.

Precondition may be weakened, postcondition may be strengthened.

# Special Commands

$$\{P\}\ \textbf{skip}\ \{P\} \qquad \{\text{true}\}\ \textbf{abort}\ \{\text{false}\}$$

- The **skip** command does not change the state; if $P$ holds before its execution, then $P$ thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
  - Axiom implies $\{P\}$ **abort** $\{Q\}$ for arbitrary $P, Q$.

Useful commands for reasoning and program transformations.

# Scalar Assignments

$$\{Q[e/x]\}\ x := e\ \{Q\}$$

- Syntax
  - Variable $x$, expression $e$.
  - $Q[e/x] \ldots Q$ where every free occurrence of $x$ is replaced by $e$.
- Interpretation
  - To make sure that $Q$ holds for $x$ after the assignment of $e$ to $x$, it suffices to make sure that $Q$ holds for $e$ before the assignment.
- Partial correctness
  - Evaluation of $e$ may abort.

$$\begin{array}{rcl} \{x+3<5\} & x := x+3 & \{x<5\} \\ \{x<2\} & x := x+3 & \{x<5\} \end{array}$$

# Array Assignments

$$\{Q[a[i \mapsto e]/a]\}\ a[i] := e\ \{Q\}$$

- An array is modelled as a function $a : I \to V$.
  - Index set $I$, value set $V$.
  - $a[i] = e$ ... array $a$ contains at index $i$ the value $e$.
- Term $a[i \mapsto e]$ ("array $a$ updated by assigning value $e$ to index $i$")
  - A new array that contains at index $i$ the value $e$.
  - All other elements of the array are the same as in $a$.
- Thus array assignment becomes a special case of scalar assignment.
  - Think of "$a[i] := e$" as "$a := a[i \mapsto e]$".

$$\{\underline{a[i \mapsto x]}[1] > 0\} \quad a[i] := x \quad \{a[1] > 0\}$$

Arrays are here considered as basic values (no pointer semantics).

# Array Assignments

How to reason about $a[i \mapsto e]$?

$$Q[\underline{a[i \mapsto e]}[j]]$$
$$\rightsquigarrow$$
$$(i = j \Rightarrow Q[e]) \land (i \neq j \Rightarrow Q[a[j]])$$

- Array Axioms

$$i = j \Rightarrow \underline{a[i \mapsto e]}[j] = e$$
$$i \neq j \Rightarrow \underline{a[i \mapsto e]}[j] = a[j]$$

$$\{\underline{a[i \mapsto x]}[1] > 0\} \quad a[i] := x \quad \{a[1] > 0\}$$
$$\{(i = 1 \Rightarrow x > 0) \land (i \neq 1 \Rightarrow a[1] > 0)\} \quad \mathrm{a[i]} := \mathrm{x} \quad \{a[1] > 0\}$$

Get rid of "array update terms" when applied to indices.

# Command Sequences

$$\frac{\{P\}\ c_1\ \{R\}\quad \{R\}\ c_2\ \{Q\}}{\{P\}\ c_1; c_2\ \{Q\}}$$

- Interpretation
  - To show that, if $P$ holds before the execution of $c_1; c_2$, then $Q$ holds afterwards, it suffices to show for some $R$ that
    - if $P$ holds before $c_1$, that $R$ holds afterwards, and that
    - if $R$ holds before $c_2$, then $Q$ holds afterwards.
- Problem: find suitable $R$.
  - Easy in many cases (see later).

$$\frac{\{x+y-1>0\}\ y := y-1\ \{x+y>0\}\quad \{x+y>0\}\ x := x+y\ \{x>0\}}{\{x+y-1>0\}\ y := y-1; x := x+y\ \{x>0\}}$$

The calculus itself does not indicate how to find intermediate property.

# Conditionals

$$\frac{\{P \wedge b\}\ c_1\ \{Q\} \quad \{P \wedge \neg b\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{\{P \wedge b\}\ c\ \{Q\} \quad (P \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

- Interpretation
  - To show that, if $P$ holds before the execution of the conditional, then $Q$ holds afterwards,
  - it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

$$\frac{\{x \neq 0 \wedge x \geq 0\}\ y := x\ \{y > 0\} \quad \{x \neq 0 \wedge x \not\geq 0\}\ y := -x\ \{y > 0\}}{\{x \neq 0\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ y := x\ \textbf{else}\ y := -x\ \{y > 0\}}$$

# Loops

$$\{\text{true}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{\{I \wedge b\}\ c\ \{I\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \wedge \neg b\}}$$

- Interpretation:
  - The **loop** command does not terminate and thus trivially satisfies partial correctness.
    - Axiom implies $\{P\}$ **loop** $\{Q\}$ for arbitrary $P, Q$.
  - If it is the case that
    - $I$ holds before the execution of the **while**-loop and
    - $I$ also holds after every iteration of the loop body,

    then $I$ holds also after the execution of the loop (together with the negation of the loop condition $b$).
    - $I$ is a loop invariant.
- Problem:
  - Rule for **while**-loop does not have arbitrary pre/post-conditions $P, Q$.

In practice, we combine this rule with the strengthening/weakening-rule.

# Loops (Generalized)

$$\frac{P \Rightarrow I \quad \{I \wedge b\}\ c\ \{I\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

- Interpretation:
  - To show that, if before the execution of a **while**-loop the property $P$ holds, after its termination the property $Q$ holds, it suffices to show for some property $I$ (the loop invariant) that
    - $I$ holds before the loop is executed (i.e. that $P$ implies $I$),
    - if $I$ holds when the loop body is entered (i.e. if also $b$ holds), that after the execution of the loop body $I$ still holds,
    - when the loop terminates (i.e. if $b$ does not hold), $I$ implies $Q$.
- Problem: find appropriate loop invariant $I$.
  - Strongest relationship between all variables modified in loop body.

The calculus itself does not indicate how to find suitable loop invariant.

# Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$

$$\frac{\begin{array}{c}(n \geq 0 \wedge s = 0 \wedge i = 1) \Rightarrow I \\ \{I \wedge i \leq n\}\ s := s+i; i := i+1\ \{I\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{n \geq 0 \wedge s = 0 \wedge i = 1\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s+i; i := i+1)\ \{s = \sum_{j=1}^{n} j\}}$$

The invariant captures the “essence” of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.

