Finite Automata over Infinite Alphabets

Irini Eleftheria Mens

Aristotle University of Thessaloniki

June 15, 2011
RISC

Motivation

Infinite Alphabets

WHY ?

Motivation

- XML documents, whose leaves are typically associated with data values from some infinite domain
- Datalog systems with infinite data domain
- Software with integer parameters

Models on infinite alphabets

- Register automata
- Pebble automata
- Data automata

- Alternating automata
- Two-way register automata
- Two-way pebble automata
- Weak pebble automata
- etc.

Problems on this models

Simplicity Criterion (efforts required in order to understand a given automaton, work with it, and implement it)

- Registers or pebbles requires a substantial modification of the transition function.
- Registers and pebbles makes the automata hard to understand and work with.
- Even basic notions of such automata cannot simply rely on familiar definitions of an *nfa*.
- Data automata need to accept several subwords going through an intermediate alphabet.
- Transducers makes data automata very complex.

Problems on this models

Compositionality Criterion (closure under the basic operations)

- Data automata and register automata are not closed under complementation.
- Register automata need fragments requirements that limit the number of registers.

Problems on this models

Computability Criterion (decidability and complexity of classical problems)

- In pebble automata the nonemptiness, universality, and containment problems are undecidable.
- In data and register automata the universality and containment problems are undecidable.
- Nonemptiness of data and register automata is decidable but with high complexity.

Variable finite automata on words

introduced in

Grumberg, O., Kupferman, O., Sheinvald, S.: Variable automata over infinite alphabets. In: Proceedings of LATA 2010. LNCS, vol. 6031, pp. 561–572. Springer, Heidelberg (2010)

Preliminaries

- Σ *alphabet*
- Σ^* free monoid generated by Σ
- ε unit element of Σ^* (*empty word*)

- $w = a_1 a_2 ... a_n$ *finite word over* Σ
- $w(i) = a_i \in \Sigma$ the *label of w at i*
- Σ^* all finite words over Σ

Preliminaries

Definition

$A = (Q, \Gamma, Q_0, \Delta, F)$ *non-deterministic finite automaton* (nfa)

- Q *finite state set*
- Γ *finite input alphabet*
- $Q_o \subseteq Q$ *initial states*
- $F \subseteq Q$ *final states*
- $\Delta \subseteq Q \times \Gamma \times Q$ *set of transitions*

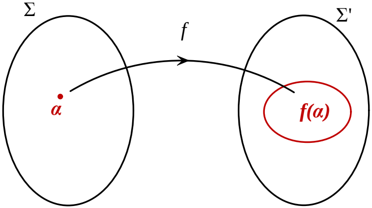
- a word $w = a_1 a_2 ... a_n \in \Gamma^*$ is *accepted by A*
  if there is a sequence $(q_0, a_1, q_1), (q_1, a_2, q_2), ..., (q_{n-1}, a_n, q_n)$ of transitions where $q_0 \in Q_0$, $q_n \in F$.
- $L(A) = \{w \in \Gamma^* \mid w \text{ accepted by } A\}$
  recognizable language
- *REC*

Definitions

relabelings on alphabets

Σ , Σ' *alphabets* (maybe *infinite*)

- A relabeling $f : \Sigma \to \mathcal{P}(\Sigma')$
- Extended to $f : \Sigma^* \to \mathcal{P}(\Sigma'^*)$



- for $w = \alpha_1 \alpha_2 \ldots \alpha_n \in \Sigma^*$, $\alpha_i \in \Sigma$, $1 \leq i \leq n$, let
  - $f(w) = \{w' = \alpha'_1 \alpha'_2 \ldots \alpha'_n \mid \alpha'_i \in f(\alpha_i)$ for $1 \leq i \leq n\}$, and
  - $f(\varepsilon) = \{\varepsilon\}$

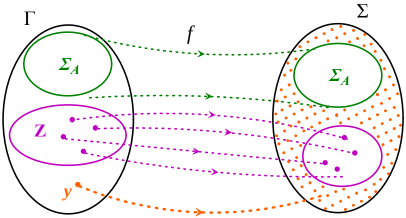
Definitions

alphabets

- Σ *infinite alphabet*
- $\Sigma_A \subseteq \Sigma$ finite subalphabet of Σ *constant alphabet*
- $Z = \{z_1, \ldots, z_k\}$, $\Sigma \cap Z = \emptyset$ *bounded variables*
- $Y = \{y\}$, $y \notin (\Sigma \cup Z)$ *free variable*

$$\Gamma = \Sigma_A \cup Z \cup \{y\}$$

$$f : \Gamma \rightarrow \mathcal{P}(\Sigma)$$



f is a *valid relabeling* on Γ if

(i) it is the identity on Σ_A,

(ii) $card(f(z)) = 1$ for every $z \in Z$,

(iii) f is injective on Z and $\Sigma_A \cap f(Z) = \emptyset$,

(iv) $f(y) = \Sigma \setminus (\Sigma_A \cup f(Z))$.

$VR(\Gamma)$ = all valid relabelings on Γ

Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$

$$\mathbf{w = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y}$$

f

$$\mathbf{w = d\ b\ c\ b\ o\ a\ k\ a\ c\ d}$$

Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\} \qquad Z = \{z_1, z_2\} \qquad \{y\}$

$$\mathbf{w = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y}$$

f

$$\mathbf{w = d\ b\ c\ b\ o\ a\ k\ a\ c\ d}$$

Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $\qquad Z = \{z_1, z_2\}$ $\qquad \{y\}$

$$\mathbf{w = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y}$$

f

$$\mathbf{w = d\ b\ c\ b\ o\ a\ k\ a\ c\ d}$$

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$

$$\mathbf{w = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y}$$

f

$$\mathbf{w = d\ b\ c\ b\ o\ a\ k\ a\ c\ d}$$

Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $Z = \{z_1, z_2\}$ $\{y\}$

$$\mathbf{w} = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y$$

f ↓

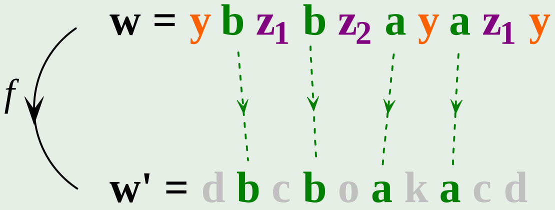
$$\mathbf{w'} = d\ b\ c\ b\ o\ a\ k\ a\ c\ d$$

Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $\quad$ $Z = \{z_1, z_2\}$ $\quad$ $\{y\}$



Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $\quad$ $Z = \{z_1, z_2\}$ $\quad$ $\{y\}$

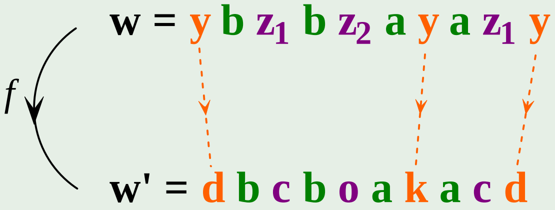
$$\mathbf{w} = y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y$$

f

$$\mathbf{w'} = d\ b\ c\ b\ o\ a\ k\ a\ c\ d$$

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$



Definitions

valid relabelings

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$

$$\mathbf{w} = \mathbf{y\ b\ z_1\ b\ z_2\ a\ y\ a\ z_1\ y}$$

f

$$\mathbf{w'} = \mathbf{d\ b\ c\ b\ o\ a\ k\ a\ c\ d}$$

Example

$\Sigma_A = \{a, b\}$ $\quad Z = \{z_1, z_2\}$ $\quad \{y\}$

w = y b z_1 b z_2 a y a z_1 y

f

w' = d b c b o a k a c d

w' = c b r b k a d a r ρ

w' = o b ñ b d a o a ñ o

Variable Finite Automaton

Definition

A *variable finite automaton* is a pair $\mathcal{A} = \langle \Sigma, A \rangle$ where

