Logic, Checking, and Proving

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1. The Language of Logic

2. The RISC Algorithm Language

3. The Art of Proving

4. The RISC ProofNavigator

The Language of Logic

Two kinds of syntactic phrases.

- Term T denoting an object.
  - Variable x
  - Object constant c
  - Function application $f(T_1, \ldots, T_n)$ (may be written infix)

    n-ary function constant f
- Formula F denoting a truth value.
  - Atomic formula $p(T_1, \ldots, T_n)$ (may be written infix)

    n-ary predicate constant p.
  - Negation $\neg F$ ("not F")
  - Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - Implication $F_1 \Rightarrow F_2$ ("if F_1, then F_2")
  - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1, then F_2, and vice versa")
  - Universal quantification $\forall x : F$ ("for all x, F")
  - Existential quantification $\exists x : F$ ("for some x, F")

Syntactic Shortcuts

- $\forall x_1, \ldots, x_n : F$
  - $\forall x_1 : \ldots : \forall x_n : F$
- $\exists x_1, \ldots, x_n : F$
  - $\exists x_1 : \ldots : \exists x_n : F$
- $\forall x \in S : F$
  - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
  - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

Examples

Terms and formulas may appear in various syntactic forms.

- Terms:

$$\begin{array}{l} \exp(x) \\ a \cdot b + 1 \\ a[i] \cdot b \\ \sqrt{\frac{x^2+2x+1}{(y+1)^2}} \end{array}$$

- Formulas:

$$\begin{array}{l} a^2 + b^2 = c^2 \\ n \mid 2n \\ \forall x \in \mathbb{N} : x \geq 0 \\ \forall x \in \mathbb{N} : 2|x \vee 2|(x+1) \\ \forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow \\ \qquad \exists z \in \mathbb{N} : x + z = y \end{array}$$

Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas

- Atomic formula $p(T_1, \ldots, T_n)$
  - True if the predicate denoted by p holds for the values of $T_1, \ldots, T_n$.
- Negation $\neg F$
  - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ ("if F_1, then F_2")
  - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1, then F_2, and vice versa")
  - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ ("for all x, F")
  - True if and only if F is true for every possible value assignment of x.
- Existential quantification $\exists x : F$ ("for some x, F")
  - True if and only if F is true for at least one value assignment of x.

Example

We assume the domain of natural numbers and the "classical" interpretation of constants 1, 2, $+$, $=$, $<$.

- $1 + 1 = 2$
  - True.
- $1 + 1 = 2 \lor 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \land 2 + 2 = 2$
  - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
  - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
  - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
  - True.

Example

- $x + 1 = 1 + x$
  - True, for every assignment of a number a to variable x.
- $\forall x : x + 1 = 1 + x$
  - True (because for every assignment a to x, $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
  - If x is assigned "one", the formula is true.
  - If x is assigned "two", the formula is false.
- $\exists x : x + 1 = 2$
  - True (because $x + 1 = 2$ is true for assignment "one" to x).
- $\forall x : x + 1 = 2$
  - False (because $x + 1 = 2$ is false for assignment "two" to x).
- $\forall x : \exists y : x < y$
  - True (because for every assignment a to x, there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
  - False (because for every assignment a to y, there is the assignment $a + 1$ to x which makes $x < y$ false).

Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \leftrightsquigarrow F_1$
- $\neg(F_1 \land F_2) \leftrightsquigarrow \neg F_1 \lor \neg F_2$
- $\neg(F_1 \lor F_2) \leftrightsquigarrow \neg F_1 \land \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \leftrightsquigarrow F_1 \land \neg F_2$
- $\neg\forall x : F \leftrightsquigarrow \exists x : \neg F$
- $\neg\exists x : F \leftrightsquigarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_1 \lor F_2$
- $F_1 \Leftrightarrow F_2 \leftrightsquigarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.

Example

- "All swans are white or black."
  - $\forall x : swan(x) \Rightarrow white(x) \lor black(x)$
- "There exists a black swan."
  - $\exists x : swan(x) \land black(x)$.
- "A swan is white, unless it is black."
  - $\forall x : swan(x) \land \neg black(x) \Rightarrow white(x)$
  - $\forall x : swan(x) \land \neg white(x) \Rightarrow black(x)$
  - $\forall x : swan(x) \Rightarrow white(x) \lor black(x)$
- "Not everything that is white or black is a swan."
  - $\neg\forall x : white(x) \lor black(x) \Rightarrow swan(x)$.
  - $\exists x : (white(x) \lor black(x)) \land \neg swan(x)$.
- "Black swans have at least one black parent".
  - $\forall x : swan(x) \land black(x) \Rightarrow \exists y : swan(y) \land black(y) \land parent(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.

The Usage of Formulas

Precise formulation of statements describing object relationships.

- Statement:

  If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y.

- Formula:

  $$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow$$
  $$\quad q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y$$

- Problem specification:

  Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y.

