Matching in Quantitative Equational Theories

Seminar: Automated Reasoning & Formal Methods II

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Contents

(Quantitative) Equational Theories

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- “Classical” setting:
  Equations $s \approx t$ between terms $s, t \in T(\Omega, X)$.

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  Indexed equations $s \approx_\varepsilon t$ for $\varepsilon \in \mathbb{Q}_{\geqslant 0}$

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- Quantitative setting (Mardare-Plotkin-Panangaden 2016):
  Indexed equations $s \approx_\varepsilon t$ for $\varepsilon \in \mathbb{Q}_{\geqslant 0}$
  - Intuition: "s is within ε of t"
    $\rightsquigarrow$ think of metric spaces: $d(s,t) \leqslant \varepsilon$
  - $s \approx_0 t$ corresponds to $s \approx t$
  - If $s \approx_\varepsilon t$, then $s \approx_\delta t$ for any $\delta > \varepsilon$
  - Transitivity has to be replaced by the triangle inequality:
    $r \approx_\varepsilon s$ and $s \approx_\delta t$ imply $r \approx_{\varepsilon+\delta} t$.

Inference rules for equational logic

$$\frac{}{E \vdash s \approx t} \text{(Ax.)} \quad \text{for } s \approx t \in E$$

$$\frac{}{E \vdash t \approx t} \text{(Refl.)} \qquad \frac{E \vdash s \approx t}{E \vdash t \approx s} \text{(Symm.)}$$

$$\frac{E \vdash s \approx t}{E \vdash s\sigma \approx t\sigma} \text{(Subst.)} \qquad \frac{E \vdash s \approx r \qquad E \vdash r \approx t}{E \vdash s \approx t} \text{(Trans.)}$$

$$\frac{E \vdash s_1 \approx t_1, \ldots, E \vdash s_n \approx t_n}{E \vdash f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)} \text{(Cong.)} \quad \text{for } f : n \in \Omega$$

Inference rules for (unconditional) quantitative equational logic

$$\frac{}{E \vdash s \approx_\varepsilon t}\ (\text{Ax.}) \quad \text{for } s \approx_\varepsilon t \in E$$

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$$\frac{E \vdash s \approx_\varepsilon t}{E \vdash s\sigma \approx_\varepsilon t\sigma}\ (\text{Subst.}) \qquad \frac{E \vdash s \approx_\varepsilon r \qquad E \vdash r \approx_\delta t}{E \vdash s \approx_{\varepsilon+\delta} t}\ (\text{Triang.})$$

$$\frac{E \vdash s_1 \approx_\varepsilon t_1, \ \dots\ , E \vdash s_n \approx_\varepsilon t_n}{E \vdash f(s_1, \dots, s_n) \approx_\varepsilon f(t_1, \dots, t_n)}\ (\text{NExp.}) \quad \text{for } f : n \in \Omega$$

$$\frac{E \vdash s \approx_\varepsilon t}{E \vdash s \approx_{\varepsilon+\delta} t}\ (\text{Max.}) \qquad \frac{E \vdash s \approx_{\varepsilon'} t \mid \varepsilon' > \varepsilon}{E \vdash s \approx_\varepsilon t}\ (\text{Cont.})$$

Semantics for equational theories

Definition

- *Ω-algebra*:
  $\mathcal{A} = (D_\mathcal{A}, \{f_\mathcal{A}\}_{f \in \Omega})$, where $D_\mathcal{A}$ is a nonempty set and for each $f : n \in \Omega$, $f_\mathcal{A}$ is a function $D_\mathcal{A}^n \to D_\mathcal{A}$.

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- *Model*:
  $\mathcal{A} \models E$ if
  $$\langle s \rangle^\alpha_\mathcal{A} = \langle t \rangle^\alpha_\mathcal{A}$$
  for every equation $s \approx t \in E$ and every variable assignment α.