- Σ infinite alphabet
- $A = (Q, \Gamma_A, Q_0, \Delta, F)$ nfa
    - $\Gamma_A = \Sigma_A \cup Z \cup \{y\}$
        - $\Sigma_A \subseteq \Sigma$ finite subalphabet
        - Z finite alphabet of bounded variables
        - y free variable

- $L(\mathcal{A}) = \bigcup_{f \in VR(\Gamma_A)} f(L(A))$ *language accepted* by $\mathcal{A}$
- $VREC(\Sigma)$ recognizable languages over Σ
- $VREC$ recognizable languages over all infinite alphabets

Variable Finite Automaton

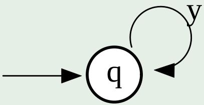
Examples

Example

$\mathcal{A} = \langle \Sigma, A \rangle$

$\Gamma_A = \{y\}$

$(\Sigma_A = \emptyset \ , \ Z = \emptyset)$



$L(A) = y^*$

$L(\mathcal{A}) = \Sigma^*$

Example

$L(\mathcal{A}) = L(A)$ if $Z = \emptyset$ and $Y = \emptyset$

Proposition

REC $\subseteq$ *VREC*.

$$L^{(1)} = L\left(\mathcal{A}^{(1)}\right) \qquad L^{(2)} = L\left(\mathcal{A}^{(2)}\right)$$

$$\mathcal{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathcal{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(1)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$\Gamma^{(1)} = \Sigma^{(1)} \cup Z^{(1)} \cup \left\{ y^{(1)} \right\} \qquad \Gamma^{(2)} = \Sigma^{(2)} \cup Z^{(2)} \cup \left\{ y^{(2)} \right\}$$

$$Q^{(1)} \cap Q^{(2)} = \emptyset \qquad Z^{(1)} \cap Z^{(2)} = \emptyset$$

$$L^{(1)} \cup L^{(2)}$$

from classical construction $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$

Problem (1)

If $y^{(1)} \neq y^{(2)} \implies$ *two free variables*

Solution

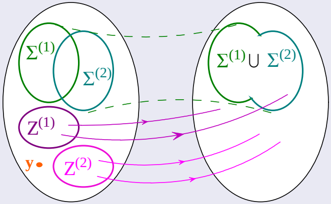
Consider one new free variable y

$$y \neq y^{(1)}\ ,\ y \neq y^{(2)}\ ,\ y \notin \Sigma \cup Z^{(1)} \cup Z^{(2)}$$

$$\Delta = \left\{ (q, a, q') \in \Delta^{(1)} \cup \Delta^{(2)} \mid a \in \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \cup Z^{(2)} \right\}$$
$$\cup \left\{ (q, y, q') \mid \left(q, y^{(1)}, q'\right) \in \Delta^{(1)} \text{ or } \left(q, y^{(2)}, q'\right) \in \Delta^{(2)} \right\}$$

Problem (2)

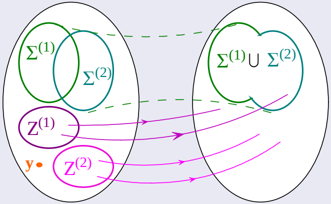
$\Gamma^{(1)} \cup \Gamma^{(2)}$



$\Sigma^{(1)} \neq \Sigma^{(2)}$

Problem (2)

$\Gamma^{(1)} \cup \Gamma^{(2)}$

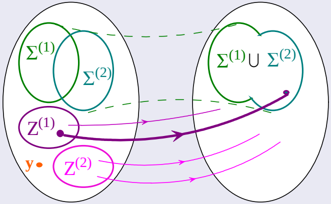


$$\Sigma^{(1)} \neq \Sigma^{(2)}$$

$$w = az_1 \in L\left(A^{(1)}\right), \quad a \in \Sigma^{(1)}, \quad z_1 \in Z^{(1)} \implies f \in VR\left(\Gamma^{(1)}\right), \quad f(a) = a, \quad f(z_1) = b \implies w' = ab \in L\left(\mathcal{A}^{(1)}\right)$$

Problem (2)

$\Gamma^{(1)} \cup \Gamma^{(2)}$



$\Sigma^{(1)} \neq \Sigma^{(2)}$

if $b \in \Sigma^{(2)}$, *then f is not a valid relabeling on* Γ

Is ab accepted by the new automaton?

Solution

$\mathcal{A} = \langle \Sigma, A \rangle$ *a vfa* $\quad f \in VR(\Gamma_A) \quad \Sigma' \subseteq \Sigma \quad \Sigma' \setminus \Sigma_A \neq \emptyset$

- $A_f = (Q_f, \Gamma_{A_f}, Q_{0_f}, \Delta_f, F_f)$ *a nfa*
  - $Q_f = \{q^f \mid q \in Q\}$
  - $Q_{0_f} = \{q^f \in Q_f \mid q \in Q_0\}$
  - $F_f = \{q^f \in Q_f \mid q \in F\}$

$$Z_f = \{z \in Z \mid f(z) \in \Sigma' \setminus \Sigma_A\}$$

  - $\Gamma_{A_f} = \Sigma_A \cup (f(Z \cup \{y\}) \cap \Sigma') \cup (Z \setminus Z_f) \cup \{y\}$
  - $\Delta_f \subseteq Q_f \times \Gamma_{A_f} \times Q_f$

$$\begin{aligned}\Delta_f \quad = \quad & \{(q_1^f, \sigma, q_2^f) \mid (q_1, \sigma, q_2) \in \Delta \textit{ and } \sigma \in (\Sigma_A) \cup (Z \setminus Z_f) \cup \{y\}\} \\ & \cup \{(q_1^f, f(z), q_2^f) \mid (q_1, z, q_2) \in \Delta \textit{ and } z \in (Z_f)\} \\ & \cup \{(q_1^f, \sigma', q_2^f) \mid (q_1, y, q_2) \in \Delta \textit{ and } \sigma' \in f(y) \cap (\Sigma' \setminus \Sigma_A)\}.\end{aligned}$$

Solution

- Σ_A , Z , Σ' *finite alphabets*
- *The automata* A_f $(f \in VR(\Gamma_A))$ *are finitely many modulo the identification of state sets*
- $V \subseteq VR(\Gamma_A)$ *finite subset determining this class*
- $A_g = (Q_g, \Gamma_{A_g}, Q_{0_g}, \Delta_g, F_g)$ $\forall g \in V$
- Q_g $(g \in V)$ *pairwise disjoint*
- $A_{(\Sigma', V)} = (Q_V, \Gamma_V, Q_{0_V}, \Delta_V, F_V)$

Solution

- $A_{(\Sigma', V)} = (Q_V, \Gamma_V, Q_{0_V}, \Delta_V, F_V)$
  - $\Gamma_V = \Sigma_A \cup \Sigma' \cup Z \cup \{y\}$
  - $Q_V = \bigcup_{g \in V} Q_g$
  - $Q_{0_V} = \bigcup_{g \in V} Q_{0_g}$
  - $F_V = \bigcup_{g \in V} F_g$
  - $\Delta_V = \bigcup_{g \in V} \Delta_g$
- $L\left(A_{(\Sigma', V)}\right) = \bigcup_{g \in V} L(A_g)$
- $L(\mathcal{A}_{(\Sigma', V)}) = \bigcup_{f \in VR(\Gamma_V)} f\left(\bigcup_{g \in V} L(A_g)\right)$

Basic Lemma

(Mens-Rahonis)

Lemma

$$L(\mathcal{A}) = L(\mathcal{A}_{(\Sigma', V)})$$

Proposition

The class $VREC(\Sigma)$ is closed under union.