# A Program Verification

- Verification of the following Hoare triple:

  $\{Input\}$ **while** $i \leq n$ **do** $(s := s + i; i := i + 1)$ $\{Output\}$

- Auxiliary predicates:

  $Input :\Leftrightarrow n \geq 0 \wedge s = 0 \wedge i = 1$
  $Output :\Leftrightarrow s = \sum_{j=1}^{n} j$
  $Invariant :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n + 1$

- Verification conditions:

  $A :\Leftrightarrow Input \Rightarrow Invariant$
  $B :\Leftrightarrow Invariant \wedge i \leq n \Rightarrow Invariant[i + 1/i][s + i/s]$
  $C :\Leftrightarrow Invariant \wedge i \not\leq n \Rightarrow Output$

If the verification conditions are valid, the Hoare triple is true.

# RISCAL: Checking Program Execution

```
val N:Nat; type number = ℕ[N]; type index = ℕ[N+1]; type result = ℕ[N·(1+N)/2];

proc summation(n:number): result
  requires n ≥ 0;
  ensures result = ∑j:number with 1 ≤ j ∧ j ≤ n. j;
{
  var s:result := 0;
  var i:index  := 1;
  while i ≤ n do
    invariant s = ∑j:number with 1 ≤ j ∧ j ≤ i-1. j;
    invariant 1 ≤ i ∧ i ≤ n+1;
  {
    s := s+i;
    i := i+1;
  }
  return s;
}
```

We check for some *N* the program execution; this implies that the invariant is not too strong.

# RISCAL: Checking Verification Conditions

```
pred Input(n:number, s:result, i:index) ⇔
  n ≥ 0 ∧ s = 0 ∧ i = 1;
pred Output(n:number, s:result) ⇔
  s = ∑j:number with 1 ≤ j ∧ j ≤ n. j;
pred Invariant(n:number, s:result, i:index) ⇔
  (s = ∑j:number with 1 ≤ j ∧ j ≤ i-1. j) ∧ 1 ≤ i ∧ i ≤ n+1;

theorem A(n:number, s:result, i:index) ⇔
  Input(n, s, i) ⇒ Invariant(n, s, i);
theorem B(n:number, s:result, i:index) ⇔
  Invariant(n, s, i) ∧ i ≤ n ⇒ Invariant(n, s+i, i+1);
theorem C(n:number, s:result, i:index) ⇔
  Invariant(n, s, i) ∧ ¬(i ≤ n) ⇒ Output(n, s);
```

We check for some $N$ that the verification conditions are valid; this also implies that the invariant is not too weak.

# Another Program Verification

Verification of the following Hoare triple:

$\{olda = a \land oldx = x\}$

$i := 0; r := -1; n = |a|$
**while** $i < n \land r = -1$ **do**
  **if** $a[i] = x$
    **then** $r := i$
    **else** $i := i + 1$

$\{a = olda \land x = oldx \land$
$((r = -1 \land \forall i : 0 \leq i < |a| \Rightarrow a[i] \neq x) \lor$
$(0 \leq r < |a| \land a[r] = x \land \forall i : 0 \leq i < r \Rightarrow a[i] \neq x))\}$

$Invariant :\Leftrightarrow olda = a \land oldx = x \land n = |a| \land$
$0 \leq i \leq n \land \forall j : 0 \leq j < i \Rightarrow a[j] \neq x \land$
$(r = -1 \lor (r = i \land i < n \land a[r] = x))$

Find the smallest index $r$ of an occurrence of value $x$ in array $a$ ($r = -1$, if $x$ does not occur in $a$).

# RISCAL: Checking Program Execution

```
val N:ℕ; val M:ℕ;
type index = ℤ[-1,N]; type elem = ℕ[M]; type array = Array[N,elem];

proc search(a:array, x:elem): index
  ensures (result = -1 ∧ ∀i:index. 0 ≤ i ∧ i < N ⇒ a[i] ≠ x) ∨
          (0 ≤ result ∧ result < N ∧
            a[result] = x ∧ ∀i:index. 0 ≤ i ∧ i < result ⇒ a[i] ≠ x);
{
  var i:index = 0;
  var r:index = -1;
  while i < N ∧ r = -1 do
    invariant 0 ≤ i ∧ i ≤ N ∧ ∀j:index. 0 ≤ j ∧ j < i ⇒ a[j] ≠ x;
    invariant r = -1 ∨ (r = i ∧ i < N ∧ a[r] = x);
  {
    if a[i] = x
      then r := i;
      else i := i+1;
  }
  return r;
}
```

We check for some $N$, $M$ the program execution.

# The Verification Conditions

$$\begin{aligned}&Input :\Leftrightarrow olda = a \wedge oldx = x \wedge n = length(a) \wedge i = 0 \wedge r = -1\\[1ex]
&Output :\Leftrightarrow a = olda \wedge x = oldx \wedge\\
&\quad ((r = -1 \wedge \forall i : 0 \le i < length(a) \Rightarrow a[i] \ne x) \vee\\
&\quad\; (0 \le r < length(a) \wedge a[r] = x \wedge \forall i : 0 \le i < r \Rightarrow a[i] \ne x))\\[1ex]
&Invariant :\Leftrightarrow olda = a \wedge oldx = x \wedge n = |a| \wedge\\
&\quad 0 \le i \le n \wedge \forall j : 0 \le j < i \Rightarrow a[j] \ne x \wedge\\
&\quad (r = -1 \vee (r = i \wedge i < n \wedge a[r] = x))\end{aligned}$$

$$\begin{aligned}&A :\Leftrightarrow Input \Rightarrow Invariant\\
&B_1 :\Leftrightarrow Invariant \wedge i < n \wedge r = -1 \wedge a[i] = x \Rightarrow Invariant[i/r]\\
&B_2 :\Leftrightarrow Invariant \wedge i < n \wedge r = -1 \wedge a[i] \ne x \Rightarrow Invariant[i+1/i]\\
&C :\Leftrightarrow Invariant \wedge \neg(i < n \wedge r = -1) \Rightarrow Output\end{aligned}$$

The verification conditions $A, B_1, B_2, C$ must be valid.

