SMT SOLVING: COMBINING DECISION PROCEDURES

Course “Computational Logic”

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Lemmas on Demand

How to decide $T \models F$ for unquantified formula F and decidable theory T?

- So far: convert F into a disjunctive normal form $C_1 \vee \ldots \vee C_n$.
  - F is T-satisfiable if and only if some C_i is T-satisfiable.
  - Problem: the number n of clauses may be exponential in the size of F.
- Better: combine the decision procedure for T with a *SAT solver*.
  - The SAT solver is applied to the propositional skeleton $\overline{F}$.
    - Every atomic formula A in F is abstracted to a propositional variable $\overline{A}$.
    - If $\overline{F}$ is unsatisfiable, F is unsatisfiable and we are done.
    - Otherwise, the SAT solver produces a satisfying assignment represented by a conjunction $\overline{L_1} \wedge \ldots \wedge \overline{L_m}$ of propositional literals.
  - The decision procedure is applied to the T-formula $L_1 \wedge \ldots \wedge L_m$.
    - Propositional variable $\overline{L_i}$ is expanded into the atomic formula L_i it represents.
    - If the formula is satisfiable, F is satisfiable and we are done.
    - Otherwise, the decision procedure determines a minimal unsatisfiable subformula C of $L_1 \wedge \ldots \wedge L_m$ and we repeat the process with $F \wedge \neg C$.

Each formula $\neg C$ produced represents a "lemma" deduced from F.

Example

E-satisfiability of $F :\Leftrightarrow x = y \land ((y = z \land x \neq z) \lor x = z)$.

- First iteration:
  - Propositional skeleton: $a \land ((b \land \neg c) \lor c)$
  - Satisfying assignment: $a \land b \land \neg c$
  - Unsatisfiable concretization: $x = y \land y = z \land x \neq z$
  - Strengthened formula: $F \land \neg(x = y \land y = z \land x \neq z)$
- Second iteration:
  - Propositional Skeleton: $a \land ((b \land \neg c) \lor c) \land \neg(a \land b \land \neg c)$
  - Satisfying assignment: $a \land b \land c$
  - Satisfiable concretization: $x = y \land y = z \land x = z$

Formula F is E-satisfiable.

Algorithm

function SAT-DECIDE(F) ▷ decides T-satisfiability of F
    $\overline{F}$:= ABSTRACT(F)
    loop
        $(sat, \overline{Ls})$:= SAT($\overline{F}$) ▷ decides satisfiability of propositional skeleton of F
        if $\neg sat$ **return false**
        Ls := CONCRETIZE($\overline{Ls}$)
        (sat, C) := DECIDE(Ls) ▷ decides T-satisfiability of Ls
        if sat **return true**
        $\overline{F}$:= $\overline{F} \wedge$ ABSTRACT($\neg C$)
    end loop
end function

This basic approach can be further optimized, e.g., by integrating the interaction with the decision procedure directly into a DPLL-based SAT solver ("lazy encoding").

Combining Decision Procedures

How to decide a conjunction of atomic formulas with operations from different decidable theories such as LRA and EUF?

$$(y \geq z) \land (x - z \geq y) \land (z \geq 0) \land (f(f(x) - f(y)) \neq f(z))$$

- Theory combination problem: decide $T_1 \cup T_2 \models F$ for formula F and theories T_1, T_2.
  - Problem: even if T_1 and T_2 are decidable, $T_1 \cup T_2$ may be undecidable.
- Definition: a theory T is stably infinite, if for every quantifier-free formula F that is T-satisfiable, there exists an infinite domain that satisfies F.
  - Theories LRA and EUF are stably infinite.
  - The theory $\{x = a \lor x = b\}$ with constants a, b is not stably infinite (why?).
- Theorem: let T_1 and T_2 be theories for which the quantifier-free fragment is decidable and that have no common constants, functions, or predicates (except for "="). If T_1 and T_2 are stably infinite, then the quantifier-free fragment of $T_1 \cup T_2$ is decidable.

Under some constraints, the theory combination problem is indeed solvable.

Formula Purification

Before proceeding, let us tidy the formula a bit.

- Purification: ensure that every atom is from only one theory.
  - Repeatedly replace in the formula each "alien" subexpression E by a fresh variable v_E and add the constraint $v_E = E$.
  - The transformation preserves the satisfiability of the formula.
- Example: $(f(x,0) \geq z) \wedge (f(y,0) \leq z) \wedge (x \geq y) \wedge (y \leq x) \wedge (z - f(x,0) \geq 1)$.

$$(v_1 \geq z) \wedge (v_2 \leq z) \wedge (x \geq y) \wedge (y \leq x) \wedge (z - v_1 \geq 1) \wedge$$
$$v_1 = f(x, v_3) \wedge v_2 = f(y, v_3) \wedge v_3 = 0$$

A preparatory step for theory combination.

The Nelson-Oppen Method (for Convex Theories)

Greg Nelson and Derek C. Oppen (1979).