Proof.

$$\mathcal{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathcal{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(1)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$Q^{(1)} \cap Q^{(2)} = \emptyset \qquad Z^{(1)} \cap Z^{(2)} = \emptyset \qquad y^{(1)} \neq y^{(2)}$$

$$A^{(1)}_{(\Sigma^{(2)}, V_1)} = \left(Q^{(1)}_{V_1}, \Gamma^{(1)} \cup \Sigma^{(2)}, Q^{(1)}_{0_{V_1}}, \Delta^{(1)}_{V_1}, F^{(1)}_{V_1} \right)$$

$$A^{(2)}_{(\Sigma^{(1)}, V_2)} = \left(Q^{(2)}_{V_2}, \Gamma^{(2)} \cup \Sigma^{(1)}, Q^{(2)}_{0_{V_2}}, \Delta^{(2)}_{V_2}, F^{(2)}_{V_2} \right)$$

$$Q^{(1)}_{V_1} \cap Q^{(2)}_{V_2} = \emptyset$$

Proof (continued).

disjoint union of $A^{(1)}_{(\Sigma^{(2)},V_1)}$ and $A^{(2)}_{(\Sigma^{(1)},V_2)}$

$$A = (Q^{(1)}_{V_1} \cup Q^{(2)}_{V_2}, \Gamma, \overline{\Delta}^{(1)}_{V_1} \cup \overline{\Delta}^{(2)}_{V_2}, Q^{(1)}_{0_{V_1}} \cup Q^{(2)}_{0_{V_2}}, F^{(1)}_{V_1} \cup F^{(2)}_{V_2})$$

- $\Gamma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \cup Z^{(2)} \cup \{y\}$
- transition set $\overline{\Delta}^{(i)}_{V_i}$ obtained from $\Delta^{(i)}_{V_i}$, $i = 1, 2$,

  by replacing every occurrence of $y^{(i)}$ with the new symbol y.

We will show that $L(\mathcal{A}) = L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$ where $\mathcal{A} = \langle \Sigma, A \rangle$

Proof (continued).

- $L(\mathcal{A}) \subseteq L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$

$w \in L(\mathcal{A})$

$$\left.\begin{array}{l} \exists u \in L(A) \\ \exists f \in VR(\Gamma) \end{array}\right\} \Longrightarrow w \in f(u)$$

$\exists u' \in L\left(A^{(1)}_{(\Sigma^{(2)}, V_1)}\right) \cup L\left(A^{(2)}_{(\Sigma^{(1)}, V_2)}\right)$ such that

u is obtained from u' by replacing every occurrence of $y^{(1)}$ and $y^{(2)}$ with y

Proof (continued).

$$u' \in L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right) \cup L\left(A^{(2)}_{(\Sigma^{(1)},V_2)}\right)$$

If $u' \in L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right)$

$$f' : \Gamma^{(1)} \cup \Sigma^{(2)} \to \Sigma$$

$$f'(\sigma) = \begin{cases} f(\sigma) & \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \\ f(y) \cup f\left(Z^{(2)}\right) & \sigma = y^{(1)} \end{cases}$$

$$\begin{aligned} f' \in VR\left(\Gamma^{(1)} \cup \Sigma^{(2)}\right) \text{ and } w \in f'(u') \quad &\Longrightarrow \quad w \in L\left(\mathcal{A}^{(1)}_{(\Sigma^{(2)},V_1)}\right) \\ &\Longrightarrow \quad w \in L\left(\mathcal{A}^{(1)}\right) \end{aligned}$$

If $u' \in L\left(A^{(2)}_{(\Sigma^{(1)},V_2)}\right)$ similarly.

Proof (continued).

- $L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \subseteq L(\mathcal{A})$

$w \in L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$

$$\text{If } w \in L\left(\mathcal{A}^{(1)}\right) \implies w \in L\left(\mathcal{A}^{(1)}_{(\Sigma^{(2)}, V_1)}\right) \implies \left.\begin{array}{l} \exists u \in L\left(A^{(1)}_{(\Sigma^{(2)}, V_1)}\right) \\ \exists f \in VR(\Gamma^{(1)} \cup \Sigma^{(2)}) \end{array}\right\}$$

$$\implies w \in f(u)$$

$$f' : \Gamma \to \Sigma$$
$$\begin{cases} f'(\sigma) = f(\sigma) & \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \\ f'(z) \in \Sigma \setminus \left(\Sigma^{(1)} \cup \Sigma^{(2)} \cup f(Z^{(1)}) \cup \Sigma''\right) & z \in Z^{(2)} \end{cases}$$

Σ'' : all labels of w obtained by replacing all occurrences of y in u.

Proof (continued).

u' the word obtained from u by replacing every occurrence of $y^{(1)}$ with y.

$u' \in L(A)$ and $w \in f'(u') \Longrightarrow w \in L(\mathcal{A})$.

If $w \in L\left(\mathcal{A}^{(2)}\right)$ similarly

$$\left.\begin{array}{l} L(\mathcal{A}) \subseteq L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \\ L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \subseteq L(\mathcal{A}) \end{array}\right\} \Longrightarrow L(\mathcal{A}) = L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$$

□

Proposition

The class $VREC(\Sigma)$ is closed under intersection.

Proof.

$$\mathcal{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathcal{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(1)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$Z^{(1)} \cap Z^{(2)} = \emptyset \qquad y^{(1)} \neq y^{(2)}$$

$$A^{(1)}_{(\Sigma^{(2)}, V_1)} = \left(Q^{(1)}_{V_1}, \Gamma^{(1)} \cup \Sigma^{(2)}, Q^{(1)}_{0_{V_1}}, \Delta^{(1)}_{V_1}, F^{(1)}_{V_1} \right)$$

$$A^{(2)}_{(\Sigma^{(1)}, V_2)} = \left(Q^{(2)}_{V_2}, \Gamma^{(2)} \cup \Sigma^{(1)}, Q^{(2)}_{0_{V_2}}, \Delta^{(2)}_{V_2}, F^{(2)}_{V_2} \right)$$

Proof (continued).

