Pattern formation in biological systems

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Main messages from the Lecture 1:

- some solutions of nonlinear partial differential equations form patterns;
- to construct the pattern-forming solutions we have to solve resonance conditions in integers;
- resonance conditions have the form:

$$\begin{cases}\omega_1 + \omega_2 = \omega_3, \\ \vec{k}_1 + \vec{k}_2 = \vec{k}_3, \end{cases} \quad \text{or} \quad \begin{cases}\omega_1 + \omega_2 = \omega_3 + \omega_4, \\ \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4, \end{cases} \tag{1}$$

  with $\vec{k}_j = (m_j, n_j), \quad m_j, n_j \in \mathbb{Z}$;
- function ω is called dispersion function; $\vec{k}$ is called wavevector; m, n are called wavenumbers or indexes of Fourier harmonics;
- the most frequently met dispersion functions depend on the modulus of the wave vector $k = |\vec{k}| = (m^2 + n^2)^{1/2}$, for instance: $\omega \sim (m^2 + n^2)^{1/4}, \quad (m^2 + n^2)^{3/4}, \quad (m^2 + n^2)^{-1/2}, \ldots.$

Analytical solution

General analytical solution does not exist, the problem is equivalent to the Hilbert's Tenth problem, it is proven to be unsolvable $\Rightarrow$:(

Brute-force numerical solution

$$\begin{cases} \omega_1 + \omega_2 = \omega_3 + \omega_4, \quad \omega = (m^2 + n^2)^{1/4}, \quad m, n \leq 10^3 \\ \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4, \end{cases} \tag{2}$$

8 integer variables of order 10^{32} $\Rightarrow$:(
2005, Warwick Mathematical School: computations for $m, n \leq 128$ took **3 DAYS** with Pentium 4.

q-class decomposition

This is specially developed method, which gives a huge computational advantage.
2007, RISC: computations for $m, n \leq 10^3$ took **15 MINUTES** with Pentium 3.

The idea of q-class decomposition

Main idea is based on two simple facts.

Fact 1: Main theorem of arithmetics

Every integer has **unique** presentation as a product of different primes in some powers, for example:

$$420 = 2^2 \cdot 3 \cdot 5 \cdot 7 = 3 \cdot 7 \cdot 2^2 \cdot 5 = 3 \cdot 4 \cdot 5 \cdot 7$$

Control question: What is wrong with the last presentation?

Fact 2: Linear independence of algebraic numbers

For rational numbers a, b, c

$$a\sqrt{3} + b\sqrt{5} = c \quad \Rightarrow \; a = b = c = 0.$$

This statement can be generalized to any finite number of terms and different roots (see next slide)

Example

$$a\sqrt{3} + b\sqrt{5} + c\sqrt{20} + d\sqrt{3} + e\sqrt{27} = g \quad \Rightarrow \tag{3}$$

$$\begin{cases} a + d + 3e = 0, \\ b + 2c = 0, \\ g = 0 \end{cases} \tag{4}$$

Control question: Add following terms into (3):

$$f \cdot \sqrt[4]{3}, \; s \cdot \sqrt[4]{48}$$

and write additional linear equation into (4).

Definition of q-class

For a given

$$c \in \mathbb{Z}, c \neq 0, 1, -1$$

consider the set of algebraic numbers

$$R_c = \pm k^{1/c}, k \in \mathbb{N}.$$

Any such number k_c has a unique representation

$$k_c = \gamma q^{1/c}, \gamma \in \mathbb{Z} \quad \text{with} \quad q = p_1^{e_1} p_2^{e_2} ... p_n^{e_n},$$

and $p_1, ...p_n$ being all different primes and the powers $e_1, ...e_n \in \mathbb{N}$ are all smaller than c.
The set of numbers from R_c having the same q is called **q-class** Cl_q.
The number q is called **class index**.

Principal example: $\omega = 1/\sqrt{m^2+n^2}$

For any wavevector $(m,n): \ m,n \in \mathbb{N}$ define q-class Cl_q as following

$$|(m,n)| = \sqrt{m^2+n^2} = \gamma\sqrt{q}, \quad \gamma,\ q \in \mathbb{N} \tag{5}$$

with square-free q.

Control question: compute q-class of the the wavevector $(1,7)$.

