Tree transformations: the equational point of view

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Material from three papers

- S. Bozapalidis, Z. Fülöp and G. Rahonis, Equational tree transformations, *Theoret. Comput. Sci.* 4012 (2011) 3676–3692.
- S. Bozapalidis, Z. Fülöp and G. Rahonis, Equational weighted tree transformations, *Acta Inform.* 49 (2012) 29–52.
- Z. Fülöp and G. Rahonis, Equational weighted tree transformations with discounting, *Lecture Notes in Comput. Sci.* 7020 (2012) 112–145.

Motivation

Engelfriet & Schmidt 1977:

- . . . In theoretical computer science there are two basic ways of describing the meaning of a *syntactical object*: *operational* and *equational*. Operational semantics is defined by some effective (eventually nondeterministic) stepwise process which, from the syntactical object, generates its meaning. Equational semantics is defined by interpreting the syntactical object as a system of equations to be solved in some space of meanings. . . .

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- *IO* (*Inside-Out*) interprets the *Call-by-Value* method of "calling procedures, functions, etc." in programming languages
- *OI* (*Outside-In*) interprets the *Call-by-Name* (or *Call-by-Reference*) method of "calling procedures, functions, etc." in programming languages

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  - for $s = \sigma(t_1, \ldots, t_k)$, $k \geq 1$, $\sigma \in \Sigma_k$,
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- $L \subseteq T_\Sigma(X_n)$: *tree language*

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- *Tarski's fixpoint theorem:* Let $(V, \leq)$ be an ω-complete poset with least element $\perp$ and $f : V \to V$ an ω-continuous mapping, i.e., $f(\sup\{a_i \mid i \geq 0\}) = \sup\{f(a_i) \mid i \geq 0\}$ for every ω-chain $a_0 \leq a_1 \leq \ldots$ in V. Then f has a least fixpoint $\mathrm{fix} f$, and $\mathrm{fix} f = \sup\{f^{(i)}(\perp) \mid i \geq 0\}$.

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  $\left\{(s,t)\left[\mathbf{r}^{(1)},\ldots,\mathbf{r}^{(n)}\right] \mid \mathbf{r}^{(i)} \in R_i^{m_i}, 1 \le i \le n\right\}$

Rairs of trees: [IO]- and OI-substitutions

Example

$\sigma \in \Sigma_3$, $\delta \in \Delta_2$, $(s,t) = (\sigma(x_1,x_1,x_3), \delta(x_3,x_1))$,

$R_1 = \{(s_1,t_1),(s_1',t_1')\}$, $R_2 = \emptyset$, $R_3 = \{(s_3,t_3)\}$

$$(s,t)\,[R_1,R_2,R_3]_{[IO]} = \{(\sigma(s_1,s_1,s_3),\delta(t_3,t_1)),(\sigma(s_1',s_1',s_3),\delta(t_3,t_1'))\}$$

$$(s,t)\,[R_1,R_2,R_3]_{OI} = \{(\sigma(s_1,s_1,s_3),\delta(t_3,t_1)),(\sigma(s_1,s_1',s_3),\delta(t_3,t_1)), (\sigma(s_1',s_1',s_3),\delta(t_3,t_1')),(\sigma(s_1',s_1,s_3),\delta(t_3,t_1'))\}$$

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Systems of equations of tree transformations

Definition

A system of equations of tree transformations over Σ and Δ is a system

$$(\mathrm{E}) \qquad x_i = R_i,\ 1 \leq i \leq n,$$

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u-equational tree transformations

Definition

$S \subseteq T_\Sigma \times T_\Delta$ is *u-equational* (u-[IO], OI) if it is the union of some components of the least u-solution of a system of equations of tree transformations over Σ and Δ.

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- $EQUT_{OI}$ the class of all OI-equational tree transformations

u-equational tree transformations - further classes

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u-equational tree transformations - further classes

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- A Σ-*algebra*

$$\mathcal{A} = (A, \Sigma^{\mathcal{A}})$$

where A is a nonempty set, called the domain set of $\mathcal{A}$, and $\Sigma^{\mathcal{A}} = (\sigma^{\mathcal{A}} \mid \sigma \in \Sigma)$ such that $\forall k \geq 0$ and $\sigma \in \Sigma_k$, we have $\sigma^{\mathcal{A}} : A^k \to A$

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    $s\left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\right]_{\mathcal{A}} =$
    $\sigma^{\mathcal{A}}\left(s_1\left[\mathbf{a}^{(1,1)}, \ldots, \mathbf{a}^{(1,n)}\right]_{\mathcal{A}}, \ldots, s_k\left[\mathbf{a}^{(k,1)}, \ldots, \mathbf{a}^{(k,n)}\right]_{\mathcal{A}}\right)$.