$((Z^{(1)} \cup Y^{(1)}) \times (Z^{(2)} \cup Y^{(2)})) \setminus \left\{ \left(y^{(1)}, y^{(2)} \right) \right\}$ alphabet

$R \subseteq ((Z^{(1)} \cup Y^{(1)}) \times (Z^{(2)} \cup Y^{(2)})) \setminus \left\{ \left(y^{(1)}, y^{(2)} \right) \right\}$ maximal subalphabet

(every element of $Z^{(1)}$ (resp. $Z^{(2)}$) occurs as a left (resp. right) coordinate in at most one pair in R)

Let $R_1, \ldots, R_m$, and for every $1 \leq j \leq m$

$$A_{R_j} = \left(Q_{V_1}^{(1)} \times Q_{V_2}^{(2)}, \Gamma_{R_j}, Q_{0_{V_1}}^{(1)} \times Q_{0_{V_2}}^{(2)}, \Delta_{R_j}, F_{V_1}^{(1)} \times F_{V_2}^{(2)} \right)$$

Proof (continued).

$$A_{R_j} = \left(Q^{(1)}_{V_1} \times Q^{(2)}_{V_2}, \Gamma_{R_j}, Q^{(1)}_{0_{V_1}} \times Q^{(2)}_{0_{V_2}}, \Delta_{R_j}, F^{(1)}_{V_1} \times F^{(2)}_{V_2} \right)$$

- $\Gamma_{R_j} = \Sigma^{(1)} \cup \Sigma^{(2)} \cup R_j \cup \left\{ \left(y^{(1)}, y^{(2)} \right) \right\}$
  - R_j set of bounded variables
  - $(y^{(1)}, y^{(2)})$ free variable
- $\Delta_{R_j} = \{((q_1^{(1)}, q_1^{(2)}), \sigma, (q_2^{(1)}, q_2^{(2)})) \mid (q_1^{(i)}, \sigma, q_2^{(i)}) \in \Delta^{(i)}_{V_i}, i = 1, 2, \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)}\}$
  $\cup \{((q_1^{(1)}, q_1^{(2)}), (x^{(1)}, x^{(2)}), (q_2^{(1)}, q_2^{(2)})) \mid (q_1^{(i)}, x^{(i)}, q_2^{(i)}) \in \Delta^{(i)}_{V_i}, i = 1, 2, (x^{(1)}, x^{(2)}) \in (R_j) \cup Y\}.$

We show that $L\left(\mathcal{A}^{(1)}\right) \cap L\left(\mathcal{A}^{(2)}\right) = L(\mathcal{A}_{R_1}) \cup \ldots \cup L(\mathcal{A}_{R_m})$ □

Emptiness Problem

Proposition

The emptiness problem is decidable in $VREC(\Sigma)$.

Proof.

$L \in VREC(\Sigma)$
$L = L(\mathcal{A})$ where $\mathcal{A} = \langle \Sigma, A \rangle$ is a vfa.

$L = \emptyset$ iff $L(A) = \emptyset$

Since the emptiness problem is decidable for recognizable languages we are done. □

Equivalence Problem

Application of a more general result for trees (Mens-Rahonis)

Theorem

The equivalence problem is decidable for

(i) *vfa whose transitions do not contain any free variable*

(ii) *vfa whose transitions do not contain any bounded variable.*

Universality Problem

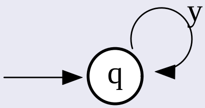
Application of a more general result for trees (Mens-Rahonis)

Proposition

The universality problem is decidable for vfa whose transitions do not contain any bounded variable.

Proof.

- the variable automaton



accepts $L(\mathcal{A}) = \Sigma^*$

- equivalence problem is decidable

□

Complementation

Proposition

The class $VREC(\Sigma)$ is not closed under complementation.

Open Problems and Future Research

- Determinization (in Grumberg et al. this problem is identified with the unambiguity)
- Extension to trees (done by Mens-Rahonis in [5])
- Extension to infinite trees (under progress by Mens-Rahonis)

Thank You!

References

1. Bojańczyk, M., David, C., Muscholl, A., Schwentick, T., Segoufin, L.:Two-variable logic on words with data. In: Proceedings of the Symposium on Logic in Computer Science (LICS'06), IEEE Computer Society Press, Los Alamitos pp. 7–16, (2006)
2. Grumberg, O., Kupferman, O., Sheinvald, S.: Variable automata over infinite alphabets. In: Proceedings of LATA 2010. LNCS, vol. 6031, pp. 561–572. Springer, Heidelberg (2010)
3. Grumberg, O., Kupferman, O., Sheinvald, S.: Variable automata over infinite alphabets. http://www.cs.huji.ac.il/~ornak/publications/lata10.pdf
4. Kaminski, M., Francez, N.: Finite-memory automata. Theoret. Comput. Sci. 134, 329–363 (1994)
5. I.E. Mens, G. Rahonis, Variable tree automata over infinite ranked alphabets, in: Proceedings of CAI 2011, LNCS, vol. 6742, pp. 247-260. Springer, Heidelberg (2011)

References

6. Neven, F., Schwentick, T., Vianu, V.: Towards regular languages over infinite alphabets. In: Proceedings of MFCS 2001. LNCS, vol. 2136, pp. 560–572. Springer, Heidelberg (2001)

7. Neven, F., Schwentick, T., Vianu, V.: Finite state machines for strings over infinite alphabets. ACM Trans. Comput. Log. 5, 403–435 (2004)

8. Shemesh, Y., Francez, N.: Finite-state unification automata and relational languages. Infom. and Comput. 114, 192–213 (1994)

9. Thatcher, J.W., Wright, J.B.: Generalized finite automata theory with application to a decision problem of second-order logic. Math. Systems Theory 2, 57–81 (1968)

Omitted Proofs - Basic Lemma

Lemma

$L(\mathcal{A}) = L(\mathcal{A}_{(\Sigma',V)})$

Proof.

Let $w = \alpha_1\alpha_2...\alpha_n \in \Sigma^*$, $\alpha_i \in \Sigma$, $1 \leq i \leq n$

- We will show $L(\mathcal{A}) \subseteq L(\mathcal{A}_{(\Sigma',V)})$

Let $w \in L(\mathcal{A})$

$$\left.\begin{array}{l} \exists u = u_1u_2...u_n \in \Gamma_A^* \ , \quad u_i \in \Gamma_A \ , \quad 1 \leq i \leq n \\ f \in VR(\Gamma_A) \end{array}\right\} \Longrightarrow \ w \in f(u)$$

By construction of $\mathcal{A}_{(\Sigma',V)}$

$$\exists g \in V \ \text{ such that } \begin{cases} g(z) = f(z) & \forall z \in Z_f \\ g(y) \cap \Sigma' = f(y) \cap \Sigma' \end{cases}$$

Proof.

We define for $1 \leq i \leq n$

$$u_i' = \begin{cases} u_i & \text{if } (u_i \in \Sigma_A \cup (Z \setminus Z_f)) \text{ or } (u_i = y \text{ and } w_i \notin \Sigma' \setminus \Sigma_A) \\ w_i & \text{if } (u_i \in Z_f) \text{ or } (u_i = y \text{ and } w_i \in \Sigma' \setminus \Sigma_A). \end{cases}$$

$$\Longrightarrow u' = u_1' u_2' \ldots u_n' \in L(A_g)$$

$f' \in VR(\Gamma_V)$

$$\begin{array}{ll} f'(z) = f(z) & \text{for every } z \in (Z \setminus Z_f) \\ f'(z) \in \Sigma \setminus (\Sigma_A \cup \Sigma' \cup f(Z \setminus Z_f)) & \text{for every } z \in (Z_f) \\ f'(y) = \Sigma \setminus (\Sigma_A \cup \Sigma' \cup f'(Z)) & \end{array}$$

$$w \in f'(u') \Longrightarrow w \in f'(L(A_g)) \Longrightarrow w \in L(\mathcal{A}_{(\Sigma', V)})$$

Omitted Proofs - Basic Lemma

Proof.

