FIRST-ORDER LOGIC: REASONING ABOUT EQUALITY

Course “Computational Logic”

Wolfgang Schreiner
Research Institute for Symbolic Computation (RISC)
Wolfgang.Schreiner@risc.jku.at

JOHANNES KEPLER UNIVERSITY LINZ

Equality

So far, the binary predicate symbol "=" has played no special role; however, due to its central role in mathematics, it deserves particular attention.

- Standard: First-Order Logic with Equality
  - Most important logic in general practice.
  - First-order logic where "=" has the fixed interpretation "equality".
    - Normal model: a structure where $=$ is interpreted as "equality".
    - Simple approach: add explicit equality axioms to every proving problem.
    - More comprehensive: extend first-order proof calculus by rules for equality.
- Alternative: Equational Logic
  - A restricted subset of predicate logic.
  - The only predicate is "=" (other predicates simulated as functions into Bool).
    - Implement special (semi-)decision procedure for this logic.

We will now sketch these alternatives in turn.

Equality Axioms

Equality is the equivalence relation that is a congruence for every predicate/function.

$$\forall x.\ x = x \tag{1}$$

$$\forall x, y.\ x = y \Rightarrow y = x \tag{2}$$

$$\forall x, y, z.\ x = y \land y = z \Rightarrow x = z \tag{3}$$

$$\forall x_1, \dots, x_n, y_1, \dots, y_n.\ x_1 = y_1 \land \dots \land x_n = y_n \Rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n) \tag{4}$$

$$\forall x_1, \dots, x_n, y_1, \dots, y_n.\ x_1 = y_1 \land \dots \land x_n = y_n \Rightarrow p(x_1, \dots, x_n) \Leftrightarrow p(y_1, \dots, y_n) \tag{5}$$

- Axioms (1-3): $=$ is reflexive, symmetric, transitive, i.e., $=$ an equivalence relation.
- Axiom schemes (4-5): $=$ is a function/predicate congruence.
  - One instance of the schemes for every function symbol f and every predicate symbol p.
- Theorem: Let Δ be a set of formulas and $eq(\Delta)$ be the equivalence relation axioms together with the instances of the congruence schemes for every function/predicate in Δ. Then Δ is satisfiable by a normal model (valid in all normal models) if and only if $\Delta \cup eq(\Delta)$ is satisfiable (valid).
  - Proof sketch: Any model of $\Delta \cup eq(\Delta)$ can be lifted to a normal model of Δ by partitioning the domain into equivalence classes according to the interpretation of $=$.

Implementation in OCaml

```
let function_congruence (f,n) = ... ;;
let predicate_congruence (p,n) = ... ;;

let equivalence_axioms =
  [<<forall x. x = x>>; <<forall x y z. x = y /\ x = z ==> y = z>>];;

let equalitize fm =
  let allpreds = predicates fm in
  if not (mem ("=",2) allpreds) then fm else
  let preds = subtract allpreds ["=",2] and funcs = functions fm in
  let axioms = itlist (union ** function_congruence) funcs
                      (itlist (union ** predicate_congruence) preds
                              equivalence_axioms) in
  Imp(end_itlist mk_and axioms,fm);;
```

A special version of transitivity that also implies symmetry.

Implementation in OCaml

```
# let ewd = equalitize
 <<(forall x. f(x) ==> g(x)) /\ (exists x. f(x)) /\ (forall x y. g(x) /\ g(y) ==> x = y)
   ==> forall y. g(y) ==> f(y)>>;;
val ewd : fol formula =
  <<(forall x. x = x) /\ (forall x y z. x = y /\ x = z ==> y = z) /\
    (forall x1 y1. x1 = y1 ==> f(x1) ==> f(y1)) /\
    (forall x1 y1. x1 = y1 ==> g(x1) ==> g(y1)) ==>
    (forall x. f(x) ==> g(x)) /\
    (exists x. f(x)) /\ (forall x y. g(x) /\ g(y) ==> x = y) ==>
    (forall y. g(y) ==> f(y))>>

# splittab ewd ;;
Searching with depth limit 0
...
Searching with depth limit 9
- : int list = [9]
```

Simple approach but not very effective in more complex examples.

