THE MATHEMATICAL THEORY OF RELATIONAL DATABASES

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The goal

In order to see the actual use of the following considerations, we take an actual SQL database as a model.

Figure 1: DDL script

```
BEGIN TRANSACTION;
CREATE TABLE IF NOT EXISTS ‘s‘ (
‘Field1‘ INTEGER,
‘Field2‘ INTEGER
);
INSERT INTO ‘s‘ VALUES (0,0);
INSERT INTO ‘s‘ VALUES (0,1);
INSERT INTO ‘s‘ VALUES (1,0);
CREATE TABLE IF NOT EXISTS ‘r‘ (
‘Field1‘ INTEGER,
‘Field2‘ INTEGER,
‘Field3‘ INTEGER
);
INSERT INTO ‘r‘ VALUES (1,1,0);
INSERT INTO ‘r‘ VALUES (0,1,0);
INSERT INTO ‘r‘ VALUES (0,0,0);
INSERT INTO ‘r‘ VALUES (1,1,1);
COMMIT;
```

The goal

Later on, we check if our algebraic approach leads to the same result as the query below.

Figure 2: Query

```
SELECT distinct *
FROM
(SELECT r.Field1 as 'a', r.Field3 as 'b'
FROM r WHERE r.Field2 = 1) as 't'
INNER JOIN s
ON s.Field1 = t.a;
```

Table 1: Output

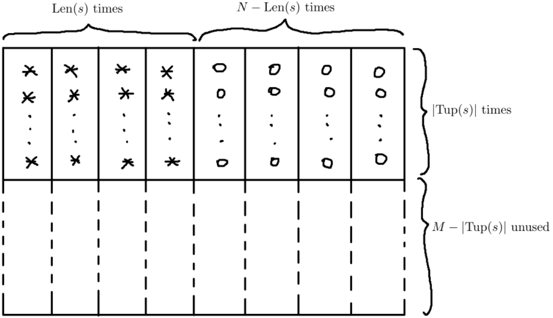
1	0	1	0
0	0	0	0
0	0	0	1
1	1	1	0

Algebra

- Theoretical foundation for the implementation in RISCAL
- The algebra we construct consists of ...
  - a domain `Relation`
  - and operations with signatures of the form $* \to \texttt{Relation}$.
- For each operation we also define suitable preconditions.

Domain

- The domain will be parametrized by constants $M, N \in \mathbb{N}$ where M is the **maximum cardinality of relations** and N the **maximum length of tuples**.
- Let `Row` be the set of all functions $\{0, \ldots, N-1\} \to \{0, 1\}$.
- The domain `Relation` consists of all $\langle n, r \rangle \in \{0, \ldots, N-1\} \times \mathcal{P}(\texttt{Row})$ that satisfy
  - $|r| \leq M$
  - and $\forall t \in r, i \in \{n, \ldots, N-1\} : t[i] = 0.$ Note that $\{n, \ldots, N-1\} = \emptyset$ for $n > N-1$.
- Notation: $\mathrm{Len}(s) := n$ and $\mathrm{Tup}(s) = r$ for $s \in \texttt{Relation}$
- Note: As a means of abstraction the "cells" of a "table" contain only bit values.


Len(s) times
N − Len(s) times
|Tup(s)| times
M − |Tup(s)| unused

Operations

- The actual operations we will construct are `cartesian`, `select`, `project`, `join`, `union`, `intersect` and `minus`.
- We will also have a `concat` function, which is not an actual operation. It will help to introduce `cartesian`.

concat

- Description: The function concatenates two rows.
- Signature: $\texttt{Row} \times \texttt{Row} \times \{0, \ldots, N\} \times \{0, \ldots, N\} \to \texttt{Row}$

Definition

$$\texttt{concat}(t_1, t_2, n_1, n_2) := n \mapsto \begin{cases} t_1(n), & \text{if } n < n_1 \\ t_2(n - n_1), & \text{if } n_1 \leq n < n_1 + n_2 \\ 0, & \text{else} \end{cases}$$

- Precondition: The parameters n_1, n_2 denote the actual length of a row. Therefore we need to ensure that $n_1 + n_2 \leq N$.

RISCAL

`cartesian`

- Description: The function constructs the cartesian product of two relations.
- Signature: $\texttt{Relation} \times \texttt{Relation} \rightarrow \texttt{Relation}$

Definition

$\texttt{cartesian}(r_1, r_2) = r :\Leftrightarrow$
$\mathsf{Tup}(r) = \{\mathrm{concat}(t_1, t_2) : t_1 \in \mathsf{Tup}(r_1), t_2 \in \mathsf{Tup}(r_2)\}$ and
$\mathsf{Len}(r) = \mathsf{Len}(r_1) + \mathsf{Len}(r_2)$.

- Precondition: The cartesian product is a relation where the rows have the length $\mathsf{Len}(r_1) + \mathsf{Len}(r_2)$, therefore we need to ensure that $\mathsf{Len}(r_1) + \mathsf{Len}(r_2) \leq N$. The maximum cardinality of this relation is $|\,\mathsf{Tup}(r_1)| \cdot |\,\mathsf{Tup}\, r_2|$, therefore we need to ensure that $|\,\mathsf{Tup}(r_1)| \cdot |\,\mathsf{Tup}\, r_2| \leq M$.