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- $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, $\mathcal{B} = (B, \Delta^{\mathcal{B}})$: arbitrary algebras
- $(s,t) \in T_\Sigma(X_n) \times T_\Delta(X_n)$ with
  $|s|_{x_i} = \lambda_i, \quad |t|_{x_i} = \mu_i, \quad m_i = \max\{\lambda_i, \mu_i\} \quad \forall 1 \leq i \leq n$
- $\mathbf{v}^{(i)} = \left(\left(a_1^{(i)}, b_1^{(i)} \right), \ldots, \left(a_{m_i}^{(i)}, b_{m_i}^{(i)} \right) \right) \in (A \times B)^{m_i}$
- $\mathbf{a}^{(i)} = \left(a_1^{(i)}, \ldots, a_{\lambda_i}^{(i)} \right)$
- $\mathbf{b}^{(i)} = \left(b_1^{(i)}, \ldots, b_{\mu_i}^{(i)} \right)$
- *evaluation of (s,t) at $(v^{(1)}, \ldots, v^{(n)})$ in $(\mathcal{A}, \mathcal{B})$:*
- $(s,t) \left[\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)} \right]_{(\mathcal{A},\mathcal{B})} =$
  $\left(s \left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)} \right]_{\mathcal{A}}, t \left[\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(n)} \right]_{\mathcal{B}} \right)$

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Systems of equations: solutions in pairs of algebras

A system of equations of tree transformations over Σ and Δ

$$(\mathrm{E}) \qquad x_i = R_i,\ 1 \leq i \leq n$$

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- $U \in \mathcal{P}(A \times B)$ is *u-equational* if it is the union of some components of the least u-solution in $(\mathcal{A}, \mathcal{B})$ of a system of equations of tree transformations

Mezei-Wright type result

Theorem

Let $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$ *and* $\mathcal{B} = (B, \Delta^{\mathcal{B}})$ *be arbitrary algebras and u=[IO], OI. A relation* $U \subseteq A \times B$ *is u-equational iff there exists a u-equational tree transformation* $S \subseteq T_\Sigma \times T_\Delta$ *such that* $H_{(\mathcal{A},\mathcal{B})}(S) = U$*, where* $H_{(\mathcal{A},\mathcal{B})}((s,t)) = (H_{\mathcal{A}}(s), H_{\mathcal{B}}(t))$ *for every* $(s,t) \in T_\Sigma \times T_\Delta$.

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- K-semimodule $(M, +, 0)$: *continuous* if $(M, +, 0)$ is continuous

Algebras

- *K-Σ-algebra*:

$$\mathcal{A} = (A, +, 0, \Sigma^{\mathcal{A}})$$

where $(A, +, 0)$ is a K-semimodule, and $\Sigma^{\mathcal{A}} = (\sigma^{\mathcal{A}} \mid \sigma \in \Sigma)$ a family of multilinear operations on A such that $\forall n \geq 0$, $\sigma \in \Sigma_n$, we have $\sigma^{\mathcal{A}} : A^n \longrightarrow A$ satisfying
$\sigma^{\mathcal{A}}(a_1, \ldots, k \bullet a_i + k' \bullet a_i', \ldots, a_n) =$
$k \bullet \sigma^{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) + k' \bullet \sigma^{\mathcal{A}}(a_1, \ldots, a_i', \ldots, a_n)$
$\forall k, k' \in K, a_1, \ldots, a_i, a_i', \ldots, a_n \in A, 1 \leq i \leq n$

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where $(A, +, 0)$ is a K-semimodule, and $\Sigma^{\mathcal{A}} = (\sigma^{\mathcal{A}} \mid \sigma \in \Sigma)$ a family of multilinear operations on A such that $\forall n \geq 0$, $\sigma \in \Sigma_n$, we have $\sigma^{\mathcal{A}} : A^n \longrightarrow A$ satisfying
$\sigma^{\mathcal{A}}(a_1, \ldots, k \bullet a_i + k' \bullet a_i', \ldots, a_n) =$
$k \bullet \sigma^{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) + k' \bullet \sigma^{\mathcal{A}}(a_1, \ldots, a_i', \ldots, a_n)$
$\forall k, k' \in K, a_1, \ldots, a_i, a_i', \ldots, a_n \in A, 1 \leq i \leq n$