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Theorem (Birkhoff 1935)

$E \models s \approx t \iff E \vdash s \approx t$.

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- *Quantitative Ω-algebra*:
  $\mathcal{A} = (D_{\mathcal{A}}, d_{\mathcal{A}}, \{f_{\mathcal{A}}\}_{f \in \Omega})$, where $(D_{\mathcal{A}}, d_{\mathcal{A}})$ is an extended metric space and each $f_{\mathcal{A}}$ is a *non-expansive* function $D_{\mathcal{A}}^n \to D_{\mathcal{A}}$.

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- *Quantitative model*:
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$$d_{\mathcal{A}}(\langle s \rangle^{\alpha}_{\mathcal{A}}, \langle t \rangle^{\alpha}_{\mathcal{A}}) \leqslant \varepsilon$$

  for every indexed equation $s \approx_{\varepsilon} t \in E$ and every variable assignment α.

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- *Semantic consequence*: As in the classical case.

Theorem (Mardare-Panangaden-Plotkin 2016)

$E \models s \approx_\varepsilon t \iff E \vdash s \approx_\varepsilon t.$

Matching Problems

Let $s, t \in T(\Omega, X)$ be terms, E a set of equations.

Matching problem: $s \lesssim^?_E t$

Find a substitution σ such that $E \vdash s\sigma \approx t$.

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Let $s, t \in T(\Omega, X)$ be terms, E a set of indexed equations, $\varepsilon \in \mathbb{Q}_{\geqslant 0}$

Quantitative matching problems

- $s \lesssim_\varepsilon^? t$: Find a substitution σ such that $E \vdash s\sigma \approx_\varepsilon t$.
- $s \lesssim_?^? t$: Find the least $\delta \in \mathbb{Q}_{\geqslant 0}$ such that there exists a substitution σ satisfying $E \vdash s\sigma \approx_\delta t$.

For this talk: Focus on the first problem ("fixed-range matching").

Assumptions

- **Running assumption:** E is finite.
  We may assume that all equations from E have indices in $\mathbb{N}_0$.
- **Notation:** Write $E = E_0 \sqcup E_+$, where

$$E_0 = \{s \approx_\varepsilon t \in E \mid \varepsilon = 0\} \qquad (\text{“crisp part”}),$$
$$E_+ = \{s \approx_\varepsilon t \in E \mid \varepsilon > 0\} \qquad (\text{“quantitative part”}).$$

- Note: E_0 can be viewed as a classical (non-quantitative) equational theory.
- Assume that E_0 has finitary unification type and that a unification algorithm for E_0 is given.

First steps towards a solution

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- Problem: $B_{\varepsilon}(t)$ need not be finite!

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Examples

1. $E = \{f(x) \approx_1 g(x, y)\}$, $t = f(a)$, where $a \in \Omega$ is a constant.
   Then: $E \vdash f(a) \approx_1 g(a, y)$ by (Subst.)
   $\Rightarrow$ every instance of $g(a, y)$ is in $B_1(f(a))$ by (Subst.)
   $\Rightarrow B_1(t)$ is infinite.
2. $E = \{x \approx_0 f(x)\}$, $t = a$ (constant).
   By (Triang.), $f^n(a) \in B_0(a)$ for every n
   $\Rightarrow B_0(t)$ is infinite.

First steps towards a solution

To guarantee finiteness, compute a finite representation $\mathcal{R}_\varepsilon(t)$ of $B_\varepsilon(t)$ that contains:

- non-ground terms from $B_\varepsilon(t)$, but not all of their instances
- representatives of terms up to E_0

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Examples, revisited

1. $E = \{f(x) \approx_1 g(x, y)\}$, $t = f(a)$, where $a \in \Omega$ is a constant. $B_1(t)$ is infinite.

   $\rightsquigarrow$ take $\mathcal{R}_1(t) = \{f(a), g(a, y)\}$ instead!
2. $E = \{x \approx_0 f(x)\}$, $t = a$ (constant). $B_0(t)$ is infinite.