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function NelsonOppen(F)            ▷ decides T_1 ∪ ... ∪ T_n-satisfiability of literal conjunction F
    F_1, ..., F_n := Purify(F)                          ▷ for convex theories T_1, ..., T_n
    loop
        if ∃i. ¬Decide_i(F_i) return false                      ▷ decide T_i-satisfiability of F_i
        if ¬∃x, y, j. Inferred_j(x, y) return true
        choose x, y, j with Inferred_j(x, y)   ▷ infer variable equality x = y not present in theory T_j
        F_j := F_j ∪ {x = y}                       ▷ propagate inferred variable equality to T_j
    end loop
end function
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$\textsc{Inferred}_j(x, y) :\Leftrightarrow \exists i.\ (\textsc{Shared}(F_i, F_j, \{x, y\})) \land \textsc{Infer}_i(F_i, (x = y)) \land \neg\textsc{Infer}_j(F_j, (x = y)))$

- $\textsc{Shared}(F_i, F_j, \{x, y\})$: variables x, y are shared by formulas F_i and F_j.
- $\textsc{Infer}_i(F_i, (x = y))$: variable equality $(x = y)$ can be inferred from F_i in theory T_i.
    - $F_i \Rightarrow x = y$ is T_i-valid ($F_i \land \neg(x = y)$ is T_i-unsatisfiable).

The iterative propagation of inferred variable equalities between theories.

Example

$$(f(x,0) \geq z) \land (f(y,0) \leq z) \land (x \geq y) \land (y \geq x) \land (z - f(x,0) \geq 1)$$

- Purified formula:

$$(v_1 \geq z) \land (v_2 \leq z) \land (x \geq y) \land (y \geq x) \land (z - v_1 \geq 1) \land$$
$$v_1 = f(x, v_3) \land v_2 = f(y, v_3) \land v_3 = 0$$

- Equality propagation:

$F_1(LRA)$		$F_2(EUF)$
$v_1 \geq z$		$v_1 = f(x, v_3)$
$v_2 \leq z$		$v_2 = f(y, v_3)$
$x \geq y$		
$y \geq x$		
$z - v_1 \geq 1$		
$v_3 = 0$		
$\underline{x = y}$	$\rightarrow$	$x = y$
$v_1 = v_2$	$\leftarrow$	$\underline{v_1 = v_2}$
$\underline{v_1 = z}$		
unsat		

Example

$$(y \geq x) \wedge (x - z \geq y) \wedge (z \geq 0) \wedge (f(f(x) - f(y)) \neq f(z))$$

- Purified formula:

$$(y \geq x) \wedge (x - z \geq y) \wedge (z \geq 0) \wedge (f(v_1) \neq f(z)) \wedge$$
$$v_1 = v_2 - v_3 \wedge v_2 = f(x) \wedge v_3 = f(y)$$

- Equality propagation:

$F_1(LRA)$		$F_2(EUF)$
$y \geq x$		$f(v_1) \neq f(z)$
$x - z \geq y$		$v_2 = f(x)$
$z \geq 0$		$v_3 = f(y)$
$v_1 = v_2 - v_3$		
$z = 0$		
$\underline{x = y}$	$\longrightarrow$	$x = y$
$v_2 = v_3$	$\longleftarrow$	$\underline{v_2 = v_3}$
$v_1 = 0$		
$\underline{v_1 = z}$	$\longrightarrow$	$v_1 = z$
		unsat

Convex Theories

- Definition: Theory T is convex, if for every formula $F := L_1 \wedge \ldots \wedge L_m$ with literals $L_1, \ldots, L_m$ the following holds (for variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$):
  - If $T \models F \Rightarrow x_1 = y_1 \vee \ldots \vee x_n = y_n$, then $T \models (F \Rightarrow x_i = y_i)$ for some $i \in \{1, \ldots, n\}$.
    - If F implies in T a disjunction of equalities, it already implies one of these equalities.
    - Thus F cannot express "real" disjunctions and it suffices to infer plain equalities.
- Examples:
  - LRA is convex: a "real" disjunction corresponds to a finite set of $n \geq 2$ geometric points; however, by a conjunction of linear inequalities (which represent intersections of half-planes), we can only define point sets that are empty, singletons, or infinite.
  - EUF is convex: if $F \Rightarrow x_1 = y_1 \vee \ldots \vee x_n = y_n$, then $F \wedge x_1 \neq y_1 \wedge \ldots \wedge x_n \neq y_n$ is unsatisfiable. The congruence closure algorithm shows this by demonstrating for some i that $F \wedge x_i \neq y_i$ is unsatisfiable, i.e., that $F \Rightarrow x_i = y_i$ is valid.
  - LIA (linear integer arithmetic) is not convex: take $F :\Leftrightarrow 1 \leq x \wedge x \leq 2 \wedge y = 1 \wedge z = 2$; then F implies $x = y \vee x = z$ but neither $x = y$ nor $x = z$.

Non-Convex Theories

How to combine with a non-convex theory T_i?

- We may infer in T_i from formula F_i only a disjunction $x_1 = y_1 \vee \ldots \vee x_n = y_n$.
  - But not any equality $x_i = y_i$ of this disjunction.
- However, this disjunction can be made minimal (strongest).
  - Start with the disjunction of all possible variable equalities.
  - If it cannot be inferred, no smaller disjunction can be inferred either.
  - Otherwise, strip every $x_i = y_i$ if this keeps the disjunction inferred.
- For each remaining $x_i = y_i$, recursively call NelsonOppen$(F \wedge x_i = y_i)$.
  - Return "true" if any call returns "true" and "false", otherwise.

Thus the Nelson-Oppen method is also applicable to non-convex theories (but with generally much greater complexity).