Lemma 1. If three vectors $(m_i,n_i),\ i=1,2,3$ give a solution of

$$1/\sqrt{m_1^2+n_1^2} + 1/\sqrt{m_2^2+n_2^2} = 1/\sqrt{m_3^2+n_3^2} \tag{6}$$

then they belong to the same class:

$$\exists q \in \mathbb{N}: \quad (m_i,n_i) \in Cl_q, \quad i = 1,2,3.$$

◀ Statement of this Lemma is equivalent to irrationality of the square root of a product of different primes. ▮

Properties of classes

Lemma 2. The following properties of classes keep true

1) $Cl_{q_1} \cap Cl_{q_2} \neq \{0\}$ **iff** $Cl_{q_1} = Cl_{q_2}$ (intersection of two classes is not empty **iff** these classes coincide);

2) card $\{Cl_q\} = \infty$ (there exists an infinite number of classes);

3) $Cl_q \neq \emptyset \quad \Rightarrow \quad$ card $Cl_q = \infty$ (every non-empty class consists of infinite number of elements);

4) $Cl_q \neq \emptyset \Leftrightarrow q = p_1 p_2 \cdots p_n$ where $p_i \in \mathbb{P}$ are different primes such that $p_i \not\equiv 3 \pmod 4$;

5) $Cl_q \neq \emptyset$ & $q > 1 \;\Rightarrow\; q = a^2 + b^2$ for some integers $a, b \in \mathbb{Z}$ (in each non-empty class, the minimal element has norm q);

6)
$Cl_1 = \{m, n : m^2 + n^2 = k^2, m = a^2 - b^2, n = 2ab, k = a^2 + b^2, a > b\}$
(elements of class Cl_1 can be parameterized by two natural parameters.)

◀ Indeed,
1) follows from definition,
2) follows from the fact that the set of primes is infinite,
3) is due to the fact that vectors (m, n) and (sm, sn) have the same index for arbitrary $s \in \mathbb{N}$.
In order to prove 4) and 5) one should use Lagrange theorem:
Natural number N, $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ *can be presented as a sum of two squares* **iff** $\{p_i \in 4\mathbb{N} + 3 \ \Rightarrow \ \alpha_i \in 2\mathbb{N}\}$.
Last property 6) is known parametrization by Pythagorean numbers. ∎

Example of 4-term resonance

Dispersion function: $\omega = (m^2 + n^2)^{1/4}$

Construction of q-classes:
for any wavevector (m, n) : $m, n \in \mathbb{N}$ define q-class Cl_q as following

$$\omega = (m^2 + n^2)^{1/4} = \gamma q^{1/4}, \quad \gamma,\ q \in \mathbb{N} \tag{7}$$

with q free of 4th powers.

Solutions from two classes are possible

Case 1: Solutions belong to one class Cl_q:

$$\gamma_1 \sqrt[4]{q} + \gamma_2 \sqrt[4]{q} = \gamma_3 \sqrt[4]{q} + \gamma_4 \sqrt[4]{q} \tag{8}$$

with $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{N}$ and q is class index.
Case 2: Solutions belong to two different classes Cl_{q_1}, Cl_{q_2}:

$$\gamma_1 \sqrt[4]{q_1} + \gamma_2 \sqrt[4]{q_2} = \gamma_1 \sqrt[4]{q_1} + \gamma_2 \sqrt[4]{q_2} \tag{9}$$

Example of 3-term resonances with empty q-classes

Dispersion function: $\omega = (m^2 + n^2)^{3/4}$

Construction of q-classes:
for any wavevector (m, n) : $m, n \in \mathbb{N}$ define q-class Cl_q as following

$$(m^2 + n^2) = \gamma^4 q, \quad \gamma,\ q \in \mathbb{N} \tag{10}$$

with q free of 4th powers.

$$\omega_1 + \omega_2 = \omega_3 \;\Rightarrow\; \gamma_1^3 q^{3/4} + \gamma_2^3 q^{3/4} = \gamma_3^3 q^{3/4} \;\Rightarrow\; \gamma_1^3 + \gamma_2^3 = \gamma_3^3$$

This is particular case of the Last Fermat Theorem, i.e. there is no integer solutions for arbitrary q: all q-classes are empty.

Control questions:

- Describe Cl_1.
- Show that Lemma 1 gives only **necessary** condition of the existence of a solution (construct an example).
- Take 1) $\omega = \sqrt{m^2 + n^2}$ and 2) $\omega = \alpha\sqrt{m^2 + n^2} + \beta$ with integer α and β. What will be changed in Lemma 1?
- What is the main difference between 3- and 4-term resonances?

Control questions:

- Show that Lemma 1 gives only **necessary** condition of the existence of a solution (construct an example).
- Take 1) $\omega = \sqrt{m^2 + n^2}$ and 2) $\omega = \sqrt{m^2 + n^2} + \beta$ with integer $\beta \neq 0$. What will be changed in the class construction and in the class properties?
  The answer: construction is the same but in the case 2) only Cl_1 can be non-empty.
- What is the main difference between 3- and 4-term resonances?
  The answer: in 3-term resonances we always have one *q*-class, in 4-term resonances two classes are possible.

What to read on the implementation of the q-class decomposition (available at RISC publication list):

- 2006, E.K., A. Kartashov (C++, 1 class)
- 2007, E.K., A. Kartashov (C++, 2 classes)
- 2007, E.K., G. Mayrhofer (Mathematica)
- 2008, E.K., C. Raab, Ch. Feurer, G. Mayrhofer, W. Schreiner (Mathematica + on-line implementation)