  - Inputs: x, y
  - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
  - Output: q
  - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : x = y \cdot q + r \wedge r < y$

Problem Specifications

- The specification of a computation problem:
  - Input: variables $x_1 \in S_1, \ldots, x_n \in S_n$
  - Input condition ("precondition"): formula $I(x_1, \ldots, x_n)$.
  - Output: variables $y_1 \in T_1, \ldots, y_m \in T_n$
  - Output condition ("postcondition"): $O(x_1, \ldots, x_n, y_1, \ldots, y_m)$.
    - $F(x_1, \ldots, x_n)$: only $x_1, \ldots, x_n$ are free in formula F.
    - x is *free* in F, if not every occurrence of x is inside the scope of a quantifier (such as $\forall$ or $\exists$) that binds x.
- An implementation of the specification:
  - A function (program) $f : S_1 \times \ldots \times S_n \to T_1 \times \ldots \times T_m$ such that

$$
\begin{aligned}
&\forall x_1 \in S_1, \ldots, x_n \in S_n : I(x_1, \ldots, x_n) \Rightarrow \\
&\qquad \textbf{let } (y_1, \ldots, y_m) = f(x_1, \ldots, x_n) \textbf{ in} \\
&\qquad O(x_1, \ldots, x_n, y_1, \ldots, y_m)
\end{aligned}
$$

  - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.

Example: A Problem Specification

Given an integer array a, a position p in a, and a length l, return the array b derived from a by removing $a[p], \ldots, a[p+l-1]$.

- Input: $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- Input condition:

  $$p + l \leq \text{length}(a)$$

- Output: $b \in \mathbb{Z}^*$
- Output condition:

  $$\begin{array}{l} \textbf{let } n = \text{length}(a) \textbf{ in} \\ \text{length}(b) = n - l \land \\ (\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \land \\ (\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i+l]) \end{array}$$

Mathematical theory:

$$\begin{array}{l} T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \to T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\} \\ \text{length} : T^* \to \mathbb{N}, \text{length}(a) = \textbf{such } i \in \mathbb{N} : a \in T^i \end{array}$$

Validating Problem Specifications

Do formal input condition $I(x)$ and output condition $O(x, y)$ really capture our informal intentions?

- Do concrete inputs/output satisfy/violate these conditions?
  - $I(a_1)$, $\neg I(a_2)$, $O(a_1, b_1)$, $\neg O(a_1, b_2)$.
- Is input condition satisfiable?
  - $\exists x : I(x)$.
- Is input condition not trivial?
  - $\exists x : \neg I(x)$.
- Is output condition satisfiable for every input?
  - $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- Is output condition for all (at least some) inputs not trivial?
  - $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.
  - $\exists x : I(x) \wedge \exists y : \neg O(x, y)$.
- Is for every legal input at most one output legal?
  - $\forall x : I(x) \Rightarrow \forall y_1, y_2 : O(x, y_1) \wedge O(x, y_2) \Rightarrow y_1 = y_2$.

Validate specification to increase our confidence in its meaning!

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2. **The RISC Algorithm Language**
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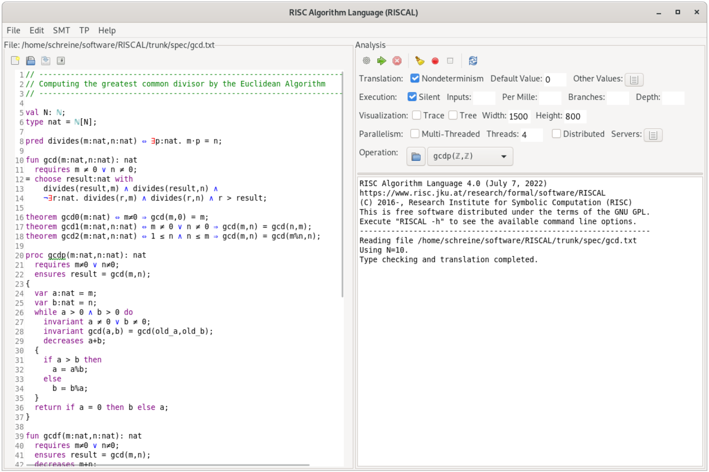
The RISC Algorithm Language (RISCAL)

- A system for formally specifying and checking algorithms.
  - Research Institute for Symbolic Computation (RISC), 2016–.
    http://www.risc.jku.at/research/formal/software/RISCAL.
  - Implemented in Java with SWT library for the GUI.
    - Tested under Linux only; freely available as open source (GPL3).
- A language for the defining mathematical theories and algorithms.
  - A static type system with only finite types (of parameterized sizes).
  - Predicates, explicitly (also recursively) and implicitly def.d functions.
  - Theorems (universally quantified predicates expected to be true).
  - Procedures (also recursively defined).
  - Pre- and post-conditions, invariants, termination measures.
- A framework for evaluating/executing all definitions.
  - Model checking: predicates, functions, theorems, procedures, annotations may be evaluated/executed for all possible inputs.
  - All paths of a non-deterministic execution may be elaborated.
  - The execution/evaluation may be visualized.

Validating algorithms by automatically verifying finite approximations.

The RISC Algorithm Language (RISCAL)

`RISCAL divide.txt &`



Using RISCAL

See also the (printed/online) "Tutorial and Reference Manual".

- Press button (or <Ctrl>-s) to save specification.
  - Automatically processes (parses and type-checks) specification.
  - Press button to re-process specification.
- Choose values for undefined constants in specification.
  - Natural number for `val` *const*: $\mathbb{N}$.
  - *Default Value*: used if no other value is specified.
  - *Other Values*: specific values for individual constants.
- Select *Operation* from menu and then press button .
  - Executes operation for chosen constant values and all possible inputs.
  - Option *Silent*: result of operation is not printed.
  - Option *Nondeterminism*: all execution paths are taken.
  - Option *Multi-threaded*: multiple threads execute different inputs.
  - Press buttton to abort execution.

During evaluation all annotations (pre/postconditions, etc.) are checked.