   $\rightsquigarrow$ take $\mathcal{R}_0(a) = \{a\}$ instead!

Compact representation of the ball

Definition

Define $\mathcal{R}_\varepsilon(x) := \{x\}$ if x is a variable, and otherwise, set

$$\mathcal{R}_\varepsilon(t) = \{t\} \cup \bigcup_{\substack{\zeta\in\mathbb{N},\\ 0<\zeta\leqslant\varepsilon,\\ t=f(t_1,\ldots,t_n),\\ s_i\in\mathcal{R}_\zeta(t_i)}} \mathcal{R}_{\varepsilon-\zeta}(f(s_1,\ldots,s_n)) \cup \bigcup_{\substack{l\approxeq_\delta r\in E_+,\\ \delta\leqslant\varepsilon,\\ \sigma\in\mathrm{mcu}_{E_0}(l,t)}} \mathcal{R}_{\varepsilon-\delta}(r\sigma),$$

where

- $l \approxeq_\delta r$ is a fresh, unoriented variant of an equation in E_+
- $\mathrm{mcu}_{E_0}(l, t)$ is a minimal complete set of E_0-unifiers of l and t

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Remarks

- $\mathcal{R}_\varepsilon(t)$ is finite and defined uniquely up to renaming variables.
- $\mathcal{R}_0(t) = \{t\}$
- If $\varepsilon \leqslant \delta$, then $\mathcal{R}_\varepsilon(t) \subseteq \mathcal{R}_\delta(t)$

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   Solve $h(x) \lesssim_2 g(b)$.

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2. $E = \{f(x,y) \approx_0 f(y,x), f(x,y) \approx_1 g(x,y)\}$.
   Solve $f(g(b,z),z) \lesssim_1 f(f(a,b),a)$.

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   Solve $f(g(b,z),z) \lesssim_1 f(f(a,b),a)$.

$$\mathcal{R}_1(f(f(a,b),a)) = \{f(f(a,b),a), f(g(a,b),a), f(g(b,a),a), g(a,f(a,b)), g(f(a,b),a)\}$$

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First results

Proposition

If $E = E_+$ is regular and t is a ground term, then $\mathcal{R}_\varepsilon(t) = B_\varepsilon(t)$.

In particular: $E \vdash s\sigma \approx_\varepsilon t \iff s\sigma \in B_\varepsilon(t) \iff s\sigma \in \mathcal{R}_\varepsilon(t)$.

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Quantitative matching algorithm 1:

Input: Regular $E = E_+$; E-matching problem $s \lesssim_\varepsilon t$ with t ground.
Output: A complete set of solutions.

1. $S \leftarrow \emptyset$
2. Compute $\mathcal{R}_\varepsilon(t)$
3. For each $u \in \mathcal{R}_\varepsilon(t)$:
4. $\quad$ $S \leftarrow S \cup \{\text{syntactic matchers of } s \text{ to } u\}$
5. Return S

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Corollary

The above algorithm is sound and complete.

Relaxing the assumptions: non-regular E_+

Consider the case where $E = E_+$ need not be regular.

Example

$E = \{f(x) \approx_1 g(x, y)\}$; solve $g(x, b) \lesssim_1^? f(a)$.
$\mathcal{R}_1(f(a)) = \{f(a), g(a, y)\}$.

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Desired Proposition

Assume that $E = E_+$ and s, t are terms, t ground. Then
$E \vdash s \approx_\varepsilon t \iff s = u\tau$ for some $u \in \mathcal{R}_\varepsilon(t)$ and some substitution τ.

Assuming the proposition, we could replace syntactic matchers by syntactic unifiers in the algorithm!

Relaxing the assumptions: non-empty E_0

Now, consider non-empty E_0.

Recall: $\mathcal{R}_\varepsilon(t)$ represents terms up to equality modulo E_0.