# RISCAL: Checking Verification Conditions

```
pred Input(i:index, r:index) ⇔ i = 0 ∧ r = -1;
pred Output(a:array, x:elem, i:index, r:index) ⇔
  (r = -1 ∧ ∀i:index. 0 ≤ i ∧ i < N ⇒ a[i] ≠ x) ∨
  (0 ≤ r ∧ r < N ∧ a[r] = x ∧ ∀i:index. 0 ≤ i ∧ i < r ⇒ a[i] ≠ x);
pred Invariant(a:array, x:elem, i:index, r:index) ⇔
  0 ≤ i ∧ i ≤ N ∧ (∀j:index. 0 ≤ j ∧ j < i ⇒ a[j] ≠ x) ∧
  (r = -1 ∨ (r = i ∧ i < N ∧ a[r] = x));

theorem A(a:array, x:elem, i:index, r:index) ⇔
  Input(i, r) ⇒ Invariant(a, x, i, r);
theorem B1(a:array, x:elem, i:index, r:index) ⇔
  Invariant(a, x, i, r) ∧ i < N ∧ r = -1 ∧ a[i] = x ⇒
    Invariant(a, x, i, i);
theorem B2(a:array, x:elem, i:index, r:index) ⇔
  Invariant(a, x, i, r) ∧ i < N ∧ r = -1 ∧ a[i] ≠ x ⇒
    Invariant(a, x, i+1, r);
theorem C(a:array, x:elem, i:index, r:index) ⇔
  Invariant(a, x, i, r) ∧ ¬(i < N ∧ r = -1) ⇒
    Output(a, x, i, r);
```

We check for some $N$, $M$ that the verification conditions are valid.

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# Backward Reasoning

Implication of rule for command sequences and rule for assignments:

$$\frac{\{P\}\ c\ \{Q[e/x]\}}{\{P\}\ c; x := e\ \{Q\}}$$

- Interpretation
  - If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
  - By multiple application, assignment sequences can be removed from the back to the front.

| | | | | |
|---|---|---|---|---|
| $\{P\}$ | $\{P\}$ | $\{P\}$ | $\{P\}$ | $P \Rightarrow x = 4$ |
| x := x+1; | x := x+1; | x := x+1; | $\{x+1=5\}$ | |
| y := 2*x; | y := 2*x; | $\{x+2x=15\}$ | $(\Leftrightarrow x=4)$ | |
| z := x+y | $\{x+y=15\}$ | $(\Leftrightarrow 3x=15)$ | | |
| $\{z=15\}$ | | $(\Leftrightarrow x=5)$ | | |

# Weakest Preconditions

A calculus for "backward reasoning" (E.W. Dijkstra).

- Predicate transformer wp
  - Function "wp" that takes a command $c$ and a postcondition $Q$ and returns a precondition.
  - Read $\mathrm{wp}(c, Q)$ as "the weakest precondition of $c$ w.r.t. $Q$".
- $\mathrm{wp}(c, Q)$ is a precondition for $c$ that ensures $Q$ as a postcondition.
  - Must satisfy $\{\mathrm{wp}(c, Q)\}\ c\ \{Q\}$.
- $\mathrm{wp}(c, Q)$ is the weakest such precondition.
  - Take any $P$ such that $\{P\}\ c\ \{Q\}$.
  - Then $P \Rightarrow \mathrm{wp}(c, Q)$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \mathrm{wp}(c, Q))$
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $P \Rightarrow \mathrm{wp}(c, Q)$ instead.

Verification is reduced to the calculation of weakest preconditions.

# Weakest Preconditions

The weakest precondition of each program construct.

$$\begin{aligned}
&\text{wp}(\textbf{skip}, Q) = Q\\
&\text{wp}(\textbf{abort}, Q) = \text{true}\\
&\text{wp}(x := e, Q) = Q[e/x]\\
&\text{wp}(c_1; c_2, Q) = \text{wp}(c_1, \text{wp}(c_2, Q))\\
&\text{wp}(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, Q) = (b \Rightarrow \text{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \text{wp}(c_2, Q))\\
&\text{wp}(\textbf{if } b \textbf{ then } c, Q) \Leftrightarrow (b \Rightarrow \text{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)\\
&\text{wp}(\textbf{while } b \textbf{ do } c, Q) = \ldots
\end{aligned}$$

Loops represent a special problem (see later).

# Forward Reasoning

Sometimes, we want to derive a postcondition from a given precondition.

$$\{P\}\ x := e\ \{\exists x_0 : P[x_0/x] \land x = e[x_0/x]\}$$

- Forward Reasoning
  - What is the maximum we know about the post-state of an assignment $x := e$, if the pre-state satisfies $P$?
  - We know that $P$ holds for some value $x_0$ (the value of $x$ in the pre-state) and that $x$ equals $e[x_0/x]$.

$$\begin{gathered}
\{x \geq 0 \land y = a\} \\
x := x + 1 \\
\{\exists x_0 : x_0 \geq 0 \land y = a \land x = x_0 + 1\} \\
(\Leftrightarrow (\exists x_0 : x_0 \geq 0 \land x = x_0 + 1) \land y = a) \\
(\Leftrightarrow x > 0 \land y = a)
\end{gathered}$$

# Strongest Postcondition

A calculus for forward reasoning.

- Predicate transformer sp
  - Function "sp" that takes a precondition $P$ and a command $c$ and returns a postcondition.
  - Read $\text{sp}(c, P)$ as "the strongest postcondition of $c$ w.r.t. $P$".
- $\text{sp}(c, P)$ is a postcondition for $c$ that is ensured by precondition $P$.
  - Must satisfy $\{P\}\ c\ \{\text{sp}(c, P)\}$.
- $\text{sp}(c, P)$ is the strongest such postcondition.
  - Take any $P, Q$ such that $\{P\}\ c\ \{Q\}$.
  - Then $\text{sp}(c, P) \Rightarrow Q$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(\text{sp}(c, P) \Rightarrow Q)$.
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $\text{sp}(c, P) \Rightarrow Q$ instead.

Verification is reduced to the calculation of strongest postconditions.

# Strongest Postconditions

The strongest postcondition of each program construct.

$$
\begin{aligned}
&\mathrm{sp}(\mathbf{skip}, P) = P\\
&\mathrm{sp}(\mathbf{abort}, P) = \mathrm{false}\\
&\mathrm{sp}(x := e, P) = \exists x_0 : P[x_0/x] \land x = e[x_0/x]\\
&\mathrm{sp}(c_1; c_2, P) = sp(c_2, sp(c_1, P))\\
&\mathrm{sp}(\mathbf{if}\ b\ \mathbf{then}\ c_1\ \mathbf{else}\ c_2, P) \Leftrightarrow \mathrm{sp}(c_1, P \land b) \lor \mathrm{sp}(c_2, P \land \neg b)\\
&\mathrm{sp}(\mathbf{if}\ b\ \mathbf{then}\ c, P) = \mathrm{sp}(c, P \land b) \lor (P \land \neg b)\\
&\mathrm{sp}(\mathbf{while}\ b\ \mathbf{do}\ c, P) = \ldots
\end{aligned}
$$

Forward reasoning as a (less-known) alternative to backward-reasoning.