Typing Mathematical Symbols

ASCII String	Unicode Character	ASCII String	Unicode Character
`Int`	$\mathbb{Z}$	`~=`	$\neq$
`Nat`	$\mathbb{N}$	`<=`	$\leq$
`:=`	$:=$	`>=`	$\geq$
`true`	$\top$	`*`	$\cdot$
`false`	$\bot$	`times`	$\times$
`~`	$\neg$	`{}`	$\emptyset$
`/\`	$\wedge$	`intersect`	$\cap$
`\/`	$\vee$	`union`	$\cup$
`=>`	$\Rightarrow$	`Intersect`	$\bigcap$
`<=>`	$\Leftrightarrow$	`Union`	$\bigcup$
`forall`	$\forall$	`isin`	$\in$
`exists`	$\exists$	`subseteq`	$\subseteq$
`sum`	$\sum$	`<<`	$\langle$
`product`	$\prod$	`>>`	$\rangle$

Type the ASCII string and press <Ctrl>-# to get the Unicode character.

Example: Quotient and Remainder

Given natural numbers n and m, we want to compute the quotient q and remainder r of n divided by m.

```
// the type of natural numbers less than equal N
val N: ℕ;
type Num = ℕ[N];

// the precondition of the computation
pred pre(n:Num, m:Num) ⇔ m ≠ 0;

// the postcondition, first formulation
pred post1(n:Num, m:Num, q:Num, r:Num) ⇔
  n = m·q + r ∧
  ∀q0:Num, r0:Num.
    n = m·q0 + r0 ⇒ r ≤ r0;

// the postcondition, second formulation
pred post2(n:Num, m:Num, q:Num, r:Num) ⇔
  n = m·q + r ∧ r < m;
```

We will investigate this specification.

Example: Quotient and Remainder

```
// for all inputs that satisfy the precondition
// both formulations are equivalent:
// ∀n:Num, m:Num, q:Num, r:Num.
//   pre(n, m) ⇒ (post1(n, m, q, r) ⇔ post2(n, m, q, r));
theorem postEquiv(n:Num, m:Num, q:Num, r:Num)
  requires pre(n, m);
⇔ post1(n, m, q, r) ⇔ post2(n, m, q, r);

// we will thus use the simpler formulation from now on
pred post(n:Num, m:Num, q:Num, r:Num) ⇔ post2(n, m, q, r);
```

Check equivalence for all values that satisfy the precondition.

Example: Quotient and Remainder

Choose e.g. value 5 for N.

- Switch option *Silent* off:

```
Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function postEquiv(0,1,0,0):
Result (0 ms): true
Run 7 of deterministic function postEquiv(1,1,0,0):
Result (0 ms): true
...
Run 1295 of deterministic function postEquiv(5,5,5,5):
Result (0 ms): true
Execution completed for ALL inputs (6314 ms, 1080 checked, 216 inadmissible).
```

- Switch option *Silent* on:

```
Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Execution completed for ALL inputs (244 ms, 1080 checked, 216 inadmissible).
```

If theorem is false for some input, an error message is displayed.

Example: Quotient and Remainder

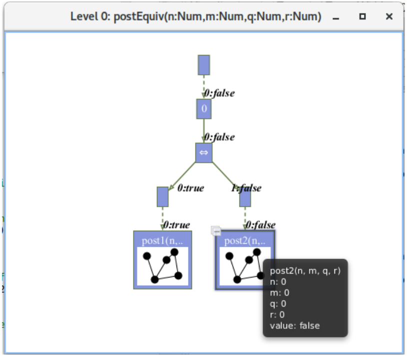
Drop precondition from theorem.

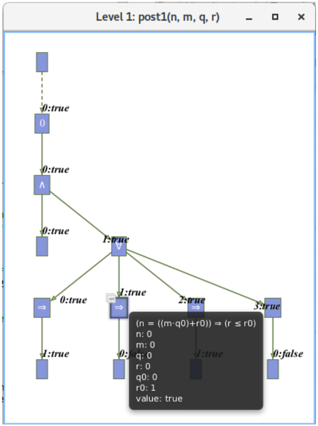
```
theorem postEquiv(n:Num, m:Num, q:Num, r:Num) ⇔
  // requires pre(n, m);
  post1(n, m, q, r) ⇔ post2(n, m, q, r);

Executing postEquiv(ℤ,ℤ,ℤ,ℤ) with all 1296 inputs.
Run 0 of deterministic function postEquiv(0,0,0,0):
ERROR in execution of postEquiv(0,0,0,0): evaluation of
  postEquiv
at line 25 in file divide.txt:
  theorem is not true
ERROR encountered in execution.
```

For $n = 0, m = 0, q = 0, r = 0$, the modified theorem is not true.

Visualizing the Formula Evaluation

Select $N = 1$ and visualization option "Tree".





Investigate the (pruned) evaluation tree to determine how the truth value of a formula was derived (double click to zoom into/out of predicates).

Example: Quotient and Remainder

Switch option "Nondeterminism" on.

```
// 1. investigate whether the specified input/output combinations are as desired
fun quotremFun(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
= choose q:Num, r:Num with post(n, m, q, r);

Executing quotremFun(ℤ,ℤ) with all 36 inputs.
Ignoring inadmissible inputs...
Branch 0:6 of nondeterministic function quotremFun(0,1):
Result (0 ms): [0,0]
Branch 1:6 of nondeterministic function quotremFun(0,1):
No more results (8 ms).
...
Branch 0:35 of nondeterministic function quotremFun(5,5):
Result (0 ms): [1,0]
Branch 1:35 of nondeterministic function quotremFun(5,5):
No more results (14 ms).
Execution completed for ALL inputs (413 ms, 30 checked, 6 inadmissible).
```

First validation by inspecting the values determined by output condition (nondeterminism may produce for some inputs multiple outputs).