RISCAL

select

- Description: The function filters out rows whose columns have a certain value.
- Signature:
  $\mathtt{Relation} \times \{0, \ldots, N-1\} \times \{0, 1\} \to \mathtt{Relation}$

Definition

$\mathtt{select}(r, a, e) := \langle \mathsf{Len}(r), \{t \in r : t(a) = e\} \rangle$

- Precondition: We need to ensure that the column indicator a is not greater or equal the length of the rows of r, i.e. we need the precondition $a < \mathsf{Len}(r)$.

RISCAL

`project`

- Description: The function can be used to create a new relation consisting of a rearrangement of certain columns of the previous relation.
- Signature:
  $\mathtt{Relation} \times \{0, \ldots, N\}^{\{0,\ldots,N-1\}} \to \mathtt{Relation}$

Definition

$\mathtt{project}(r, c) = s :\Leftrightarrow \mathrm{Len}(s) = |\{i \in \{0, \ldots, N-1\} : c(i) \neq N\}|$
and
$\forall t_r \in \mathrm{Tup}(r) \exists t_s \in \mathrm{Tup}(s) \forall i \in \{0, \ldots, \mathrm{Len}(s) - 1\} : t_s(i) = t_r(c(i))$

- Precondition: The parameter c should denote a choice of valid column indices in a certain order. A convenient precondition is given by

$$\exists i \in \{0, \ldots, N-1\} \forall j \in \{0, \ldots, N-1\} :$$

$$(j > i \Rightarrow c(i) = N) \wedge (j \leq i \Rightarrow c(i) < \mathrm{Len}(r))$$

RISCAL

join

- Description: The function filters out all rows in the cartesian product that have matching values in two certain columns.
- Signature: $\texttt{Relation}^2 \times \{0, \dots, N-1\}^2 \to \texttt{Relation}$

Definition

$\texttt{join}(r_1, r_2, n_1, n_2) = s :\Leftrightarrow \mathsf{Len}(s) = \mathsf{Len}(r_1) + \mathsf{Len}(r_2)$ and $Tup(s) = \{\texttt{concat}(t_1, t_2, \mathsf{Len}(r_1), \mathsf{Len}(r_2)) : t_1 \in \mathsf{Tup}(r_1), t_2 \in \mathsf{Tup}(r_2), t_1(n_1) = t_2(n_2)\}$

- Precondition: Firstly n_1, n_2 need to denote valid columns, therefore we need a precondition $n_1 < \mathsf{Len}(r_1), n_2 < \mathsf{Len}(r_2)$. Secondly, just as in the cartesian product we need the preconditions $\mathsf{Len}(r_1) + \mathsf{Len}(r_2) \leq N$ and $|\mathsf{Tup}(r_1)| \cdot |\mathsf{Tup}(r_2)| \leq M$.

RISCAL

Set operations

- Description: The functions perform the regular set operations on relations.
- Signature: $\texttt{Relation} \times \texttt{Relation} \to \texttt{Relation}$

Definition

$$\texttt{union}(r_1, r_2) := \langle \mathsf{Len}(r_1), \mathsf{Tup}(r_1) \cup \mathsf{Tup}(r_2) \rangle$$
$$\texttt{intersect}(r_1, r_2) := \langle \mathsf{Len}(r_1), \mathsf{Tup}(r_1) \cap \mathsf{Tup}(r_2) \rangle$$
$$\texttt{minus}(r_1, r_2) := \langle \mathsf{Len}(r_1), \mathsf{Tup}(r_1) \setminus \mathsf{Tup}(r_2) \rangle$$

- For each of the three operations the relations r_1, r_2 need to be *union-compatible*, i.e. $\mathsf{Len}(r_1) = \mathsf{Len}(r_2)$. In case of $\texttt{union}$ we additionally have to ensure that $|\mathsf{Tup}(r_1)| + |\mathsf{Tup}(r_2)| \leq M$.

RISCAL

Verification idea

1. Encoding of ...
   - the database
   - and the query

   ... in a single RISCAL procedure.
2. We prove as a theorem, that our model produces the same output as the query.

Encoding of the query

Figure 3: RISCAL procedure query()

```
proc query():Relation {
var dum:Map[Attribute,Element] := Map[Attribute,Element](0);

var r1:Relation := ⟨len: 3, tup: choose s:Set[Row] with |s|=0⟩;
var r2:Relation := ⟨len: 2, tup: choose s:Set[Row] with |s|=0⟩;

r1.tup := r1.tup ∪ {dum};
r2.tup := r2.tup ∪ {dum};
dum[1] := 1;
r1.tup := r1.tup ∪ {dum};
r2.tup := r2.tup ∪ {dum};
dum[0] := 1;
r1.tup := r1.tup ∪ {dum};
dum[1] := 0;
r2.tup := r2.tup ∪ {dum};
dum[1] := 1;
dum[2] := 1;
r1.tup := r1.tup ∪ {dum};
print r1;
print r2;

var columns:Array[N,Length] := Array[N,Length](N);
columns[0] := 0;
columns[1] := 2;
print columns;

return join(project2(select(r1,1,1),columns),r2,0,0);
}
```