- We will show $L(\mathcal{A}_{(\Sigma',V)}) \subseteq L(\mathcal{A})$

Let $w \in L(\mathcal{A}_{(\Sigma',V)})$

$$\left.\begin{array}{l} \exists\ u = u_1 u_2 \ldots u_n \in L(A_{(\Sigma',V)}),\ u_i \in \Gamma_V \quad 1 \leq i \leq n \\ \exists\ f \in VR\left(\Gamma_V\right) \end{array}\right\} \Longrightarrow w \in f(u)$$

$$\exists\ g \in V : u \in L\left(A_g\right) \Longrightarrow w \in f(L(A_g))$$

By construction of A_g
$u' = u'_1 u'_2 \ldots u'_n \in L(A) : u \in g'(u')$

Proof.

where $g' : \Gamma_A \to \mathcal{P}(\Sigma) \cup Z \cup \{y\}$

$$g'(\sigma) = \begin{cases} g(\sigma) & \sigma \in \Sigma_A \\ g(\sigma) & \sigma \in Z_g \\ \sigma & \sigma \in Z \setminus Z_g \\ g(y) \cup \{y\} & \sigma = y \end{cases}$$

Now we consider $f' : \Gamma_A \to \Sigma$

$$\begin{cases} f'(a) = a & a \in \Sigma_A \\ f'(z) = f(z) & z \in Z \setminus Z_g \\ f'(z) = g(z) & z \in Z_g \\ f'(y) = f(y) \cup ((g(y) \cap \Sigma') \setminus g(Z_g)) & \end{cases}$$

$$\left.\begin{array}{c} f' \in VR(\Gamma_A) \\ w \in f'(u') \end{array}\right\} \Longrightarrow w \in L(\mathcal{A})$$

□

Omitted Proofs - Union

Proposition

The class $VREC(\Sigma)$ is closed under union.

Proof.

$$\mathcal{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathcal{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(1)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$Q^{(1)} \cap Q^{(2)} = \emptyset \qquad Z^{(1)} \cap Z^{(2)} = \emptyset \qquad y^{(1)} \neq y^{(2)}$$

$$A^{(1)}_{(\Sigma^{(2)}, V_1)} = \left(Q^{(1)}_{V_1}, \Gamma^{(1)} \cup \Sigma^{(2)}, Q^{(1)}_{0_{V_1}}, \Delta^{(1)}_{V_1}, F^{(1)}_{V_1} \right)$$

$$A^{(2)}_{(\Sigma^{(1)}, V_2)} = \left(Q^{(2)}_{V_2}, \Gamma^{(2)} \cup \Sigma^{(1)}, Q^{(2)}_{0_{V_2}}, \Delta^{(2)}_{V_2}, F^{(2)}_{V_2} \right)$$

$$Q^{(1)}_{V_1} \cap Q^{(2)}_{V_2} = \emptyset$$

□

Omitted Proofs - Union

Proof.

disjoint union of $A^{(1)}_{(\Sigma^{(2)},V_1)}$ and $A^{(2)}_{(\Sigma^{(1)},V_2)}$

$$A = (Q^{(1)}_{V_1} \cup Q^{(2)}_{V_2}, \Gamma, \overline{\Delta}^{(1)}_{V_1} \cup \overline{\Delta}^{(2)}_{V_2}, Q^{(1)}_{0_{V_1}} \cup Q^{(2)}_{0_{V_2}}, F^{(1)}_{V_1} \cup F^{(2)}_{V_2})$$

- $\Gamma = \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \cup Z^{(2)} \cup \{y\}$
- transition set $\overline{\Delta}^{(i)}_{V_i}$ obtained from $\Delta^{(i)}_{V_i}$, $i = 1, 2$,

  by replacing every occurrence of $y^{(i)}$ with the new symbol y.

We will show that $L(\mathcal{A}) = L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$ where $\mathcal{A} = \langle \Sigma, A \rangle$

Omitted Proofs - Union

Proof.

- $L(\mathcal{A}) \subseteq L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$

$w \in L(\mathcal{A})$

$$\left.\begin{array}{l} \exists u \in L(A) \\ \exists f \in VR(\Gamma) \end{array}\right\} \Longrightarrow w \in f(u)$$

$\exists u' \in L\left(A^{(1)}_{(\Sigma^{(2)}, V_1)}\right) \cup L\left(A^{(2)}_{(\Sigma^{(1)}, V_2)}\right)$ such that

u is obtained from u' by replacing every occurrence of $y^{(1)}$ and $y^{(2)}$ with y

Omitted Proofs - Union

Proof.

$u' \in L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right) \cup L\left(A^{(2)}_{(\Sigma^{(1)},V_2)}\right)$

If $u' \in L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right)$

$$f' : \Gamma^{(1)} \cup \Sigma^{(2)} \to \Sigma$$

$$f'(\sigma) = \begin{cases} f(\sigma) & \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \\ f(y) \cup f\left(Z^{(2)}\right) & \sigma = y^{(1)} \end{cases}$$

$$\begin{aligned} f' \in VR\left(\Gamma^{(1)} \cup \Sigma^{(2)}\right) \text{ and } w \in f'(u') &\implies w \in L\left(\mathcal{A}^{(1)}_{(\Sigma^{(2)},V_1)}\right) \\ &\implies w \in L\left(\mathcal{A}^{(1)}\right) \end{aligned}$$

If $u' \in L\left(A^{(2)}_{(\Sigma^{(1)},V_2)}\right)$ similarly.

Omitted Proofs - Union

Proof.

- $L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \subseteq L(\mathcal{A})$

$$w \in L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$$

$$\text{If } w \in L\left(\mathcal{A}^{(1)}\right) \Longrightarrow w \in L\left(\mathcal{A}^{(1)}_{(\Sigma^{(2)},V_1)}\right) \Longrightarrow \left.\begin{array}{l} \exists u \in L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right) \\ \exists f \in VR(\Gamma^{(1)} \cup \Sigma^{(2)}) \end{array}\right\}$$

$$\Longrightarrow w \in f(u)$$

$$f' : \Gamma \rightarrow \Sigma$$

$$\begin{cases} f'(\sigma) = f(\sigma) & \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)} \cup Z^{(1)} \\ f'(z) \in \Sigma \setminus \left(\Sigma^{(1)} \cup \Sigma^{(2)} \cup f(Z^{(1)}) \cup \Sigma''\right) & z \in Z^{(2)} \end{cases}$$

Σ'' : all labels of w obtained by replacing all occurrences of y in u.

Omitted Proofs - Union

Proof.

u' the word obtained from u by replacing every occurrence of $y^{(1)}$ with y.

$u' \in L(A)$ and $w \in f'(u') \implies w \in L(\mathcal{A})$.

If $w \in L\left(\mathcal{A}^{(2)}\right)$ similarly

$$\left.\begin{array}{l} L(\mathcal{A}) \subseteq L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \\ L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right) \subseteq L(\mathcal{A}) \end{array}\right\} \implies L(\mathcal{A}) = L\left(\mathcal{A}^{(1)}\right) \cup L\left(\mathcal{A}^{(2)}\right)$$

□

Omitted Proofs - Intersection

Proposition

The class $VREC(\Sigma)$ is closed under intersection.

Proof.

$$\mathcal{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathcal{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(1)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$Z^{(1)} \cap Z^{(2)} = \emptyset \qquad y^{(1)} \neq y^{(2)}$$

$$A^{(1)}_{(\Sigma^{(2)}, V_1)} = \left(Q^{(1)}_{V_1}, \Gamma^{(1)} \cup \Sigma^{(2)}, Q^{(1)}_{0_{V_1}}, \Delta^{(1)}_{V_1}, F^{(1)}_{V_1} \right)$$

$$A^{(2)}_{(\Sigma^{(1)}, V_2)} = \left(Q^{(2)}_{V_2}, \Gamma^{(2)} \cup \Sigma^{(1)}, Q^{(2)}_{0_{V_2}}, \Delta^{(2)}_{V_2}, F^{(2)}_{V_2} \right)$$

Omitted Proofs - Intersection

Proof.