By assumption, we know how to solve unification in E_0. Can we just replace syntactic unification by unification modulo E_0 to solve the matching problem in E?

Example 1

$E = \{f(a, x) \approx_1 g(x, a), a \approx_0 b\}$.
Solve $f(b, y) \lesssim_1 g(c, b)$.

$\mathcal{R}_1(g(c, b)) = \{g(c, b), f(a, c)\}$.

$\sigma = \{y \mapsto c\}$ is an E_0-unifier of $f(b, y)$ and $f(a, c)$.

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Example 2

$E = \{f(a, x) \approx_0 g(x), a \approx_1 b\}$; solve $f(b, y) \lesssim_1 g(a)$.
Then $\mathcal{R}_1(g(a)) = \{g(a), g(b)\}$.

There is no E_0-unifier!

To find the solution, one would also need to compute $\tilde{\mathcal{R}}_1(f(b, y)) = \{f(b, y), f(a, y)\}$

Outlook

Possible future work:

- Results for matching in the more general cases: non-regular E_+, non-empty E_0
- Different (e.g., rule-based) approaches for quantitative matching
- Matching in conditional theories
- Other equational problems in the quantitative setting (unification, anti-unification)
- Different versions of quantitative equational reasoning, e.g. Gavazzo-Di Florio (2023)

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Inference rules for equational logic

(Refl) $\emptyset \vdash s \approx s \in \mathcal{U}$

(Symm) $\{s \approx t\} \vdash t \approx s \in \mathcal{U}$

(Trans) $\{s \approx t, t \approx u\} \vdash s \approx u \in \mathcal{U}$

(Cong) $\{s_1 \approx t_1, \ldots, s_n \approx t_n\} \vdash f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n) \in \mathcal{U}$ for any $f : n \in \Omega$.

(Subst) If $\Gamma \vdash \Delta \in \mathcal{U}$, then $\Gamma\sigma \vdash \Delta\sigma \in \mathcal{U}$ for any substitution σ

(Assum) If $s \approx t \in E$, then $E \vdash s \approx t \in \mathcal{U}$

(Cut) If $\Gamma \vdash \Delta \in \mathcal{U}$ and $\Delta \vdash \Theta \in \mathcal{U}$, then $\Gamma \vdash \Theta \in \mathcal{U}$.

Inference rules for quantitative equational logic

(Refl) $\emptyset \vdash s \approx_\varepsilon s \in \mathcal{U}$

(Symm) $\{s \approx_\varepsilon t\} \vdash t \approx_\varepsilon s \in \mathcal{U}$

(Triang) $\{s \approx_\varepsilon t, t \approx_\delta u\} \vdash s \approx_{\varepsilon+\delta} u \in \mathcal{U}$

(Max) $\{s \approx_\varepsilon t\} \vdash s \approx_\delta t \in \mathcal{U}$ for every $\delta \geqslant \varepsilon$

(NExp) $\{s_1 \approx_\varepsilon t_1, \ldots, s_n \approx_\varepsilon t_n\} \vdash f(s_1, \ldots, s_n) \approx_\varepsilon f(t_1, \ldots, t_n) \in \mathcal{U}$ for any $f : n \in \Omega$.

(Cont) $\{s \approx_{\varepsilon'} t \mid \varepsilon' > \varepsilon\} \vdash s \approx_\varepsilon t$.

(Subst) If $\Gamma \vdash \Delta \in \mathcal{U}$, then $\Gamma\sigma \vdash \Delta\sigma \in \mathcal{U}$ for any substitution σ

(Assum) If $s \approx_\varepsilon t \in E$, then $E \vdash s \approx_\varepsilon t \in \mathcal{U}$

(Cut) If $\Gamma \vdash \Delta \in \mathcal{U}$ and $\Delta \vdash \Theta \in \mathcal{U}$, then $\Gamma \vdash \Theta \in \mathcal{U}$.