# Hoare Calc. and Predicate Transformers

In practice, often a combination of the calculi is applied.

$$\{P\}\ c_1; \textbf{while}\ b\ \textbf{do}\ c; c_2\ \{Q\}$$

- Assume $c_1$ and $c_2$ do not contain loop commands.
- It suffices to prove

$$\{\mathrm{sp}(P, c_1)\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{\mathrm{wp}(c_2, Q)\}$$

Predicate transformers are applied to reduce the verification of a program to the Hoare-style verification of loops.

# Weakest Liberal Preconditions for Loops

Why not apply predicate transformers to loops?

$$\text{wp}(\textbf{loop}, Q) = \text{true}$$
$$\text{wp}(\textbf{while } b \textbf{ do } c, Q) = L_0(Q) \wedge L_1(Q) \wedge L_2(Q) \wedge \ldots$$

$$L_0(Q) = \text{true}$$
$$L_{i+1}(Q) = (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$$

- Interpretation
  - Weakest precondition that ensures that loops stops in a state satisfying $Q$, unless it aborts or runs forever.
- Infinite sequence of predicates $L_i(Q)$:
  - Weakest precondition that ensures that after less than $i$ iterations the state satisfies $Q$, unless the loop aborts or does not yet terminate.
- Alternative view: $L_i(Q) = \text{wp}(\text{if}_i, Q)$

$$\text{if}_0 = \textbf{loop}$$
$$\text{if}_{i+1} = \textbf{if } b \textbf{ then } (c; \text{if}_i)$$

# Example

$$\mathrm{wp}(\textbf{while } i < n \textbf{ do } i := i+1, Q)$$

$$\begin{aligned}
L_0(Q) &= \text{true} \\
L_1(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, \text{true})) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{true}) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \\
L_2(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, i \not< n \Rightarrow Q)) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow (i+1 \not< n \Rightarrow Q[i+1/i])) \\
L_3(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, \\
&\qquad (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow (i+1 \not< n \Rightarrow Q[i+1/i])))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow ((i+1 \not< n \Rightarrow Q[i+1/i]) \wedge \\
&\qquad\qquad (i+1 < n \Rightarrow (i+2 \not< n \Rightarrow Q[i+2/i]))))
\end{aligned}$$

# Weakest Liberal Preconditions for Loops

- Sequence $L_i(Q)$ is monotonically increasing in strength:
  - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$.
- The weakest precondition is the "lowest upper bound":
  - $\forall i \in \mathbb{N} : \text{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : P \Rightarrow L_i(Q)) \Rightarrow (P \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q))$.
- We can only compute weaker approximation $L_i(Q)$.
  - $\text{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
- We want to prove $\{P\}$ **while** $b$ **do** $c$ $\{Q\}$.
  - This is equivalent to proving $P \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - Thus $P \Rightarrow L_i(Q)$ must hold as well.
- If we can prove $\neg(P \Rightarrow L_i(Q))$, ...
  - $\{P\}$ **while** $b$ **do** $c$ $\{Q\}$ does not hold.
  - If we fail, we may try the easier proof $\neg(P \Rightarrow L_{i+1}(Q))$.

Falsification is possible by use of approximation $L_i$, but verification is not.

# Preconditions for Loops with Invariants

$$\begin{aligned}
&\mathrm{wp}(\textbf{while } b \textbf{ do invariant } I;\ c^{x,\cdots}, Q) = \\
&\quad \textbf{let } oldx = x, \ldots \textbf{ in} \\
&\quad I \land (\forall x, \ldots : I \land b \Rightarrow wp(c, I)) \land \\
&\quad (\forall x, \ldots : I \land \neg b \Rightarrow Q)
\end{aligned}$$

- Loop body $c$ only modifies variables $x, \ldots$
- Loop is annotated with invariant $I$.
  - May refer to new values $x, \ldots$ of variables after every iteration.
  - May refer to original values $oldx, \ldots$ when loop started execution.
- Generated verification condition ensures:
  1. $I$ holds in the initial state of the loop.
  2. $I$ is preserved by the execution of the loop body $c$.
  3. When the loop terminates, $I$ ensures postcondition $Q$.

This precondition is only "weakest" relative to the invariant.

# Example

$$\textbf{while } i \leq n \textbf{ do } (s := s + i; i := i + 1)$$

$$c^{s,i} := (s := s + i; i := i + 1)$$
$$I :\Leftrightarrow s = olds + \left(\sum_{j=oldi}^{i-1} j\right) \wedge oldi \leq i \leq n + 1$$

- Weakest precondition:

$$\begin{aligned}&\text{wp}(\textbf{while } i \leq n \textbf{ do invariant } I;\ c^{s,i}, Q) = \\ &\qquad \textbf{let } olds = s, oldi = i \textbf{ in} \\ &\qquad I \wedge (\forall s, i : I \wedge i \leq n \Rightarrow I[i+1/i][s+i/s]) \wedge \\ &\qquad (\forall s, i : I \wedge \neg(i \leq n) \Rightarrow Q)\end{aligned}$$

- Verification condition:

$$n \geq 0 \wedge i = 1 \wedge s = 0 \Rightarrow \text{wp}(\ldots, s = \textstyle\sum_{j=1}^{n} j)$$

Many verification systems implement (a variant of) this calculus.

# Termination

Hoare rules for **loop** and **while** are replaced as follows:

$$\{\text{false}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{I \Rightarrow t \geq 0 \quad \{I \land b \land t = N\}\ c\ \{I \land t < N\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \land \neg b\}}$$

$$\frac{P \Rightarrow I \quad I \Rightarrow t \geq 0 \quad \{I \land b \land t = N\}\ c\ \{I \land t < N\} \quad (I \land \neg b) \Rightarrow Q}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of $c$ starts in a state where $P$ holds, then execution terminates in a state where $Q$ holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The **loop** command thus does not satisfy total correctness.
- Termination measure $t$ (term type-checked to denote an integer).
  - Becomes smaller by every iteration of the loop.
  - But does not become negative.
  - Consequently, the loop must eventually terminate.