Results

Figure 4: RISCAL procedure `result()`

```
proc result():Relation {
var dum:Map[Attribute,Element] := Map[Attribute,Element](0);
var r:Relation := ⟨len: 4, tup: choose s:Set[Row] with |s|=0⟩;

r.tup := r.tup ∪ {dum};

dum[3] := 1;
r.tup := r.tup ∪ {dum};

dum[3] := 0;
dum[0] := 1;
dum[2] := 1;
r.tup := r.tup ∪ {dum};

dum[1] := 1;
r.tup := r.tup ∪ {dum};

return r;
}

theorem correct_result() ⇔ query() = result();
```

In RISCAL it can be verified that the theorem above is true.

concat Go back

Figure 5: RISCAL implementation of concat

```
fun concat1(t1:Row, t2:Row, n1:Length, n2:Length):Row
requires n1 + n2 ≤ N;
= choose t:Row with ∀ i:Attribute. (
if i < n1 then t[i] = t1[i]
else if i ≥ n1 ∧ i < n1+n2 then t[i] = t2[i-n1]
else t[i] = 0
);

proc concat2(t1:Row, t2:Row, n1:Length, n2:Length):Row
requires n1 + n2 ≤ N; {
var t:Row = Array[N,Element](0);
for var i:Length:=0; i<n1; i:=i+1 do {
t[i] := t1[i];
}
for var i:Length:=n1; i<n1+n2; i:=i+1 do {
t[i] := t2[i-n1];
}
return t;
}

theorem concat_equiv(t1:Row, t2:Row, n1:Length, n2:Length)
requires n1 + n2 ≤ N; ⇔
concat1(t1,t2,n1,n2) = concat2(t1,t2,n1,n2);
```

cartesian Go back

Figure 6: RISCAL implementation of `cartesian`

```
fun cartesian(r1:Relation, r2:Relation):Relation
requires r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
= ⟨len: r1.len+r2.len, tup: concat1(t1,t2,r1.len,r2.len) | t1∈r1.tup, t2∈r2.tup⟩;
```

select Go back

Figure 7: RISCAL implementation of select

```
fun select(r:Relation, a:Attribute, e:Element):Relation
requires a < r.len;
= ⟨len: r.len, tup: t | t∈r.tup with t[a] = e⟩;
```

project Go back

Figure 8: RISCAL implementation of project

```
fun project1(r:Relation, columns:Array[N,Length]):Relation
requires (∃ i:Attribute. ∀ j:Attribute.
(j>i ⇒ columns[j] = N) ∧ (j≤i ⇒ columns[j] < r.len));
= choose s:Relation with s.len = |i | i:Attribute with columns[i] ≠ N| ∧
(∀ tr:Row. tr∈r.tup ⇒
∃ ts:Row. ts∈s.tup ∧ ∀ i:Attribute. i < s.len ⇒ ts[i]=tr[columns[i]]);

proc project2(r:Relation, columns:Array[N,Length]):Relation
requires (∃ i:Attribute. ∀ j:Attribute.
(j>i ⇒ columns[j] = N) ∧ (j≤i ⇒ columns[j] < r.len)); {

var l:Length := |i | i:Attribute with columns[i] ≠ N|;
var q:Relation := ⟨len: l, tup: choose s:Set[Row] with |s|=0⟩;
var s:Set[Row] := r.tup;

choose t ∈ s do {
s := s \ {t};

var tn:Row := Array[N,Element](0);

var j:Length := 0;
for var i:Length := 0; i<N; i:=i+1 do {
if columns[i] ≠ N then {
tn[j] := t[columns[i]];
j := j+1;
}
}
q.tup := q.tup ∪ {tn};
}

return q;
}
```

join

Go back

Figure 9: RISCAL implementation of join

```
fun join(r1:Relation, r2:Relation, n1:Attribute, n2:Attribute):Relation
requires n1<r1.len ∧ n2<r2.len ∧ r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
= ⟨len: r1.len+r2.len,
tup: concat1(t1,t2,r1.len,r2.len) | t1∈r1.tup, t2∈r2.tup with t1[n1] = t2[n2]⟩;
```

Set operations Go back

Figure 10: RISCAL implementation of the set operation

```
pred union_compatible(r1:Relation, r2:Relation) ⇔ r1.len=r2.len;

fun rUnion(r1:Relation, r2:Relation):Relation
requires union_compatible(r1,r2) ∧ |r1.tup| + |r2.tup| ≤ M;
= ⟨len: r1.len, tup: r1.tup ∪ r2.tup⟩;

fun rIntersect(r1:Relation, r2:Relation):Relation
requires union_compatible(r1,r2);
= ⟨len: r1.len, tup: r1.tup ∩ r2.tup⟩;

fun rMinus(r1:Relation, r2:Relation):Relation
requires union_compatible(r1,r2);
= ⟨len: r1.len, tup: r1.tup \ r2.tup⟩;
```