Example: Quotient and Remainder

```
// 2. check that some but not all inputs are allowed
theorem someInput() ⇔ ∃n:Num, m:Num. pre(n, m);
theorem notEveryInput() ⇔ ∃n:Num, m:Num. ¬pre(n, m);

Executing someInput().
Execution completed (0 ms).
Executing notEveryInput().
Execution completed (0 ms).
```

A very rough validation of the input condition.

Example: Quotient and Remainder

```
// 3. check whether for all inputs that satisfy the precondition
// there are some outputs that satisfy the postcondition
theorem someOutput(n:Num, m:Num)
  requires pre(n, m);
⇔ ∃q:Num, r:Num. post(n, m, q, r);

// 4. check that not every output satisfies the postcondition
theorem notEveryOutput(n:Num, m:Num)
  requires pre(n, m);
⇔ ∃q:Num, r:Num. ¬post(n, m, q, r);

Executing someOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).
Executing notEveryOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (5 ms, 30 checked, 6 inadmissible).
```

A very rough validation of the output condition.

Example: Quotient and Remainder

```
// 5. check that the output is uniquely defined
// (optional, need not generally be the case)
theorem uniqueOutput(n:Num, m:Num)
  requires pre(n, m);
⇔
  ∀q:Num, r:Num. post(n, m, q, r) ⇒
  ∀q0:Num, r0:Num. post(n, m, q0, r0) ⇒
    q = q0 ∧ r = r0;

Executing uniqueOutput(ℤ,ℤ) with all 36 inputs.
Execution completed for ALL inputs (18 ms, 30 checked, 6 inadmissible).
```

The output condition indeed determines the outputs uniquely.

Example: Quotient and Remainder

```
// 6. check whether the algorithm satisfies the specification
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures let q=result.1, r=result.2 in post(n, m, q, r);
{
  var q: Num = 0;
  var r: Num = n;
  while r ≥ m do
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

Check whether the algorithm satisfies the specification.

Example: Quotient and Remainder

```
Executing quotRemProc(ℤ,ℤ) with all 36 inputs.
Ignoring inadmissible inputs...
Run 6 of deterministic function quotRemProc(0,1):
Result (0 ms): [0,0]
Run 7 of deterministic function quotRemProc(1,1):
Result (0 ms): [1,0]
...
Run 31 of deterministic function quotRemProc(1,5):
Result (1 ms): [0,1]
Run 32 of deterministic function quotRemProc(2,5):
Result (0 ms): [0,2]
Run 33 of deterministic function quotRemProc(3,5):
Result (0 ms): [0,3]
Run 34 of deterministic function quotRemProc(4,5):
Result (0 ms): [0,4]
Run 35 of deterministic function quotRemProc(5,5):
Result (1 ms): [1,0]
Execution completed for ALL inputs (161 ms, 30 checked, 6 inadmissible).
```

A verification of the algorithm by checking all possible executions.

Example: Quotient and Remainder

```
proc quotRemProc(n:Num, m:Num): Tuple[Num,Num]
  requires pre(n, m);
  ensures post(n, m, result.1, result.2);
{
  var q: Num = 0;
  var r: Num = n;
  while r > m do // error!
  {
    r := r-m;
    q := q+1;
  }
  return ⟨q,r⟩;
}
```

```
Executing quotRemProc(ℤ,ℤ) with all 36 inputs.
ERROR in execution of quotRemProc(1,1): evaluation of
  ensures let q = result.1, r = result.2 in post(n, m, q, r);
at line 65 in file divide.txt:
  postcondition is violated by result [0,1]
ERROR encountered in execution.
```

A falsification of an incorrect algorithm.

Example: Sorting an Array

```
val N:Nat; val M:Nat;
type nat = Nat[M]; type array = Array[N,nat]; type index = Nat[N-1];

proc sort(a:array): array
  ensures ∀i:nat. i < N-1 ⇒ result[i] ≤ result[i+1];
  ensures ∃p:Array[N,index].
            (∀i:index,j:index. i ≠ j ⇒ p[i] ≠ p[j]) ∧
            (∀i:index. a[i] = result[p[i]]);
{
  var b:array = a;
  for var i:Nat[N]:=1; i<N; i:=i+1 do {
    var x:nat := b[i];
    var j:Int[-1,N] := i-1;
    while j ≥ 0 ∧ b[j] > x do {
      b[j+1] := b[j];
      j := j-1;
    }
    b[j+1] := x;
  }
  return b;
}
```

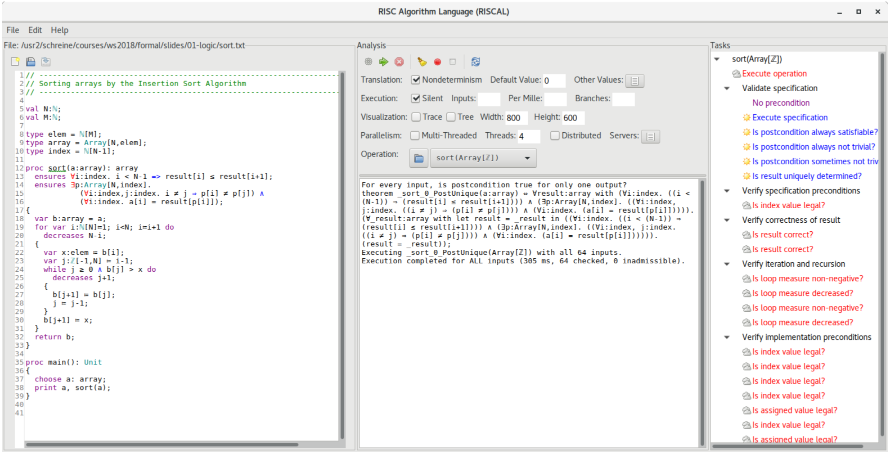
Example: Sorting an Array