The initial value of $t$ limits the number of loop iterations.

Any *well-founded ordering* may be used as the domain of $t$.

# Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge 1 \leq i \leq n+1$$
$$t := n - i + 1$$

$$\frac{\begin{array}{c}(n \geq 0 \wedge i = 1 \wedge s = 0) \Rightarrow I \quad I \Rightarrow n - i + 1 \geq 0 \\ \{I \wedge i \leq n \wedge n - i + 1 = N\}\ s := s + i; i := i + 1\ \{I \wedge n - i + 1 < N\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{n \geq 0 \wedge i = 1 \wedge s = 0\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s + i; i := i + 1)\ \{s = \sum_{j=1}^{n} j\}}$$

In practice, termination is easy to show (compared to partial correctness).

# Termination in RISCAL

```
while i ≤ n do
  invariant s = ∑j:number with 1 ≤ j ∧ j ≤ i-1. j;
  invariant 1 ≤ i ∧ i ≤ n+1;
  decreases n+1-i;
{
  s := s+i;
  i := i+1;
}

fun Termination(n:number, s:result, i:index): number =
  n+1-i;
theorem T(n:number, s:result, i:index) ⇔
  Invariant(n, s, i) ⇒ Termination(n, s, i) ≥ 0;
theorem B(n:number, s:result, i:index) ⇔
  Invariant(n, s, i) ∧ i ≤ n ⇒
    Invariant(n, s+i, i+1) ∧
    Termination(n, s+i, i+1) < Termination(n, s, i);
```

# Termination in RISCAL

```
while i < N ∧ r = -1 do
  invariant 0 ≤ i ∧ i ≤ N;
  invariant ∀j:index. 0 ≤ j ∧ j < i ⇒ a[j] ≠ x;
  invariant r = -1 ∨ (r = i ∧ i < N ∧ a[r] = x);
  decreases if r = -1 then N-i else 0;
{
  if a[i] = x
    then r := i;
    else i := i+1;
}

fun Termination(a:array, x:elem, i:index, r:index): index =
  if r = -1 then N-i else 0;
theorem T(a:array, x:elem, i:index, r:index) ⇔
  Invariant(a, x, i, r) ⇒ Termination(a, x, i, r) ≥ 0;
theorem B1(a:array, x:elem, i:index, r:index) ⇔
  Invariant(a, x, i, r) ∧ i < N ∧ r = -1 ∧ a[i] = x ⇒
    Invariant(a, x, i, i) ∧
    Termination(a, x, i, i) < Termination(a, x, i, r);
theorem B2(a:array, x:elem, i:index, r:index) ⇔ ...
```

# Weakest Preconditions for Loops

$$\text{wp}(\textbf{loop}, Q) = \text{false}$$
$$\text{wp}(\textbf{while } b \textbf{ do } c, Q) = L_0(Q) \lor L_1(Q) \lor L_2(Q) \lor \ldots$$

$$L_0(Q) = \text{false}$$
$$L_{i+1}(Q) = (\neg b \Rightarrow Q) \land (b \Rightarrow \text{wp}(c, L_i(Q)))$$

- New interpretation
  - Weakest precondition that ensures that the loop terminates in a state in which $Q$ holds, unless it aborts.
- New interpretation of $L_i(Q)$
  - Weakest precondition that ensures that the loop terminates after less than $i$ iterations in a state in which $Q$ holds, unless it aborts.
- Preserves property: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$
  - Now for total correctness interpretation of Hoare calculus.
- Preserves alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

$$\text{if}_0 = \textbf{loop}$$
$$\text{if}_{i+1} = \textbf{if } b \textbf{ then } (c; \text{if}_i)$$

# Example

$$\mathrm{wp}(\textbf{while } i < n \textbf{ do } i := i + 1, Q)$$

$$\begin{aligned}
L_0(Q) &= \text{false} \\
L_1(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ wp(i := i + 1, L_0(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ \text{false}) \\
&\Leftrightarrow i \not< n \wedge Q \\
L_2(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ wp(i := i + 1, L_1(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow (i + 1 \not< n \wedge Q[i + 1/i])) \\
L_3(Q) &= (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \ \mathrm{wp}(i := i + 1, L_2(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow \ ((i + 1 \not< n \Rightarrow Q[i + 1/i]) \wedge \\
&\qquad\qquad (i + 1 < n \Rightarrow (i + 2 \not< n \wedge Q[i + 2/i]))))
\end{aligned}$$

...

# Weakest Preconditions for Loops

- Sequence $L_i(Q)$ is now monotonically decreasing in strength:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q)$.
- The weakest precondition is the "greatest lower bound":
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : L_i(Q) \Rightarrow P) \Rightarrow (\mathrm{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow P)$.
- We can only compute a stronger approximation $L_i(Q)$.
  - $L_i(Q) \Rightarrow \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q)$.
- We want to prove $\{P\}\ c\ \{Q\}$.
  - It suffices to prove $P \Rightarrow \mathrm{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - It thus also suffices to prove $P \Rightarrow L_i(Q)$.
  - If proof fails, we may try the easier proof $P \Rightarrow L_{i+1}(Q)$

However, verifications are typically not successful with any finite approximation of the weakest precondition.

# Weakest Precondition with Measures

$$\begin{array}{l} \text{wp}(\textbf{while } b \textbf{ do invariant } I; \textbf{decreases } t;\ c^{x,\cdots}, Q) = \\ \quad \textbf{let } oldx = x, \ldots \textbf{ in} \\ \quad I \wedge (\forall x, \ldots : I \wedge b \Rightarrow wp(c, I)) \wedge \\ \quad (\forall x, \ldots : I \wedge \neg b \Rightarrow Q) \wedge \\ \quad (\forall x, \ldots : I \Rightarrow t \geq 0) \wedge \\ \quad (\forall x, \ldots : I \wedge b \Rightarrow \texttt{let } T = t \texttt{ in } wp(c, t < T)) \end{array}$$

- Loop body $c$ only modifies variables $x, \ldots$
- Loop is annotated with termination measure (term) $t$.
  - May refer to new values $x, \ldots$ of variables after every iteration.
- Generated verification condition ensures:
  1. $t$ is non-negative before/after every loop iteration.
  2. $t$ is decremented by the execution of the loop body $c$.

