Towards Coalgebraic Specification

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Outline

Motivation

Modelling

data and **processes**

in one uniform (formal) framework.

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Historical remarks (1)

- **Computability:** Turing machines etc.
  Important results before computers existed!

- **Processes:** modelled by
  - (various types of) automata,
  - finite/abstract state machines,
  - Petri nets,
  - (labeled) transition systems,
  - ...

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Modeling data

- For basic results only the data type "natural numbers" was necessary.
- For real life applications, advanced algorithms, ... richer data structures have to be specified.
- For modeling *Abstract Data Types (ADTs)* we need:
  - Base sets (sorts of objects)
  - Basic functions and predicates
  - Constraints (axioms)
- Thesis: ADTs are (classes of) **algebras**
- Gave raise to "Algebraic Specification" (of ADTs).

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Algebraic Specification (of ADTs)

- 1972: First paper by Sir C.A.R: Hoare
- 1974-76: ADJ-group, Guttag/Horning, Liskov/Zilles, . . .
- 1983: B. Kutzler, F. Lichtenberger:
  "Bibliography of Abstract Data Types"
  More than 500 references!
- Several AlgSpec languages developed:
  OBJ3, ASL, ACT ONE/TWO, Larch, . . .
- AlgSpec concepts used in CA-Systems: Scratchpad, Axiom, Magma, (Reduce 4), . . .
- CoFI: Common Framework for Algebraic Specification and Development, EU-Project, started 1995.
- 2003: CASL - Common Algebraic Specification Language

Semantics of ADT specifications

Concepts from **category theory** are ubiquitous

- Loose, initial, final, . . . semantics
- Free algebras, galois connexion for (equational) definability
- Free functors, natural transformations for parametrized specifications
- Pushout, pullback in parameter passing
- . . .

Categories

Definition

A category C consists of

- a class of objects,
- a set of morphisms (arrows) between each of two objects,
- a composition of morphisms which is associative,
- an identity morphism for each object.

Remark

Many notions can be defined on this "categorical level", like products and coproducts, mono-, epi-, isomorphisms, initial and final objects, pullbacks and pushouts, limits and colimits, etc. etc.

These notions come "in pairs", i.e. are dual to each other.

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Initial and final objects

Definition

- An object $\mathbf{1}$ is called **final** in a category C, iff
  for every object X there
  exists a unique morphism $X \rightarrow \mathbf{1}$
- An object $\mathbf{0}$ is called **initial** in a category C, iff
  for every object X there
  exists a unique morphism $\mathbf{0} \rightarrow X$

Remark

These notions are **dual**, i.e.

- *initial* is *co-final*
- *final* is *co-initial*

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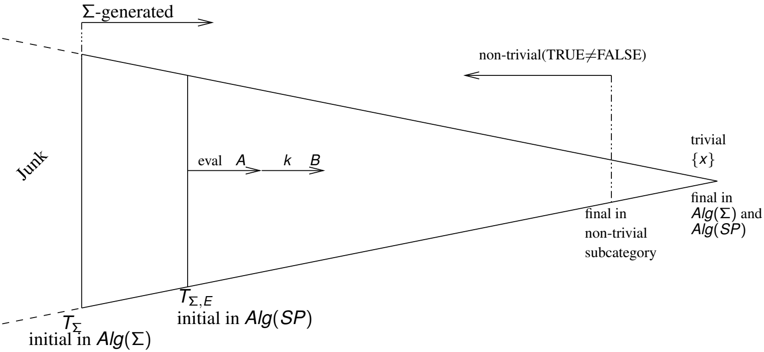
The structure of $Alg(\Sigma)$ and $Alg(SP)$

- The structure of $Alg(\Sigma)$ and $Alg(SP)$, $\Sigma = (S, OP)$ a signature, $SP = (\Sigma, E)$ a specification



Initial-algebra semantics: no junk, no confusion (Rod Burstall)

Functors

Remark

A *functor* is a "Homomorphism between categories".

Definition

Let C and D categories.
A **functor** $F : C \to D$ is a pair of maps, i. e. it maps

- objects of C to objects of D, and
- morphisms $f : X \to X'$ (for $X, X' \in C$) to morphisms $F(f) : F(X) \to F(X')$ in D,

such that

$$F(g \circ f) = F(g) \circ F(f)$$

and

$$F(id_X) = id_{F(X)}$$

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The category SET

- We work in the category SET (objects are sets, morphisms are (total) functions, usual composition of functions).
- We use the following operations on sets:
  - Product:
  $$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}$$
  - Coproduct (direct sum):
  $$X + Y = \{ \langle 0, x\rangle \mid x \in X \} \cup \{ \langle 1, y\rangle \mid y \in Y \}$$
  - Powerset:
  $$\mathcal{P}(X) = \{ Y \mid Y \subseteq X \}$$
  - Function space:
  $$X^Y = \{ f \mid f : Y \to X \}$$

Remark

These operations are **functorial**, i.e., can be lifted from sets to functions between sets, thus forming functors from SET to SET.