- $\mathcal{A} = (A, +, 0, \Sigma^{\mathcal{A}})$: *continuous* if the K-semimodule $(A, +, 0)$ is continuous

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- if $\varphi_0 \leq \varphi_1 \leq \ldots$ is an ω-chain in $K\langle\langle T_\Sigma(X_n)\rangle\rangle$, then

$$\varphi_k = \sum_{0 \leq i \leq k} \rho_i,$$

where $\rho_i \in K\langle\langle T_\Sigma(X_n)\rangle\rangle$, $(i \geq 0)$, and thus

$$\sup_{k \geq 0}(\varphi_k) = \sum_{i \geq 0} \rho_i.$$

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  where $\mathbf{s}^{(i)} = \left(s_1^{(i)}, \ldots, s_{\lambda_i}^{(i)}\right) \in T_\Sigma(X_n)^{\lambda_i}$,
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- $(K\langle\langle T_\Sigma(X_n) \times T_\Delta(X_n)\rangle\rangle, +, \widetilde{0})$: continuous naturally ordered K-semimodule

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Weighted tree transformations: systems of equations

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A *system of equations of weighted tree transformations over* (Σ, Δ, K) is a system

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- $(\tau_1, \ldots, \tau_n) \in (K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n$ *least u-solution* of (E) if $\tau_i \leq \tau_i'$ $(1 \leq i \leq n)$ for every other *u*-solution $(\tau_1', \ldots, \tau_n')$ of (E)

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A *system of equations of weighted tree transformations over* (Σ, Δ, K) is a system

$$(\mathrm{E}) \qquad x_i = \rho_i,\ 1 \leq i \leq n,$$

$\rho_1, \ldots, \rho_n \in K\langle T_\Sigma(X_n) \times T_\Delta(X_n)\rangle$ polynomials.

- (E) : *variabe identical* if ρ_i is variable identical $\forall 1 \leq i \leq n$
- $(\tau_1, \ldots, \tau_n) \in (K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n$ *u-solution* of (E) if $\tau_i = \rho_i[\tau_1, \ldots, \tau_n]_u \ \ \forall 1 \leq i \leq n$
- $(\tau_1, \ldots, \tau_n) \in (K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n$ *least u-solution* of (E) if $\tau_i \leq \tau_i'$ $(1 \leq i \leq n)$ for every other u-solution $(\tau_1', \ldots, \tau_n')$ of (E)
- Existence of the least u-solution of (E): $(K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n$ is ω-complete and $F_{E,u} : (K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n \to (K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle)^n$

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Equational weighted tree transformations

Definition

$\tau \in K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ *u-equational* (u=[IO], OI) if it is a component of the least u-solution of a system of equations of weighted tree transformations over (Σ, Δ, K).

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Weighted bimorphisms

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- $\varphi \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$
- h, h', and φ define a weighted tree transformation over (Σ, Δ, K) by:

$$\langle h, h' \rangle (\varphi) = \sum_{u \in T_\Gamma(X_n)} (\varphi, u).(h(u), h'(u))$$

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- A *weighted bimorphism over* $(\Gamma, \Sigma, \Delta, K)$:

$$(h, \varphi, h')$$

where $\varphi \in K\langle\langle T_\Gamma \rangle\rangle$ is a recognizable tree series,
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- $\langle h, h' \rangle (\varphi)$: the *weighted tree transformation computed by* (h, φ, h')

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- $B(uc(H, H))$: the class of all weighted tree transformations computed by weighted bimorphisms whose input and output homomorphism constitute an *ultimately nondeleting pair* of tree homomorphisms, i.e., if $h : T_\Gamma(X_n) \to T_\Sigma(X_n)$ and $h' : T_\Gamma(X_n) \to T_\Delta(X_n)$, then $var(h_k(\gamma)) \cup var(h'_k(\gamma)) = \{\xi_1, \ldots, \xi_k\}$ for every $\gamma \in \Gamma_k$, $k \geq 0$