Also here any well-founded ordering may be used as the domain of $t$.

# Example

$$\textbf{while } i \le n \textbf{ do } (s := s + i; i := i + 1)$$

$c^{s,i} := (s := s + i; i := i + 1)$

$I :\Leftrightarrow s = olds + \left(\sum_{j=oldi}^{i-1}\right) \wedge oldi \le i \le n + 1$

$t := n + 1 - i$

- Weakest precondition:

$$\begin{aligned}
&\text{wp}(\textbf{while } i \le n \textbf{ do invariant } I;\ c^{s,i}, Q) = \\
&\quad \textbf{let } olds = s, oldi = i \textbf{ in} \\
&\quad I \wedge (\forall s, i : I \wedge i \le n \Rightarrow I[s+i/s, i+1/i]) \wedge \\
&\quad (\forall s, i : I \wedge \neg(i \le n) \Rightarrow Q) \wedge \\
&\quad (\forall s, i : I \Rightarrow t \ge 0) \wedge \\
&\quad (\forall s, i : I \wedge i \le n \Rightarrow \texttt{let } T = n+1-i \texttt{ in } n+1-(i+1) < T)
\end{aligned}$$

- Verification condition:

$$n \ge 0 \wedge i = 1 \wedge s = 0 \Rightarrow \text{wp}(\ldots, s = \textstyle\sum_{j=1}^{n} j)$$

1. The Hoare Calculus
2. Checking Verification Conditions
3. Predicate Transformers
4. Termination
5. **Abortion**
6. Generating Verification Conditions
7. Proving Verification Conditions
8. Procedures

# Abortion

New rules to prevent abortion.

$$\{\text{false}\}\ \textbf{abort}\ \{\text{true}\}$$
$$\{Q[e/x] \land D(e)\}\ x := e\ \{Q\}$$
$$\{Q[a[i \mapsto e]/a] \land D(e) \land D(i) \land 0 \leq i < \text{length}(a)\}\ a[i] := e\ \{Q\}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of $c$ starts in a state, in which property $P$ holds, then it does not abort and eventually terminates in a state in which $Q$ holds.
- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

$D(e)$ makes sure that every subexpression of $e$ is well defined.

# Definedness of Expressions

$$
\begin{aligned}
&D(0) = \text{true}.\\
&D(1) = \text{true}.\\
&D(x) = \text{true}.\\
&D(a[i]) = D(i) \wedge 0 \le i < \text{length}(a).\\
&D(e_1 + e_2) = D(e_1) \wedge D(e_2).\\
&D(e_1 * e_2) = D(e_1) \wedge D(e_2).\\
&D(e_1 / e_2) = D(e_1) \wedge D(e_2) \wedge e_2 \neq 0.\\
&D(\text{true}) = \text{true}.\\
&D(\text{false}) = \text{true}.\\
&D(\neg b) = D(b).\\
&D(b_1 \wedge b_2) = D(b_1) \wedge D(b_2).\\
&D(b_1 \vee b_2) = D(b_1) \wedge D(b_2).\\
&D(e_1 < e_2) = D(e_1) \wedge D(e_2).\\
&D(e_1 \le e_2) = D(e_1) \wedge D(e_2).\\
&D(e_1 > e_2) = D(e_1) \wedge D(e_2).\\
&D(e_1 \ge e_2) = D(e_1) \wedge D(e_2).
\end{aligned}
$$

Assumes that expressions have already been type-checked.

# Abortion

Slight modification of existing rules.

$$\frac{P \Rightarrow D(b) \quad \{P \land b\}\ c_1\ \{Q\} \quad \{P \land \neg b\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{P \Rightarrow D(b) \quad \{P \land b\}\ c\ \{Q\} \quad (P \land \neg b) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

$$\frac{I \Rightarrow (t \geq 0 \land D(b)) \quad \{I \land b \land t = N\}\ c\ \{I \land t < N\}}{\{I\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{I \land \neg b\}}$$

Expressions must be defined in any context.

# Abortion

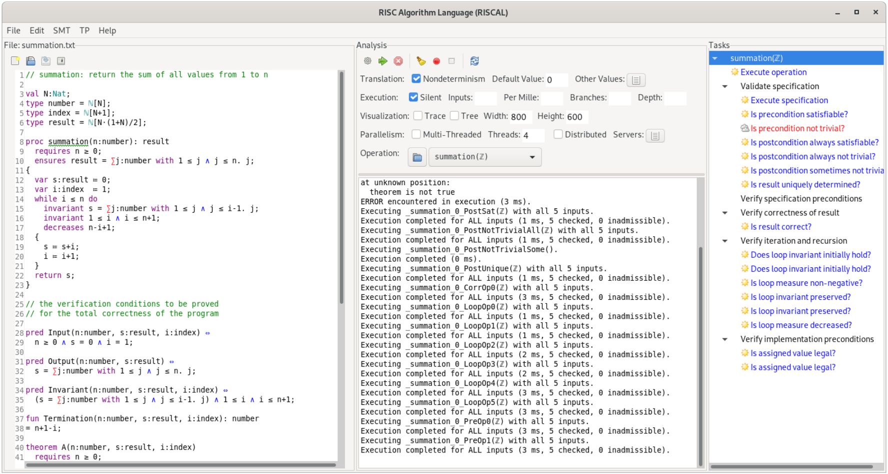
Similar modifications of weakest preconditions.

$$\begin{aligned}
&\mathrm{wp}(\textbf{abort}, Q) = \text{false} \\
&\mathrm{wp}(x := e, Q) = Q[e/x] \land D(e) \\
&\mathrm{wp}(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, Q) = \\
&\quad D(b) \land (b \Rightarrow \mathrm{wp}(c_1, Q)) \land (\neg b \Rightarrow \mathrm{wp}(c_2, Q)) \\
&\mathrm{wp}(\textbf{if } b \textbf{ then } c, Q) = D(b) \land (b \Rightarrow \mathrm{wp}(c, Q)) \land (\neg b \Rightarrow Q) \\
&\mathrm{wp}(\textbf{while } b \textbf{ do } c, Q) = (L_0(Q) \lor L_1(Q) \lor L_2(Q) \lor \ldots)
\end{aligned}$$

$$\begin{aligned}
&L_0(Q) = \text{false} \\
&L_{i+1}(Q) = D(b) \land (\neg b \Rightarrow Q) \land (b \Rightarrow \mathrm{wp}(c, L_i(Q)))
\end{aligned}$$

$\mathrm{wp}(c, Q)$ now makes sure that the execution of $c$ does not abort but eventually terminates in a state in which $Q$ holds.