$((Z^{(1)} \cup Y^{(1)}) \times (Z^{(2)} \cup Y^{(2)})) \setminus \left\{ \left(y^{(1)}, y^{(2)} \right) \right\}$ alphabet

$R \subseteq ((Z^{(1)} \cup Y^{(1)}) \times (Z^{(2)} \cup Y^{(2)})) \setminus \left\{ \left(y^{(1)}, y^{(2)} \right) \right\}$ maximal subalphabet

(every element of $Z^{(1)}$ (resp. $Z^{(2)}$) occurs as a left (resp. right) coordinate in at most one pair in R)

Let $R_1, \dots, R_m$, and for every $1 \leq j \leq m$, we consider

$$A_{R_j} = \left(Q^{(1)}_{V_1} \times Q^{(2)}_{V_2}, \Gamma_{R_j}, Q^{(1)}_{0_{V_1}} \times Q^{(2)}_{0_{V_2}}, \Delta_{R_j}, F^{(1)}_{V_1} \times F^{(2)}_{V_2} \right)$$

Omitted Proofs - Intersection

Proof.

$$A_{R_j} = \left(Q^{(1)}_{V_1} \times Q^{(2)}_{V_2}, \Gamma_{R_j}, Q^{(1)}_{0_{V_1}} \times Q^{(2)}_{0_{V_2}}, \Delta_{R_j}, F^{(1)}_{V_1} \times F^{(2)}_{V_2}\right)$$

- $\Gamma_{R_j} = \Sigma^{(1)} \cup \Sigma^{(2)} \cup R_j \cup \left\{\left(y^{(1)}, y^{(2)}\right)\right\}$
  - R_j set of bounded variables
  - $(y^{(1)}, y^{(2)})$ free variable
- $\Delta_{R_j} = \{((q_1^{(1)}, q_1^{(2)}), \sigma, (q_2^{(1)}, q_2^{(2)})) \mid (q_1^{(i)}, \sigma, q_2^{(i)}) \in \Delta^{(i)}_{V_i},$
  $i = 1, 2,\ \sigma \in \Sigma^{(1)} \cup \Sigma^{(2)}\}$
  $\cup\{((q_1^{(1)}, q_1^{(2)}), (x^{(1)}, x^{(2)}), (q_2^{(1)}, q_2^{(2)})) \mid (q_1^{(i)}, x^{(i)}, q_2^{(i)}) \in \Delta^{(i)}_{V_i},$
  $i = 1, 2,\ (x^{(1)}, x^{(2)}) \in (R_j) \cup Y\}.$

We show that $L\left(\mathcal{A}^{(1)}\right) \cap L\left(\mathcal{A}^{(2)}\right) = L(\mathcal{A}_{R_1}) \cup \ldots \cup L(\mathcal{A}_{R_m})$

Omitted Proofs - Intersection

Proof.

$$L\left(\mathcal{A}^{(1)}\right) \cap L\left(\mathcal{A}^{(2)}\right) \subseteq L(\mathcal{A}_{R_1}) \cup \ldots \cup L(\mathcal{A}_{R_m})$$

$$w \in L\left(\mathcal{A}^{(1)}\right) \cap L\left(\mathcal{A}^{(2)}\right)$$

$$\begin{matrix} \exists u_1 \in L\left(A^{(1)}_{(\Sigma^{(2)}, V_1)}\right), f^{(1)} \in VR(\Gamma^{(1)} \cup \Sigma^{(2)}) \\ \exists u_2 \in L\left(A^{(2)}_{(\Sigma^{(1)}, V_2)}\right), f^{(2)} \in VR(\Gamma^{(2)} \cup \Sigma^{(1)}) \end{matrix} \Rightarrow w \in f^{(1)}(u_1) \cap f^{(2)}(u_2)$$

We get $dom(u_1) = dom(u_2) = dom(w)$.

$$\forall i \in dom(w) \begin{cases} w(i) \in \Sigma^{(1)} \cup \Sigma^{(2)} \Longrightarrow u_1(i) = u_2(i) = w(i) \\ \text{or} \\ w(i) \in \Sigma \setminus (\Sigma^{(1)} \cup \Sigma^{(2)}) \end{cases}$$

Omitted Proofs - Intersection

Proof.

The latter case implies:

- there are bounded variables $z^{(1)} \in Z^{(1)}, z^{(2)} \in Z^{(2)}$ such that $u_1(i) = z^{(1)},\ u_2(i) = z^{(2)}$ and $f^{(1)}(z^{(1)}) = f^{(2)}(z^{(2)}) = w(i)$, or
- there is a bounded variable $z^{(1)} \in Z^{(1)}$ such that $u_1(i) = z^{(1)},\ u_2(i) = y^{(2)}$ and $f^{(1)}(z^{(1)}) = w(i) \in f^{(2)}(y^{(2)})$, or
- there is a bounded variable $z^{(2)} \in Z^{(2)}$ such that $u_2(i) = z^{(2)}, u_1(i) = y^{(1)}$ and $f^{(2)}(z^{(2)}) = w(i) \in f^{(1)}(y^{(1)})$, or
- $u_1(i) = y^{(1)},\ u_2(i) = y^{(2)}$, and $w(i) \in f^{(1)}(y^{(1)}) \cap f^{(2)}(y^{(2)})$.

Omitted Proofs - Intersection

Proof.

By definition of the alphabets $R_1, \ldots, R_m$, $\exists\ 1 \le j \le m$:

$\{(u_1(i), u_2(i)) \mid i \in dom(w) \text{ and } w(i) \in \Sigma \setminus (\Sigma^{(1)} \cup \Sigma^{(2)})\} \subseteq R_j \cup Y$

We define $f \in VR(\Gamma_{R_j})$ by letting

$$f((x^{(1)}, x^{(2)})) =$$
$$\begin{cases} f^{(1)}(x^{(1)}) & \text{if } (x^{(1)}, x^{(2)}) \in \Big(Z^{(1)} \times (Z^{(2)} \cup Y^{(2)})\Big) \cap R_j \\ f^{(2)}(x^{(2)}) & \text{if } (x^{(1)}, x^{(2)}) \in \Big(Y^{(1)} \times Z^{(2)}\Big) \cap R_j \end{cases}$$

Consider the word u, with $dom(u) = dom(w)$
defined by $u(i) = \begin{cases} w(i) & \text{if } w(i) \in \Sigma^{(1)} \cup \Sigma^{(2)} \\ (u_1(i), u_2(i)) & \text{otherwise} \end{cases}$

Trivially $u \in L(A_{R_j}) \Longrightarrow w \in f(u) \Longrightarrow w \in L(\mathcal{A}_{R_j})$.

Omitted Proofs - Intersection

Proof.