```
Using N=5.
Using M=5.
Type checking and translation completed.
Executing sort(Array[ℤ]) with all 7776 inputs.
1223 inputs (1223 checked, 0 inadmissible, 0 ignored)...
2026 inputs (2026 checked, 0 inadmissible, 0 ignored)...
...
5114 inputs (5114 checked, 0 inadmissible, 0 ignored)...
5467 inputs (5467 checked, 0 inadmissible, 0 ignored)...
5792 inputs (5792 checked, 0 inadmissible, 0 ignored)...
6118 inputs (6118 checked, 0 inadmissible, 0 ignored)...
6500 inputs (6500 checked, 0 inadmissible, 0 ignored)...
6788 inputs (6788 checked, 0 inadmissible, 0 ignored)...
7070 inputs (7070 checked, 0 inadmissible, 0 ignored)...
7354 inputs (7354 checked, 0 inadmissible, 0 ignored)...
7634 inputs (7634 checked, 0 inadmissible, 0 ignored)...
Execution completed for ALL inputs (32606 ms, 7776 checked, 0 inadmissible).
Not all nondeterministic branches may have been considered.
```

Also this algorithm can be automatically checked.

Example: Sorting an Array

Select operation `sort` and press the button “Show/Hide Tasks”.



Automatically generated formulas to validate procedure specifications.

Example: Sorting an Array

Right-click to print definition of a formula, double-click to check it.

```
For every input, is postcondition true for only one output?

theorem _sort_0_PostUnique(a:array) ⇔
  ∀result:array with
    (∀i:index. ((i < (N-1)) ⇒ (result[i] ≤ result[i+1]))) ∧
    (∃p:Array[N,index]. ((∀i:index, j:index. ((i ≠ j) ⇒ (p[i] ≠ p[j]))) ∧
      (∀i:index. (a[i] = result[p[i]])))).
    (∀_result:array with let result = _result in #
      ((∀i:index. ((i < (N-1)) ⇒ (result[i] ≤ result[i+1]))) ∧
       (∃p:Array[N,index]. ((∀i:index, j:index. ((i ≠ j) ⇒ (p[i] ≠ p[j]))) ∧
         (∀i:index. (a[i] = result[p[i]]))))).
       (result = _result));