# RISCAL and Verification Conditions




RISCAL implements Dijkstra's calculus for VC generation.

# RISCAL Verification Conditions

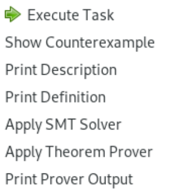
RISCAL splits Dijkstra's single condition $Input \Rightarrow wp(C, Output)$ into many "fine-grained" verification conditions:

- Is result correct?
  - One condition for every `ensures` clause.
- Does loop invariant initially hold? Is loop invariant preserved?
  - Partial correctness.
  - One condition for every `invariant` clause.
- Is loop measure non-negative? Is loop measure decreased?
  - Termination.
  - One condition for every `decreases` clause.
- Specification and implementation preconditions
  - Well-definedness of formulas and commands (later).
  - One condition for every partial function/predicate application.

Click on a condition to see the affected commands; if the procedure contains conditionals, a condition is generated for each execution branch.

# Checking Verification Conditions

- Double-click a condition to have it checked.
  - Checked conditions turn from red to blue.
- Right-click a condition to see a pop-up menu.
  - Check verification condition (same as double-click)
  - Show variable values that invalidate condition.
  - Print relevant program information (e.g. invariant).
  - Print verification condition itself.
  - Apply SMT solver for faster checking (see menu "SMT").




Example: is loop invariant preserved?

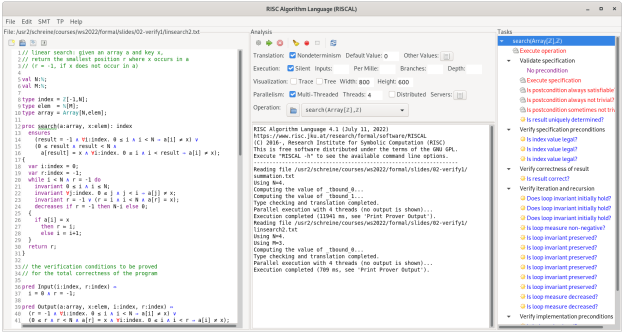
```
s = (∑j:number with (1 ≤ j) ∧ (j ≤ (i-1)). j)

theorem _summation_0_LoopOp3(n:number)
requires n ≥ 0;
 ⇔ ∀s:result,i:index. ((((s = (∑j:number with (1 ≤ j)∧(j ≤ (i-1)). j))
    ∧ ((1 ≤ i) ∧ (i ≤ (n+1)))) ∧ (i ≤ n)) ⇒
       (let s = s+i in (let i = i+1 in
         (s = (∑j:number with (1 ≤ j) ∧ (j ≤ (i-1)). j)))));
```

Important: check models with <u>small</u> type sizes.

# Proving Verification Conditions

RISCAL also provides an interface to automated theorem provers.

- Menu "TP" and menu entry "Apply Theorem Prover"
  - Tries to prove condition for <u>arbitrary</u> type sizes.
  - "Print Prover Output:" shows details of proof attempt.
  - "Apply Prover to All Theorems:" multiple proofs (in parallel).




Many (but typically not all) automatic proof attempts may succeed.

# RISC ProofNavigator: A Theory of Arrays

```
% constructive array definition
newcontext "arrays2";

% the types
INDEX: TYPE = NAT;
ELEM:  TYPE;
ARR:   TYPE =
  [INDEX, ARRAY INDEX OF ELEM];

% error constants
any:      ARRAY INDEX OF ELEM;
anyelem:  ELEM;
anyarray: ARR;

% a selector operation
content:
  ARR -> (ARRAY INDEX OF ELEM) =
  LAMBDA(a:ARR): a.1;
```

```
% the array operations
length: ARR -> INDEX =
  LAMBDA(a:ARR): a.0;
new: INDEX -> ARR =
  LAMBDA(n:INDEX): (n, any);
put: (ARR, INDEX, ELEM) -> ARR =
 LAMBDA(a:ARR, i:INDEX, e:ELEM):
   IF i < length(a)
     THEN (length(a),
           content(a) WITH [i]:=e)
     ELSE anyarray
   ENDIF;
get: (ARR, INDEX) -> ELEM =
  LAMBDA(a:ARR, i:INDEX):
    IF i < length(a)
      THEN content(a)[i]
      ELSE anyelem ENDIF;
```

# Proof of Fundamental Array Properties

```
% the classical array axioms as formulas to be proved
length1: FORMULA
  FORALL(n:INDEX): length(new(n)) = n;

length2: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => length(put(a, i, e)) = length(a);

get1: FORMULA
  FORALL(a:ARR, i:INDEX, e:ELEM):
    i < length(a) => get(put(a, i, e), i) = e;

get2: FORMULA
  FORALL(a:ARR, i, j:INDEX, e:ELEM):
    i < length(a) AND j < length(a) AND
    i /= j =>
      get(put(a, i, e), j) = get(a, j);
```

[adu]: expand length, get, put, content
- [c3b]: scatter
  - [qid]: proved (CVCL)

# The Verification Conditions

```
newcontext
  "linsearch";

% declaration
% of arrays
...

a: ARR;
olda: ARR;
x: ELEM;
oldx: ELEM;
i: NAT;
n: NAT;
r: INT;
```

```
Input: BOOLEAN = olda = a AND oldx = x AND
  n = length(a) AND i = 0 AND r = -1;

Output: BOOLEAN = a = olda AND
   ((r = -1 AND
      (FORALL(j:NAT): j < length(a) =>
         get(a,j) /= x)) OR
    (0 <= r AND r < length(a) AND get(a,r) = x AND
      (FORALL(j:NAT):
        j < r => get(a,j) /= x)));

Invariant: (ARR, ELEM, NAT, NAT, INT) -> BOOLEAN =
  LAMBDA(a: ARR, x: ELEM, i: NAT, n: NAT, r: INT):
    olda = a AND oldx = x AND
    n = length(a) AND i <= n AND
    (FORALL(j:NAT): j < i => get(a,j) /= x) AND
    (r = -1 OR (r = i AND i < n AND get(a,r) = x));
...
```