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- $B(uc(l\text{-}H, l\text{-}H))$

Equational weighted tree transformations: results

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- $\langle c\text{-}H, c\text{-}H \rangle \, (vs\text{-}EQUT_{OI}) = vi\text{-}EQUT_{[IO]}$

Rule-like equational weighted tree transformations

Definition

A system

$$\text{(E)} \quad x_i = \rho_i,\ 1 \leq i \leq n,$$

of equations of weighted tree transformations is called *rule-like* if each pair $(s,t) \in \text{supp}(\rho_i)$ $(1 \leq i \leq n)$ has the form $(\sigma(x_{i_1},\ldots,x_{i_k}),t)$, where $k \geq 0$, $\sigma \in \Sigma_k$, $\sigma(x_{i_1},\ldots,x_{i_k})$ is linear, and $t \in T_\Delta(\{x_{i_1},\ldots,x_{i_k}\})$ or the form (x_j,x_j).

rl-vi-EQUT$_u$: the class of all weighted tree transformations obtained as components of the least *u*-solutions of rule-like variable identical systems of equations of weighted tree transformations

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Equational weighted tree transformations: results

Corollary

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  - $s\left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\right]_{\mathcal{A}} =$ $\sigma^{\mathcal{A}}\left(s_1\left[\mathbf{a}^{(1,1)}, \ldots, \mathbf{a}^{(1,n)}\right]_{\mathcal{A}}, \ldots, s_k\left[\mathbf{a}^{(k,1)}, \ldots, \mathbf{a}^{(k,n)}\right]_{\mathcal{A}}\right)$

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Mezei-Wright result

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A system of equations of weighted tree transformations over (Σ, Δ, K)

$$(\mathrm{E}) \quad x_i = \rho_i, \ 1 \leq i \leq n$$

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- $\theta \in K\langle\langle A \times B \rangle\rangle$ *u-equational* if it a component of the least u-solution in $(\mathcal{A}, \mathcal{B}, K)$ of a system of equations of weighted tree transformations

Mezei-Wright result

Theorem (Mezei-Wright)

A weighted transformation $\theta \in K\langle\langle A \times B \rangle\rangle$ *is u-equational iff there exists a u-equational weighted tree transformation* $\tau \in K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ *such that* $H_{(\mathcal{A},\mathcal{B})}(\tau) = \theta$*, where* $H_{(\mathcal{A},\mathcal{B})}(\tau) = \sum_{(s,t) \in T_\Sigma \times T_\Delta} (\tau, (s,t)).(H_{\mathcal{A}}(s), H_{\mathcal{B}}(t))$*.*

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- $0 \leq d < 1$ a *discounting parameter*

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- Otherwise, if $var(s) = \{x_{i_1}, \ldots, x_{i_k}\}$, then we let

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- A *weighted tree transformation over* $(\Sigma, \Delta, \mathbb{R}_{\max})$

$$\tau : T_{\Sigma}(X_n) \times T_{\Delta}(X_n) \to \mathbb{R}_{\max}$$

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- max, scalar sum, partial order on $\mathbb{R}_{\max}\langle\langle T_\Sigma(X_n) \times T_\Delta(X_n)\rangle\rangle$ elementwise

Weighted tree transformations: substitutions

- $(s, t) \in T_\Sigma(X_n) \times T_\Delta(X_n)$, $\tau_1, \ldots, \tau_n \in \mathbb{R}_{\max}\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ bounded tree transformations,

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- $(s, t)[\tau_1, \dots, \tau_n]^d_{[IO]} = \sup\limits_{\substack{(s_i, t_i) \in T_\Sigma \times T_\Delta \\ 1 \leq i \leq n}} (d/k((\tau_{i_1}, (s_{i_1}, t_{i_1})) + \dots +$

  $(\tau_{i_k}, (s_{i_k}, t_{i_k}))) . (s[s_1, \dots, s_n], t[t_1, \dots, t_n]))$

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- $|s|_{x_i} = \lambda_i, \;\; |t|_{x_i} = \mu_i, \;\; m_i = \max\{\lambda_i, \mu_i\} \;\; \forall 1 \leq i \leq n$