Using N=3.
Using M=3.
Type checking and translation completed.
Executing _sort_0_PostUnique(Array[ℤ]) with all 64 inputs.
Execution completed for ALL inputs (529 ms, 64 checked, 0 inadmissible).
```

The output is indeed uniquely defined by the output condition.

Model Checking versus Proving

Two fundamental techniques for the verification of computer programs.

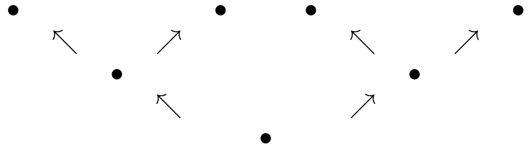
- Checking Program Executions
  - Enumeration of all possible executions and evaluation of formulas (e.g. postconditions) on the resulting states.
  - Fully automatic, no human interaction is required.
  - Only possible if there are only finitely many executions (and finitely many values for the quantified variables in the formulas).
  - State space explosion: “finitely many” means “not too many”.
- Proving Verification Conditions
  - Logic formulas that are valid if and only if program is correct with respect to its specification.
  - Also possible if there are infinitely many excutions and infinitely many values for the quantified variables.
  - Many conditions can be automatically proved (automated reasoners); in general interaction with human is required (proof assistants).

General verification requires the proving of logic formulas.

Proofs

A proof is a structured argument that a formula is true.

- A tree whose nodes represent proof situations (states).



- Each proof situation consists of knowledge and a goal.
  - $K_1, \ldots, K_n \vdash G$
  - Knowledge $K_1, \ldots, K_n$: formulas assumed to be true.
  - Goal G: formula to be proved relative to knowledge.
- The root of the tree is the initial proof situation.
  - $K_1, \ldots, K_n$: axioms of mathematical background theories.
  - G: formula to be proved.

Proof Rules

A proof rules describes how a proof situation can be reduced to zero, one, or more "subsituations".

$$\frac{\ldots \vdash \ldots \qquad\qquad \ldots \vdash \ldots}{K_1, \ldots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
  - Zero subsituations: G has been proved, (sub)proof is closed.
  - One or more subsituations: G is proved, if all subgoals are proved.
- Top-down rules: focus on G.
  - G is decomposed into simpler goals $G_1, G_2, \ldots$
- Bottom-up rules: focus on $K_1, \ldots, K_n$.
  - Knowledge is extended to $K_1, \ldots, K_n, K_{n+1}$.

In each proof situation, we aim at showing that the goal is "apparently" true with respect to the given knowledge.

Conjunction $F_1 \land F_2$

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \land G_2} \qquad \frac{\ldots, K_1 \land K_2, K_1, K_2 \vdash G}{\ldots, K_1 \land K_2 \vdash G}$$

- Goal $G_1 \land G_2$.
  - Create two subsituations with goals G_1 and G_2.

    We have to show $G_1 \land G_2$.
    - *We show G_1: ...(proof continues with goal G_1)*
    - *We show G_2: ...(proof continues with goal G_2)*
- Knowledge $K_1 \land K_2$.
  - Create one subsituation with K_1 and K_2 in knowledge.

    We know $K_1 \land K_2$. We thus also know K_1 and K_2.
    (proof continues with current goal and additional knowledge K_1 and K_2)

Disjunction $F_1 \vee F_2$

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots, K_1 \vdash G \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$.
  - Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

    We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2. (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- Knowledge $K_1 \vee K_2$.
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \vee K_2$. We thus proceed by case distinction:
    - *Case K_1: ... (proof continues with current goal and additional knowledge K_1).*
    - *Case K_2: ... (proof continues with current goal and additional knowledge K_2).*

Implication $F_1 \Rightarrow F_2$

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2.
    (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$.
    - *We show K_1: ...(proof continues with goal K_1)*
    - *We know K_2: ...(proof continues with current goal and additional knowledge K_2).*

Equivalence $F_1 \Leftrightarrow F_2$

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots, (\neg)K_2 \vdash G}{\ldots, K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$.

    - *We show $G_1 \Rightarrow G_2$: ...(proof continues with goal $G_1 \Rightarrow G_2$)*
    - *We show $G_2 \Rightarrow G_1$: ...(proof continues with goal $G_2 \Rightarrow G_1$)*
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$.

    - *We show $(\neg)K_1$: ...(proof continues with goal $(\neg)K_1$)*
    - *We know $(\neg)K_2$: ...(proof continues with current goal and additional knowledge $(\neg)K_2$)*

Universal Quantification $\forall x : F$

$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G} \;(x_0 \text{ new for } K, G) \qquad \frac{\ldots, \forall x : K, K[T/x] \vdash G}{\ldots, \forall x : K \vdash G}$$

- Goal $\forall x : G$
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x_0/x]$.

    We have to show $\forall x : G$. Take arbitrary x_0.
    We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)
- Knowledge $\forall x : K$
  - Choose term T to create one subsituation with formula $K[T/x]$ added to the knowledge.

    We know $\forall x : K$ and thus also $K[T/x]$.
    (proof continues with current goal and additional knowledge $K[T/x]$)

Existential Quantification $\exists x : F$

$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \qquad \frac{\ldots, K[x_0/x] \vdash G}{\ldots, \exists x : K \vdash G} \; (x_0 \text{ new for } K, G)$$

- Goal $\exists x : G$
  - Choose term T to create one subsituation with goal $G[T/x]$.

    We have to show $\exists x : G$. It suffices to show $G[T/x]$.
    (proof continues with goal $G[T/x]$)
- Knowledge $\exists x : K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K[x_0/x]$.

    We know $\exists x : K$. Let x_0 be such that $K[x_0/x]$.
    (proof continues with current goal and additional knowledge $K[x_0/x]$)

Example

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

Take arbitrary y_0. We show

(c) $\exists x : P(x, y_0)$

From (1) we know for some x_0

(2) $\forall y : P(x_0, y)$

From (2) we know

(3) $P(x_0, y_0)$

From (3), we know (c). QED.

Example

We show

(a) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y)) \Rightarrow (\exists x, y : q(x,y))$

We assume

(1) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y))$

and show

(b) $\exists x, y : q(x,y)$

From (1), we know

(2) $\exists x : p(x)$
(3) $\forall x : p(x) \Rightarrow \exists y : q(x,y)$

From (2) we know for some x_0

(4) $p(x_0)$

...

Example (Contd)

...

From (3), we know

$$(5)\ p(x_0) \Rightarrow \exists y : q(x_0, y)$$

From (4) and (5), we know

$$(6)\ \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7)\ q(x_0, y_0)$$

From (7), we know (b). QED.

Indirect Proofs

$$\frac{K, \neg G \vdash \text{false}}{K \vdash G} \qquad \frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G} \qquad \frac{\ldots, \neg G \vdash \neg K}{\ldots, K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.
  - Prove that "false" is true.
  - Prove that a formula F is true and also prove that it is false.
  - Prove that some knowledge K is false, i.e. that $\neg K$ is true.
    - Switches goal G and knowledge K (negating both).

Sometimes simpler than a direct proof.

Example

We show

(a) $(\exists x : \forall y : P(x,y)) \Rightarrow (\forall y : \exists x : P(x,y))$

We assume

(1) $\exists x : \forall y : P(x,y)$

and show

(b) $\forall y : \exists x : P(x,y)$

We assume

(2) $\neg\forall y : \exists x : P(x,y)$

and show a contradiction.

...

Example

...

From (2), we know

(3) $\exists y : \forall x : \neg P(x, y)$

Let y_0 be such that

(4) $\forall x : \neg P(x, y_0)$

From (1) we know for some x_0

(5) $\forall y : P(x_0, y)$

From (5) we know

(6) $P(x_0, y_0)$

From (4), we know

(7) $\neg P(x_0, y_0)$

From (6) and (7), we have a contradiction. QED.

1. The Language of Logic
2. The RISC Algorithm Language
3. The Art of Proving
4. **The RISC ProofNavigator**

The RISC ProofNavigator

- An interactive proving assistant for program verification.
  - Research Institute for Symbolic Computation (RISC), 2005–.
    http://www.risc.jku.at/research/formal/software/ProofNavigator.
  - Development based on prior experience with PVS (SRI, 1993–).
  - Kernel and GUI implemented in Java.
  - Uses external SMT (satisfiability modulo theories) solver.
    - CVCL (Cooperating Validity Checker Lite) 2.0, CVC3, CVC4 1.4.
  - Runs under Linux (only); freely available as open source (GPL).
- A language for the definition of logical theories.
  - Based on a strongly typed higher-order logic (with subtypes).
  - Introduction of types, constants, functions, predicates.
- Computer support for the construction of proofs.
  - Commands for basic inference rules and combinations of such rules.
  - Applied interactively within a sequent calculus framework.
  - Top-down elaboration of proof trees.

Designed for simplicity of use; applied to non-trivial verifications.

Using the Software

For survey, see "Program Verification with the RISC ProofNavigator". For details, see "The RISC ProofNavigator: Tutorial and Manual".

- Develop a theory.
  - Text file with declarations of types, constants, functions, predicates.
  - Axioms (propositions assumed true) and formulas (to be proved).
- Load the theory.
  - File is read; declarations are parsed and type-checked.
  - Type-checking conditions are generated and proved.
- Prove the formulas in the theory.
  - Human-guided top-down elaboration of proof tree.
  - Steps are recorded for later replay of proof.
  - Proof status is recorded as "open" or "completed".
- Modify theory and repeat above steps.
  - Software maintains dependencies of declarations and proofs.
  - Proofs whose dependencies have changed are tagged as "untrusted".

Starting the Software

- Starting the software:

  ```
  module load ProofNavigator
  ```
  (users at RISC)

  ```
  ProofNavigator &
  ```

- Command line options:

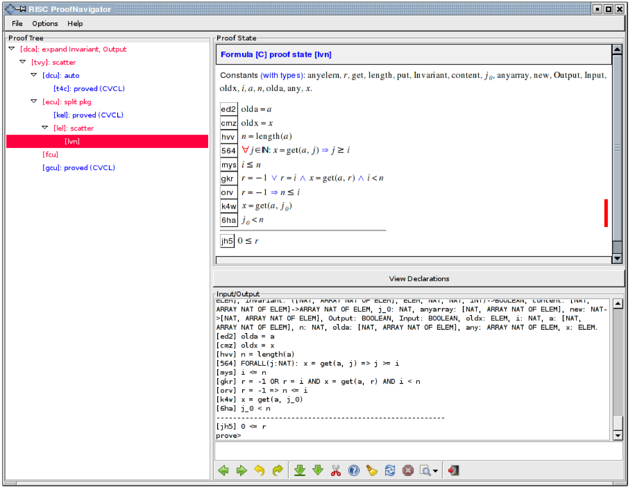
  ```
  Usage: ProofNavigator [OPTION]... [FILE]
  FILE:   name of file to be read on startup.
  OPTION: one of the following options:
          -n, --nogui:  use command line interface.
          -c, --context NAME: use subdir NAME to store context.
          --cvcl PATH: PATH refers to executable "cvcl".
          -s, --silent: omit startup message.
          -h, --help:   print this message.
  ```

- Repository stored in subdirectory of current working directory:

  ```
  ProofNavigator/
  ```

  - Option `-c` *dir* or command `newcontext "`*dir*`"`:
    - Switches to repository in directory *dir*.

The Graphical User Interface



A Theory

```
% switch repository to "sum"
newcontext "sum";

% the recursive definition of the sum from 0 to n
sum: NAT->NAT;
S1: AXIOM sum(0)=0;
S2: AXIOM FORALL(n:NAT): n>0 => sum(n)=n+sum(n-1);