# The Verification Conditions (Contd)

```
...

A: FORMULA
  Input => Invariant(a, x, i, n, r);

B1: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) = x
    => Invariant(a,x,i,n,i);

B2: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) /= x
    => Invariant(a,x,i+1,n,r);

C: FORMULA
  Invariant(a, x, i, n, r) AND NOT(i < n AND r = -1)
    => Output;
```

# The Proofs

A:

[bca]: expand Input, Invariant
  [fuo]: scatter
    [bxg]: proved (CVCL)

(2 user actions)

B1:

[p1b]: expand Invariant
  [lf6]: proved (CVCL)

(1 user action)

B2:

[q1b]: expand Invariant in 6kv
  [slx]: scatter
    [a1y]: auto
      [cch]: proved (CVCL)
    [b1y]: proved (CVCL)
    [c1y]: proved (CVCL)
    [d1y]: proved (CVCL)
    [e1y]: proved (CVCL)

(3 user actions)

C:

[dca]: expand Invariant, Output in zfg
  [tvy]: scatter
    [dcu]: auto
      [t4c]: proved (CVCL)
    [ecu]: split pkg
      [kel]: proved (CVCL)
      [lel]: scatter
        [lvn]: auto
          [lap]: proved (CVCL)
    [fcu]: auto
      [blt]: proved (CVCL)
    [gcu]: proved (CVCL)

(6 user actions)

# Termination

```
Termination: (ARR, ELEM, NAT, NAT, INT) -> INT =
  LAMBDA(a: ARR, x: ELEM, i: NAT, n: NAT, r: INT):
    IF r=-1 THEN n-i ELSE 0 ENDIF;

T: FORMULA
  Invariant(a, x, i, n, r) => Termination(a, x, i, n, r) >= 0;

B1: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) = x AND
  Termination(a, x, i, n, r) = N
    => Invariant(a,x,i,n,i) AND Termination(a,x,i,n,i) < N;

B2: FORMULA
  Invariant(a, x, i, n, r) AND i < n AND r = -1 AND get(a,i) /= x AND
  Termination(a, x, i, n, r) = N
    => Invariant(a,x,i+1,n,r) AND Termination(a,x,i+1,n,r) < N;
```

1. The Hoare Calculus
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# Procedure Specifications

global $g$;
requires $Pre$;
ensures $Post$;
$o := p(i) \{ c \}$

- Specification of a procedure $p$ implemented by a command $c$.
  - Input parameter $i$, output parameter $o$, global variable $g$.
    - Command $c$ may read/write $i$, $o$, and $g$.
  - Precondition $Pre$ (may refer to $i, g$).
  - Postcondition $Post$ (may refer to $i, o, g, g_0$).
    - $g_0$ denotes the value of $g$ before the execution of $p$.
- Proof obligation

$$\{Pre \wedge i_0 = i \wedge g_0 = g\}\ c\ \{Post[i_0/i]\}$$

Proof of the correctness of the implementation of a procedure with respect to its specification.

# Example

- Procedure specification:

  global $g$
  requires $g \geq 0 \land i > 0$
  ensures $g_0 = g \cdot i + o \land 0 \leq o < i$
  $o := p(i)\ \{\ o := g\%i;\ g := g/i\ \}$

- Proof obligation:

  $\{g \geq 0 \land i > 0 \land i_0 = i \land g_0 = g\}$
  $o := g\%i;\ g := g/i$
  $\{g_0 = g \cdot i_0 + o \land 0 \leq o < i_0\}$

A procedure that divides $g$ by $i$ and returns the remainder.

# Procedure Calls

A call of $p$ provides actual input argument $e$ and output variable $x$.

$$x := p(e)$$

Similar to assignment statement; we thus first give an alternative (equivalent) version of the assignment rule.

- Original:

$$\{D(e) \wedge Q[e/x]\}$$
$$x := e$$
$$\{Q\}$$

- Alternative:

$$\{D(e) \wedge \forall x' : x' = e \Rightarrow Q[x'/x]\}$$
$$x := e$$
$$\{Q\}$$

The new value of $x$ is given name $x'$ in the precondition.

# Procedure Calls

From this, we can derive a rule for the correctness of procedure calls.

$$
\begin{array}{c}
\{D(e) \wedge Pre[e/i] \wedge \\
\forall x', g' : Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]\} \\
x := p(e) \\
\{Q\}
\end{array}
$$

- $Pre[e/i]$ refers to the values of the actual argument $e$ (rather than to the formal parameter $i$).
- $x'$ and $g'$ denote the values of the vars $x$ and $g$ after the call.
- $Post[\ldots]$ refers to the argument values before and after the call.
- $Q[x'/x, g'/g]$ refers to the argument values after the call.

Modular reasoning: rule only relies on the *specification* of $p$, not on its implementation.

# Corresponding Predicate Transformers

$$
\begin{aligned}
&\mathrm{wp}(x = p(e), Q) = \\
&\quad D(e) \wedge Pre[e/i] \wedge \\
&\quad \forall x', g' : \\
&\quad\quad Post[e/i, x'/o, g/g_0, g'/g] \Rightarrow Q[x'/x, g'/g]
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{sp}(P, x = p(e)) = \\
&\quad \exists x_0, g_0 : \\
&\quad\quad P[x_0/y, g_0/g] \wedge \\
&\quad\quad (Pre[e[x_0/x, g_0/g]/i, g_0/g] \Rightarrow Post[e[x_0/x, g_0/g]/i, x/o])
\end{aligned}
$$

Explicit naming of old/new values required.

# Example

- Procedure specification:

  global $g$
  requires $g \geq 0 \wedge i > 0$
  ensures $g_0 = g \cdot i + o \wedge 0 \leq o < i$
  $o = p(i)\ \{\ o := g\%i;\ \ g := g/i\ \}$

- Procedure call:

  $\{g \geq 0 \wedge g = N \wedge b \geq 0\}$
  $x = p(b+1)$
  $\{g \cdot (b+1) \leq N < (g+1) \cdot (b+1)\}$

- To be proved:

$$
\begin{array}{l}
g \geq 0 \wedge g = N \wedge b \geq 0 \Rightarrow \\
\quad D(b+1) \wedge g \geq 0 \wedge b+1 > 0 \wedge \\
\quad \forall x', g' : \\
\qquad g = g' \cdot (b+1) + x' \wedge 0 \leq x' < b+1 \Rightarrow \\
\qquad\quad g' \cdot (b+1) \leq N < (g'+1) \cdot (b+1)
\end{array}
$$