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- *OI-d-substitution of* $\tau_1, \ldots, \tau_n$ *in* (s, t):
- $(s, t)\,[\tau_1, \ldots, \tau_n]^d_{OI} =$

$$\sup_{\substack{\mathbf{r}^{(i)} \in (T_\Sigma \times T_\Delta)^{m_i} \\ 1 \leq i \leq n}} \left(d / m\left(\left(\tau_1, \mathbf{r}^{(1)}\right) + \ldots + \left(\tau_n, \mathbf{r}^{(n)}\right)\right).(s,t)[\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(n)}]\right)$$

where $m = m_1 + \ldots + m_n$,

$$\mathbf{r}^{(i)} = \left(\left(s_1^{(i)}, t_1^{(i)}\right), \ldots, \left(s_{m_i}^{(i)}, t_{m_i}^{(i)}\right)\right),$$

$$\left(\tau_i, \mathbf{r}^{(i)}\right) = \left(\tau_i, \left(s_1^{(i)}, t_1^{(i)}\right)\right) + \ldots + \left(\tau_i, \left(s_{m_i}^{(i)}, t_{m_i}^{(i)}\right)\right), \text{ and}$$

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- $\tau[\tau_1, \ldots, \tau_n]_u^d = \max\limits_{(s,t)\in supp(\tau)} \Big((\tau, (s,t)) + (s,t)[\tau_1, \ldots, \tau_n]_u^d\Big)$.

Weighted tree transformations: systems of equations

Definition

A *system of equations of weighted tree transformations over* $(\Sigma, \Delta, \mathbb{R}_{\max})$ is a system

$$(\mathrm{E}) \qquad x_i = \rho_i, \ 1 \leq i \leq n,$$

$\rho_1, \ldots, \rho_n \in \mathbb{R}_{\max}\langle T_\Sigma(X_n) \times T_\Delta(X_n)\rangle$ polynomials.

- (E) : *variabe identical* if ρ_i is variable identical $\forall 1 \leq i \leq n$
- $(\tau_1, \ldots, \tau_n) \in (\mathbb{R}_{\max}\langle\langle T_\Sigma \times T_\Delta\rangle\rangle)^n$ *u-d-solution* of (E) if $\tau_i = \rho_i[\tau_1, \ldots, \tau_n]_u \ \ \forall 1 \leq i \leq n$
- $(\tau_1, \ldots, \tau_n) \in (\mathbb{R}_{\max}\langle\langle T_\Sigma \times T_\Delta\rangle\rangle)^n$ *least u-d-solution* of (E) if $\tau_i \leq \tau'_i$ $(1 \leq i \leq n)$ for every other u-d-solution $(\tau'_1, \ldots, \tau'_n)$ of (E)

Weighted tree transformations: systems of equations

Define the *u-d-approximation sequence* $(\tau_{1,k}, \ldots, \tau_{n,k})_{k \geq 0} \in (\mathbb{R}_{\max}\langle T_\Sigma \times T_\Delta \rangle)^n$ by

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- $\tau_{i,0} = \widetilde{-\infty}$, for $1 \leq i \leq n$, and
- $\tau_{i,k+1} = \rho_i \left[\tau_{1,k}, \ldots, \tau_{n,k}\right]_u^d$, for $1 \leq i \leq n$ and $k \geq 0$

Weighted tree transformations: systems of equations

Theorem

Let

$$(\mathrm{E}) \qquad x_i = \rho_i,\ 1 \leq i \leq n,$$

be a system of equations of weighted tree transformations , u=[IO] or u=OI and let $(\tau_{1,k}, \ldots, \tau_{n,k})_{k\geq 0} \in (\mathbb{R}_{\max}\langle T_\Sigma \times T_\Delta \rangle)^n$ *be the u-d-approximation sequence of (E). Then* $\lim_{k\to\infty} (\tau_{1,k}, \ldots, \tau_{n,k})$ *exists and it is the least u-d-solution of (E).*

d-equational weighted tree transformations

Definition

$\tau \in \mathbb{R}_{\max}\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ *u-d-equational* (u=[IO], OI) if it is a component of the least u–d-solution of a system of equations of weighted tree transformations over $(\Sigma, \Delta, \mathbb{R}_{\max})$.

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- $\langle c\text{-}H, c\text{-}H\rangle \left(vs\text{-}EQUT^{d}_{OI}\right) = vi\text{-}EQUT^{d}_{[IO]}$
- *A Mezei-Wright result holds for u-d-equational tree transformations*

Thank you