Initial and final sets

We write $1 = \{*\}$ for a singleton set (with typical elem. $*$).
There is exactly one function $X \to 1$ for any set X.
Thus 1 is final (or terminal) in SET.

We write 0 for the empty set.
There is exactly one function $0 \to X$ for any set X.
Thus 0 is initial in SET.

Some useful isomorphisms in SET

$$
\begin{array}{cc}
X \times Y \cong Y \times X & X + Y \cong Y + X \\
1 \times X \cong X & 0 + X \cong X \\
X \times (Y \times Z) \cong (X \times Y) \times Z & X + (Y + Z) \cong (X + Y) + Z \\
X \times 0 \cong 0 & X \times (Y + Z) \cong (X \times Y) + (X \times Z)
\end{array}
$$

We shall usually work “up to” these isomorphisms, so we can simply write for n-ary products:

$$X_1 \times X_2 \times \cdots \times X_n$$

without bothering about bracketing.

Polynomial Functors (1)

Remark

We use two trivial functors as well:

1. id : SET $\to$ SET (the identity functor)
2. For a constant set C we have the functorial operation $X \mapsto C$; a function $f : X \to X'$ is mapped to the identity function $id_C : C \to C$.

We will often say things like: consider the functor

$$T(X) = X + (C \times X),$$

i.e., we give the action only on sets, here

$$X \mapsto X + (C \times X).$$

Polynomial Functors (2)

Since all operations are functorial, the action on a function $f : X \to X'$ is derived:

$$T(f) : T(X) \to T(X')$$

explicitly, $T(f)$ in our example is

$$f + (id_C \times f) : X + (C \times X) \to X' + (C \times X')$$

given by:

$$w \mapsto \begin{cases} \langle 0, f(x) \rangle & \text{if } w = \langle 0, x \rangle \\ \langle 1, (c, f(x)) \rangle & \text{if } w = \langle 1, (c, x) \rangle \end{cases}$$

In the sequel we shall use only such **polynomial** functors built up with constants, identity functor, products, coproducts, and -later- also (finite) powersets and function spaces X^A (for constant sets A).

Polynomial Functors (2)

Since all operations are functorial, the action on a function $f : X \to X'$ is derived:

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Algebras of a functor

Example

Let T be the polynomial functor

$$T(X) = 1 + X + (X \times X).$$

For a set U, a function $a : T(U) \to U$ is a 3-cotuple $[a_1, a_2, a_3]$ of maps
$a_1 : 1 \to U$,
$a_2 : U \to U$,
$a_3 : U \times U \to U$.
The shape of the functor T determines a **signature**, here of a group:
For a carrier set G and unit element $e : 1 \to G$, inverse $i : G \to G$, multiplication $m : G \times G \to G$,
we get by cotupelling an **algebra of** T:

$$(G, [e, i, m] : T(G) \to G).$$

Example

Similarly, the algebras of the functor

$$T(X) = 1 + X \times X$$

have a **monoid** signature.

Definition

Let T be a functor. An **algebra** of T (or T-algebra) is a pair consisting of set U and a function $a : T(U) \to U$.
We call U the **carrier set**, a the **algebra structure** or **operation** of the algebra.

Example

Natural numbers $\mathbb{N}$

zero	and	successor functions
$0 : 1 \to \mathbb{N}$		$S : \mathbb{N} \to \mathbb{N}$

form an algebra $(\mathbb{N}, [0, S] : 1 + \mathbb{N} \to \mathbb{N})$ of the functor $T(X) = 1 + X$.

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Example

A-labelled binary trees: $Tree(A)$

operations:

$$\text{nil} : 1 \to Tree(A)$$

$$\text{node} : Tree(A) \times A \times Tree(A) \to Tree(A)$$

form an algebra of the polynomial functor

$$T(X) = 1 + (X \times A \times X)$$

isomorphic to

$$T(X) = 1 + A \times X^2$$

Remark

This polynomial (functor) is a precise and very concise specification of A-labelled binary trees. The *semantics* is the initial algebra of the category of T- algebras and homomorphisms between T-algebras.