% proof that explicit form is equivalent to recursive definition
S: FORMULA FORALL(n:NAT): sum(n) = (n+1)*n/2;
```

Declarations written with an external editor in a text file.

Proving a Formula

When the file is loaded, the declarations are pretty-printed:

$$\mathrm{sum} \in \mathbb{N} \to \mathbb{N}$$

$$\textbf{axiom}\ S1 \equiv \mathrm{sum}(0) = 0$$

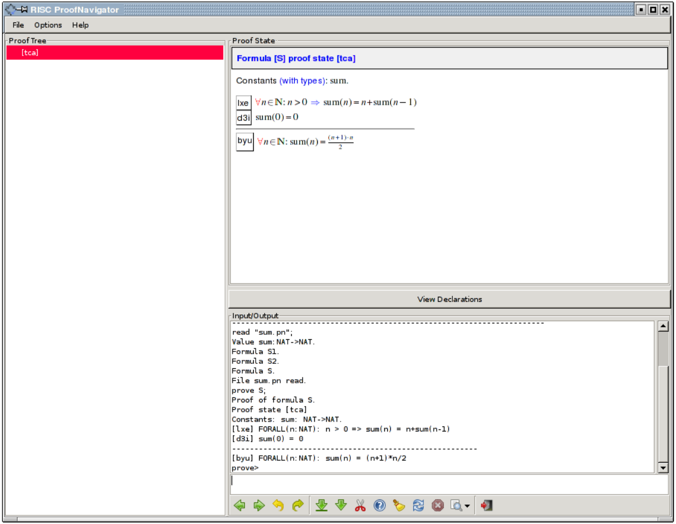
$$\textbf{axiom}\ S2 \equiv \forall n \in \mathbb{N} : n > 0 \Rightarrow \mathrm{sum}(n) = n + \mathrm{sum}(n-1)$$

$$S \equiv \forall n \in \mathbb{N} : \mathrm{sum}(n) = \frac{(n+1) \cdot n}{2}$$

The proof of a formula is started by the `prove` command.

Formula S
prove S: Construct Proof
proof S: Show Proof
formula S: Print Formula

Proving a Formula



Proving a Formula

- Proof of formula F is represented as a tree.
  - Each tree node denotes a proof state (*goal*).
    - Logical sequent:
      $A_1, A_2, \ldots \vdash B_1, B_2, \ldots$.
    - Interpretation:
      $(A_1 \wedge A_2 \wedge \ldots) \Rightarrow (B_1 \vee B_2 \vee \ldots)$
  - Initially single node $Axioms \vdash F$.

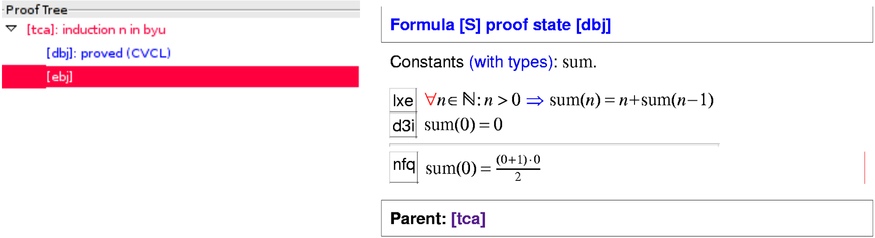
Constants: $x_0 \in S_0, \ldots$

$$
\begin{array}{ll}
[L_1] & A_1 \\
\ldots & \\
[L_n] & A_n \\
\hline
[L_{n+1}] & B_1 \\
\ldots & \\
[L_{n+m}] & B_m
\end{array}
$$

- The tree must be expanded to completion.
  - Every leaf must denote an obviously valid formula.
    - Some A_i is false or some B_j is true.
- A proof step consists of the application of a proving rule to a goal.
  - Either the goal is recognized as true.
  - Or the goal becomes the parent of a number of children (subgoals).

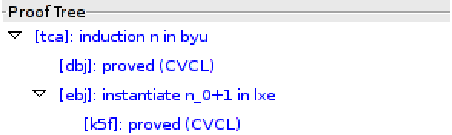
    The conjunction of the subgoals implies the parent goal.

An Open Proof Tree



Closed goals are indicated in blue; goals that are open (or have open subgoals) are indicated in red. The red bar denotes the “current” goal.

A Completed Proof Tree



The visual representation of the complete proof structure; by clicking on a node, the corresponding proof state is displayed.

Navigation Commands

Various buttons support navigation in a proof tree.

- : `prev`
  - Go to previous open state in proof tree.
- : `next`
  - Go to next open state in proof tree.
- : `undo`
  - Undo the proof command that was issued in the parent of the current state; this discards the whole proof tree rooted in the parent.
- : `redo`
  - Redo the proof command that was previously issued in the current state but later undone; this restores the discarded proof tree.

Single click on a node in the proof tree displays the corresponding state; double click makes this state the current one.

Proving Commands

The most important proving commands can be also triggered by buttons.

- (`scatter`)
  - Recursively applies decomposition rules to the current proof state and to all generated child states; attempts to close the generated states by the application of a validity checker.
- (`decompose`)
  - Like `scatter` but generates a single child state only (no branching).
- (`split`)
  - Splits current state into multiple children states by applying rule to current goal formula (or a selected formula).
- (`auto`)
  - Attempts to close current state by instantiation of quantified formulas.
- (`autostar`)
  - Attempts to close current state and its siblings by instantiation.

Automatic decomposition of proofs and closing of proof states.

Proving Commands

More commands can be selected from the menus.

- `assume`
  - Introduce a new assumption in the current state; generates a sibling state where this assumption has to be proved.
- `case`:
  - Split current state by a formula which is assumed as true in one child state and as false in the other.
- `expand`:
  - Expand the definitions of denoted constants, functions, or predicates.
- `lemma`:
  - Introduce another (previously proved) formula as new knowledge.
- `instantiate`:
  - Instantiate a universal assumption or an existential goal.
- `induction`:
  - Start an induction proof on a goal formula that is universally quantified over the natural numbers.

Here the creativity of the user is required!

Auxiliary Commands

Some buttons have no command counterparts.

- : `counterexample`
  - Generate a "counterexample" for the current proof state, i.e. an interpretation of the constants that refutes the current goal.
- 
  - Abort current prover activity (proof state simplification or counterexample generation).
- 
  - Show menu that lists all commands and their (optional) arguments.
- 
  - Simplify current state (if automatic simplification is switched off).

More facilities for proof control.

Proving Strategies

- Initially: semi-automatic proof decomposition.
  - `expand` expands constant, function, and predicate definitions.
  - `scatter` aggressively decomposes a proof into subproofs.
  - `decompose` simplifies a proof state without branching.
  - `induction` for proofs over the natural numbers.
- Later: critical hints given by user.
  - `assume` and `case` cut proof states by conditions.
  - `instantiate` provide specific formula instantiations.
- Finally: simple proof states are yielded that can be automatically closed by the validity checker.
  - `auto` and `autostar` may help to close formulas by the heuristic instantiation of quantified formulas.

Appropriate combination of semi-automatic proof decomposition, critical hints given by the user, and the application of a validity checker is crucial.