Let $1 \leq j \leq m$ and $w \in L(\mathcal{A}_{R_j})$

$$\left.\begin{array}{l} \exists u \in L(A_{R_j}) \\ \exists f \in VR(\Gamma_{R_j}) \end{array}\right\} \Longrightarrow w \in f(u)$$

We consider $f^{(1)} \in VR(\Gamma^{(1)} \cup \Sigma^{(2)})$ and $f^{(2)} \in VR(\Gamma^{(2)} \cup \Sigma^{(1)})$ defined as follows:

- $f^{(1)}(z^{(1)}) = f((z^{(1)}, x^{(2)}))$
  $\forall z^{(1)} \in Z^{(1)} : \exists x^{(2)} \in Z^{(2)} \cup Y^{(2)}$ with $(z^{(1)}, x^{(2)}) \in R_j$
- $f^{(2)}(z^{(2)}) = f((x^{(1)}, z^{(2)}))$
  $\forall z^{(2)} \in Z^{(2)} : x^{(1)} \in Z^{(1)} \cup Y^{(1)}$ with $(x^{(1)}, z^{(2)}) \in R_j$
- $f^{(i)}\left(z^{(i)}\right) \in \Sigma \setminus \left(\Sigma^{(1)} \cup \Sigma^{(2)} \cup f(R_j) \cup (f(Y) \cap u(dom(u)))\right)$
  for every remaining bounded variable $z^{(i)} \in Z^{(i)}, \quad i = 1, 2$

Omitted Proofs - Intersection

Proof.

Let u_1 and u_2 be the projections of u on $\Gamma^{(1)} \cup \Sigma^{(2)}$ and $\Gamma^{(2)} \cup \Sigma^{(1)}$, respectively

By construction of the nfa A_{R_j}, we get that

$$u_1 \in L\left(A^{(1)}_{(\Sigma^{(2)}, V_1)}\right) \text{ and } u_2 \in L\left(A^{(2)}_{(\Sigma^{(1)}, V_2)}\right)$$

By definition of $f^{(1)}$ and

$$f^{(2)} \Longrightarrow w \in f^{(1)}(u_1) \cap f^{(2)}(u_2) \Longleftrightarrow w \in L\left(\mathcal{A}^{(1)}\right) \cap L\left(\mathcal{A}^{(2)}\right)$$

We complete our proof by taking into account closure under union.

□

Omitted Proofs - Equivalence Problem

Theorem

The equivalence problem is decidable for

(i) *vfa whose transitions do not contain any free variable*

(ii) *vfa whose transitions do not contain any bounded variable.*

Proof.

Let

$$\mathrm{A}^{(1)} = \left\langle \Sigma, A^{(1)} \right\rangle \qquad \mathrm{A}^{(2)} = \left\langle \Sigma, A^{(2)} \right\rangle$$

$$A^{(i)} = (Q^{(1)}, \Gamma^{(1)}, Q_0^{(1)}, \Delta^{(1)}, F^{(1)}) \qquad A^{(2)} = (Q^{(2)}, \Gamma^{(2)}, Q_0^{(2)}, \Delta^{(2)}, F^{(2)})$$

$$\Gamma^{(1)} = \Sigma^{(1)} \cup Z^{(1)} \cup \left\{ y^{(1)} \right\} \qquad \Gamma^{(2)} = \Sigma^{(2)} \cup Z^{(2)} \cup \left\{ y^{(2)} \right\}$$

$$Z^{(1)} \cap Z^{(2)} = \emptyset \qquad y^{(1)} \neq y^{(2)}$$

Proof.

Case (i) The transitions of $\mathrm{A}^{(1)}$ and $\mathrm{A}^{(2)}$ do not contain free variables

Consider

$$A^{(1)}_{(\Sigma^{(2)},V_1)} = (Q^{(1)}_{V_1}, \Gamma^{(1)} \cup \Sigma^{(2)}, Q^{(1)}_{0_{V_1}}, \Delta^{(1)}_{V_1}, F^{(1)}_{V_1})$$

$$A^{(2)}_{(\Sigma^{(1)},V_2)} = \left(Q^{(2)}_{V_2}, \Gamma^{(2)} \cup \Sigma^{(1)}, Q^{(2)}_{0_{V_2}}, \Delta^{(2)}_{V_2}, F^{(2)}_{V_2}\right)$$

We consider the alphabet

$$\Theta = (\Sigma^{(1)} \cup \Sigma^{(2)}) \cup (Z^{(1)} \times Z^{(2)})$$

$pr_i : \Theta \longrightarrow (\Sigma^{(1)} \cup \Sigma^{(2)}) \cup Z^{(i)}$ the projection relabelings, $i = 1, 2$

$$L_1 = L\left(A^{(1)}_{(\Sigma^{(2)},V_1)}\right) \qquad L_2 = \left(A^{(2)}_{(\Sigma^{(1)},V_2)}\right)$$

$$L'_1 = pr_1^{-1}(L_1) \qquad L'_2 = pr_2^{-1}(L_2)$$

Omitted Proofs - Equivalence Problem

Proof.

Let G be a maximal subalphabet of Θ

(Every element of $Z^{(1)}$ (resp. $Z^{(2)}$) occurs as a left (resp. right) coordinate in at most one pair in G)

Let $G_1, \ldots, G_m$ be an enumeration of all maximal subalphabets of Θ

Let $L = (L_1' \cap L_2') \cap (G_1^* \cup \ldots \cup G_m^*)$ □

Lemma

The following statements are equivalent

(i) $pr_1(L) = L_1$ *and* $pr_2(L) = L_2$.

(ii) $L(A^{(1)}) = L\left(\mathcal{A}^{(2)}\right)$.

Proof.

Assume that (i) holds.

Let $w \in L(\mathrm{A}^{(1)})$

then, $\left.\begin{array}{l} \exists\ u_1 \in L_1 \\ \exists\ f^{(1)} \in VR(\Gamma^{(1)} \cup \Sigma^{(2)}) \end{array}\right\} \Longrightarrow w \in f^{(1)}(u_1)$

Let $u \in L$ such that $pr_1(u) = u_1$ and let $pr_2(u) = u_2$
where $dom(u_1) = dom(u) = dom(u_2)$

Let also $u \in G_j^*$, $1 \leq j \leq m$

We define $f^{(2)} \in VR(\Gamma^{(2)} \cup \Sigma^{(1)})$:

$$f^{(2)}(z^{(2)}) = w(i) = f^{(1)}(z^{(1)}) \quad \forall i \in dom(u) \text{ with } u(i) = (z^{(1)}, z^{(2)}), \\ z^{(1)} \in Z^{(1)},\ z^{(2)} \in Z^{(2)}$$

Omitted Proofs - Equivalence Problem

Proof.

We obtain that $w \in f^{(2)}(u_2)$ and since $u_2 \in L_2$, we get $w \in L(\mathrm{A}^{(2)})$.

$$\left.\begin{array}{l} \text{Thus } L(\mathcal{A}^{(1)}) \subseteq L(\mathcal{A}^{(2)}) \\ L(\mathcal{A}^{(2)}) \subseteq L(\mathcal{A}^{(1)}) \text{ (in a similar way)} \end{array}\right\} \Longrightarrow L(\mathrm{A}^{(1)}) = L(\mathrm{A}^{(2)})$$

Assume next that (ii) holds

and we know that $pr_1(L) \subseteq L_1$ and $pr_2(L) \subseteq L_2$.

We will show that $pr_1(L) \supseteq L_1$ and $pr_2(L) \supseteq L_2$.