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Historical remarks (2)

Later developments in Algebraic Specification:

- Modules
- Objects and components
- Concurrency
- etc. etc. . . .
- Specification of entire software systems

DEAD END STREET!

because some of the phenomena are intrinsically non-algebraic

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Coalgebras

Definition

For a functor T, a **coalgebra** (or a T-coalgebra) is a pair (U, c) consisting of a set U and a function $c : U \to T(U)$.

Like for algebras, we call U the **carrier** and c the **structure** or **operation** of (U, c). U is often called the **state space**.

Compare:

algebra: $T(U) \to U$
operation **into** the carrier U, describes **construction** of elements of U.

coalgebra: $U \to T(U)$
operation **out of** the carrier U, describes **observations** about elements of U.

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Coalgebraic phenomena (1)

Example (Black-box machine with one button and one light.)

- performs an action only if button is pressed
- light goes on only if the machine stops operating (i.e., has reached a final state)

X - the (unknown) "state space"
Describe the machine by a function:

$$\text{button} : X \to \{*\} \cup X,$$

where $* \notin X$ is a new symbol.

- The pair $(X, \text{button} : X \to \{*\} \cup X)$ is an example of a coalgebra.
- Observable behavior: an element of $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ describing the number of times the button has to be pressed until the light goes on.
- Mathematically, $\text{beh} : X \to \bar{\mathbb{N}}$, turns out to be a **final coalgebra** of the functor $T(X) = 1 + X$.

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Coalgebraic phenomena (2)

Example (Similar machine with two buttons **value** and **next**)

Described by a coalgebra:

$$(X, \langle \mathrm{value}, \mathrm{next} \rangle : X \to A \times X),$$

where A is the set of observable values.

- Observable behavior: an infinite sequence

  $$(a_0, a_1, a_2, \dots) \in A^{\mathbb{N}},$$

  where a_i is the **value** after processing **next** i-times.
- Observing this behavior for every state $s \in X$ gives a function

  $$\mathrm{beh} : X \to A^{\mathbb{N}}$$

  which is the final coalgebra of $T(X) = A \times X$.

Coalgebraic phenomena (3)

Example (transition systems)

Machine with two buttons (second example)

$$\langle \text{value}, \text{next} \rangle : X \to A \times X$$

can be understood as **deterministic** transition system. We write

$$s \xrightarrow{a} s' \text{ iff } \text{value}(s) = a \text{ and } \text{next}(s) = s'.$$

The trace $\text{Tr}(s)$ of observations of state $s \in X$:

$$\text{Tr}(s) = (a_1, a_2, \dots) \text{ where } s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots,$$

which is the observable behavior $\text{beh}(s) \in A^{\mathbb{N}}$.

Example (transition systems)

A **non deterministic** transition system

$$(X, A, \rightarrow), \text{ where } \rightarrow \subseteq X \times A \times X$$

can be described in coalgebra form as a function using a powerset:

$$\alpha : X \rightarrow \mathcal{P}(A \times X),$$

where $\alpha(s)$ is the "successor set" of $s \in X$.

Remark

Finding the right domain for the observable behavior is non-trivial here.

Summing up the examples of coalgebras:

- We have a state space X about which we make no assumptions.
- On X a function is defined (often consisting of different components) which allows us
  - to observe some aspect directly or
  - move on to next states.
- We can describe just the **behavior** by making successive observations.
- This **behavior** typically is the **final coalgebra** of a (polynomial) functor.
- This also leads to the notion of **bisimilarity**, i.e., two states (which need not be equal as elements of X) cannot be distinguished via the operations at our disposal, i.e., are "equal as far as we can see."
- Bisimilarity is an important and typically coalgebraic concept.

Data and Processes

- Data are modeled by algebras
  Semantics: initial algebra
- Processes are modeled by coalgebras
  Semantics: final coalgebra

They are dual to each other:

- data are co-processes
- processes are co-data

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Bialgebraic Modeling

In practice, algebraic and coalgebraic aspects interact on different hierarchical layers, for example

- start with algebraically specifying ones application domain
- describe dynamical systems (processes) as coalgebras, using the algebras above as codomains of observer functions,
- such coalgebraic systems may exist in an algebra of processes.

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Topics for discussion

From theoretical . . .

- Adjoint functors
- Monads/ strong monads/ comonads
- Kleisli categories
- Non-wellfounded sets
- equivalence in algebras vs bisimulation in coalgebras

- ’Added value’ of using coalgebras instead of, say, ASMs
- Monads and Kleisli Triples in functional programming
- Continuation Monad (and other monads)
- Final coalgebra semantics of specification languages
- Paper: "A Coalgebra as an Intrusion Detection System"
- Practical proof schemes for coinduction
- Proof by coinduction and bisimulation

. . . to practical