Let $u_1 \in L_1$ and consider

$$\left.\begin{array}{l} f^{(1)} \in VR(\Gamma^{(1)} \cup \Sigma^{(2)}) \\ \text{and } w \in f^{(1)}(u_1) \end{array}\right\} \Longrightarrow w \in L(\mathrm{A}^{(1)}) \Longrightarrow w \in L(\mathrm{A}^{(2)})$$

Omitted Proofs - Equivalence Problem

Proof.

$$w \in L(\mathrm{A}^{(2)}) \Longrightarrow \left.\begin{array}{l} \exists\ u_2 \in L_2 \\ \exists\ f^{(2)} \in VR(\Gamma^{(2)} \cup \Sigma^{(1)}) \end{array}\right\} \Longrightarrow w \in f^{(2)}(u_2)$$

We claim that
there exists a word $u \in L_1' \cap L_2' : pr_1(u) = u_1$ and $pr_2(u) = u_2$

IF NOT, then there is a position $i \in dom(u_1) = dom(u_2)$ such that

$$\left.\begin{array}{l} u_1(i) \in \Sigma^{(1)} \cup \Sigma^{(2)} \text{ and } u_2(i) \in Z^{(2)} \\ \text{or} \\ u_1(i) \in Z^{(1)} \text{ and } u_2(i) \in \Sigma^{(1)} \cup \Sigma^{(2)} \end{array}\right\} \begin{array}{c} \text{contradiction because} \\ w \in f^{(1)}(u_1) \cap f^{(2)}(u_2) \end{array}$$

Now we claim that $u \in G_1^* \cup \ldots \cup G_m^*$

Omitted Proofs - Equivalence Problem

Proof.

Assume that $u \notin G_1^* \cup \ldots \cup G_m^*$

Then there are two positions $i, j \in dom(u)$ and bounded variables $z^{(1)}, \overline{z}^{(1)} \in Z^{(1)}$, $z^{(2)}, \overline{z}^{(2)} \in Z^{(2)}$ such that

(1) $u(i) = (z^{(1)}, z^{(2)})$ and $u(j) = (z^{(1)}, \overline{z}^{(2)})$, or
(2) $u(i) = (z^{(1)}, z^{(2)})$ and $u(j) = (\overline{z}^{(1)}, z^{(2)})$.

If (1) holds. Then

$$\left.\begin{array}{l} u_1(i) = u_1(j) = z^{(1)}, \text{ and} \\ u_2(i) = z^{(2)}, u_2(j) = \overline{z}^{(2)} \end{array}\right\} \Longrightarrow \begin{array}{l} f^{(1)}(u_1(i)) = f^{(1)}(u_1(j)) \\ f^{(2)}(u_2(i)) \neq f^{(2)}(u_2(j)) \end{array}$$

contradiction because $w \in f^{(1)}(u_1) \cap f^{(2)}(u_2)$.

The case (2) contradicts to our assumption, similarly

Omitted Proofs - Equivalence Problem

Proof.

Thus, we get $u \in L$ which implies $L_1 \subseteq pr_1(L)$ □

Proof.

$$\left.\begin{array}{l} L_1, L_2 \in REC \Longrightarrow L'_1, L'_2 \in REC \\ G_1^* \cup \ldots \cup G_m^* \in REC \Rightarrow L \in REC \\ pr_1(L),\ pr_2(L) \in REC \\ \text{The equality of recognizable} \\ \quad\quad \text{languages is decidable} \end{array}\right\} \Rightarrow \begin{array}{l} \text{the equalities} \\ pr_1(L) = L_1 \text{ and} \\ pr_2(L) = L_2 \\ \text{are decidable} \end{array}$$

Hence $L(\mathrm{A}^{(1)}) = L\left(\mathcal{A}^{(2)}\right)$ □

Omitted Proofs - Equivalence Problem

Proof.

Case (ii) The transitions of $A^{(1)}$ and $A^{(2)}$ without bounded variables

Without any loss $\Sigma^{(1)} = \Sigma^{(2)}$

if not, then on the vfa $A^{(1)}$ (similarly for $A^{(2)}$)

$\forall\ (q, y, p) \in \Delta^{(1)} \Longrightarrow$ we add the transition $(q, \sigma, p),\ \forall \sigma \in \Sigma^{(2)} \setminus \Sigma^{(1)}$

- Consider the alphabet $\Theta = \Sigma^{(1)} \cup \left\{ (y^{(1)}, y^{(2)}) \right\}$
- the projection relabelings $pr_i : \Theta \longrightarrow \Sigma^{(1)} \cup \left\{ y^{(i)} \right\}$, $i = 1, 2$

$$L_1 = L\left(A^{(1)}\right),\ L_2 = \left(A^{(2)}\right),\ L_1' = pr_1^{-1}(L_1),\ L_2' = pr_2^{-1}(L_2)$$

$$L' = L_1' \cap L_2'$$

Lemma

The following statements are equivalent

(i) $pr_1(L') = L_1$ *and* $pr_2(L') = L_2$.

(ii) $L(A^{(1)}) = L\left(\mathcal{A}^{(2)}\right)$.

Proof.

Assume that (i) holds.

Let $w \in L(\mathrm{A}^{(1)})$, then $\left.\begin{array}{l} \exists\ u_1 \in L_1 \\ \exists\ f^{(1)} \in VR(\Gamma^{(1)}) \end{array}\right\} \Longrightarrow w \in f^{(1)}(u_1)$

Let $u \in L'$ such that $pr_1(u) = u_1$ and let $pr_2(u) = u_2$ where $dom(u_1) = dom(u) = dom(u_2)$

Omitted Proofs - Equivalence Problem

Proof.

We define $f^{(2)} \in VR(\Gamma^{(2)})$:

$$f^{(2)}(y^{(2)}) = f^{(1)}(y^{(1)}) \quad \forall j \in dom(u) \text{ with } u(j) = (y^{(1)}, y^{(2)}),$$

We obtain that $w \in f^{(2)}(u_2)$ and since $u_2 \in L_2$, we get $w \in L(\mathrm{A}^{(2)})$.

$$\left.\begin{array}{l} \text{Thus } L(\mathcal{A}^{(1)}) \subseteq L(\mathcal{A}^{(2)}) \\ L(\mathcal{A}^{(2)}) \subseteq L(\mathcal{A}^{(1)}) \text{ (in a similar way)} \end{array}\right\} \Longrightarrow L(\mathrm{A}^{(1)}) = L(\mathrm{A}^{(2)})$$

Next, assume that (ii) holds

We will show that $pr_1(L') = L_1$ and $pr_2(L') = L_2$

Omitted Proofs - Equivalence Problem

Proof.

let $u_1 \in L_1$

$$\left.\begin{array}{l}\text{Consider an } f^{(1)} \in VR(\Gamma^{(1)}) \\ \text{and a word } w \in f^{(1)}(u_1)\end{array}\right\} \Longrightarrow w \in L(\mathsf{A}^{(1)}) \Longrightarrow w \in L(\mathsf{A}^{(2)}) \Longrightarrow$$

$$\left.\begin{array}{c}\exists u_2 \in L_2 \\ \exists\ f^{(2)} \in VR(\Gamma^{(2)})\end{array}\right\} \Longrightarrow w \in f^{(2)}(u_2)$$

We claim that there exists a word $u \in L'$

such that $pr_1(u) = u_1$ and $pr_2(u) = u_2$

If not, then there is a position $j \in dom(u_1) = dom(u_2)$ such that

$$\left.\begin{array}{c}u_1(j) \in \Sigma^{(1)} \text{ and } u_2(j) = y^{(2)} \\ \text{or} \\ u_1(j) = y^{(1)} \text{ and } u_2(j) \in \Sigma^{(1)}\end{array}\right\} \Longrightarrow \begin{array}{c}\text{contradiction because} \\ w \in f^{(1)}(u_1) \cap f^{(2)}(u_2)\end{array}$$

Hence, $L_1 \subseteq pr_1^{-1}(L')$

and $L_2 \subseteq pr_2^{-1}(L')$ is shown similarly □

Omitted Proofs - Equivalence Problem

Proof.

(ii) We use the notations of Lemma and similar arguments with the proof of Statement (i) in this theorem. □

The End!