Sequent Calculus and Equality

We may extend the sequent calculus by the “core” of the equality axioms.

$$\frac{\Gamma, x = y \Rightarrow F[x] \Leftrightarrow F[y] \vdash \Delta}{\Gamma \vdash \Delta}\ \text{(SUBST)} \qquad \frac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta}\ \text{(REFL)}$$

- Rule (SUBST) represents Leibnitz’s law (the principle of substitutivity):
  - Formula $F[y]$ is identical to $F[x]$ except that *any* (not necessarily *all*) free occurrences of x may be replaced by y (which must remain free in F).
- Rule (SUBST) is equivalent to the more special congruence rules:

$$\frac{\Gamma, t_1 = u_1 \land \ldots \land t_n = u_n \Rightarrow f(t_1, \ldots, t_n) = f(u_1, \ldots, u_n) \vdash \Delta}{\Gamma \vdash \Delta}\ \text{(CONGF)}$$

$$\frac{\Gamma, t_1 = u_1 \land \ldots \land t_n = u_n \Rightarrow p(t_1, \ldots, t_n) \Leftrightarrow p(u_1, \ldots, u_n) \vdash \Delta}{\Gamma \vdash \Delta}\ \text{(CONGP)}$$

- From rules (SUBST) and (REFL), also symmetry and transitivity can be derived.

The extended calculus is sound and complete (with respect to normal models) but very inefficient to implement automatically.

First-Order Tableaux and Equality

The method of firder-order tableaux extended by the following rules:

$$\frac{\begin{array}{c} t = u \\ F[t] \end{array}}{F[u]} \qquad \frac{}{t = t}$$

- Replacement: If a branch contains the equality $t = u$ and the formula $F[t]$ with an occurrence of term t that is not in the scope of any quantifier, the branch can be extended by $F[u]$ which is a duplicate of $F[t]$ except that the occurrence of t in $F[t]$ has been replaced by term u in $F[u]$.
- Reflexivity: We may add to any branch the equality $t = t$ for an arbitrary term t.

The extended calculus is sound and complete: if a closed tableau can be derived, its root formula is not satisfiable by any normal model, and vice versa.

Example

Proof of $\forall x.\ \forall y.\ \forall z.\ x = y \wedge y = z \Rightarrow x = z$:

1.	$\neg\forall x.\ \forall y.\ \forall z.\ x = y \wedge y = z \Rightarrow x = z$	
2.	$\neg\forall y.\ \forall z.\ c = y \wedge c = z \Rightarrow c = z$	(1)
3.	$\neg\forall z.\ c = d \wedge d = z \Rightarrow c = z$	(2)
4.	$\neg(c = d \wedge d = e \Rightarrow c = e)$	(3)
5.	$c = d \wedge d = e$	(4)
6.	$\neg(c = e)$	(4)
7.	$c = d$	(5)
8.	$d = e$	(5)
9.	$c = e$	(7,8)
	(6,9)	

Proof of $\forall x.\ \forall y.\ x = y \Rightarrow y = x$:

1.	$\neg\forall x.\ \forall y.\ x = y \Rightarrow y = x$	
2.	$\neg\forall y.\ c = y \Rightarrow y = c$	(1)
3.	$\neg(c = d \Rightarrow d = c)$	(2)
4.	$c = d$	(3)
5.	$\neg(d = c)$	(3)
6.	$\neg(d = d)$	(4,5)
7.	$d = d$	
	(6,7)	

Free-Variable Tableaux and Equality

The method of free-variable tableaux extended by the following rules:

$$\frac{\begin{array}{c} t = u \\ F[t'] \end{array}}{F[u]} \qquad \frac{}{x = x} \qquad \frac{}{f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)}$$

- MGU Replacement: if $t = u$ and $F[t']$ occur in the same branch of tableau T and σ is a most general unifier of t and t', then we may replace tableau T by $T'\sigma$ where T' is identical to T except that $F[u]$ has been added to the branch.
- Reflexivity: We may add to every branch the equality $x = x$ where x is a fresh variable.
- Function Reflexivity: We may add to every branch the equality $f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ where f is an n-ary function symbol and $x_1, \ldots, x_n$ are fresh variables.

The extended calculus is sound and complete: if a closed tableau can be derived, its root formula is not satisfiable by any normal model, and vice versa.

Example

Proof of $\forall x.\ \exists y.\ (y = f(x) \land \forall z.\ (z = f(x) \Rightarrow y = z))$:

1.	$\neg\forall x.\ \exists y.\ (y = f(x) \land \forall z.\ (z = f(x) \Rightarrow y = z))$	
2.	$\neg\exists y.\ (y = f(c) \land \forall z.\ (z = f(c) \Rightarrow y = z))$	(1)
3.	$\neg(y_1 = f(c) \land \forall z.\ (z = f(c) \Rightarrow y_1 = z))$	(2)

4.	$\neg(y_1 = f(c))$	(3)
5.	$y_2 = y_2$	
	(4,5)	

4.	$\neg\forall z.\ (z = f(c) \Rightarrow f(c) = z)$	(3)
5.	$\neg(d = f(c) \Rightarrow f(c) = d)$	(4)
6.	$d = f(c)$	(5)
7.	$\neg(f(c) = d)$	(5)
8.	$\neg(f(c) = f(c))$	(6,7)
9.	$y_3 = y_3$	
	(8,9)	

Tableau closed with $\sigma = [y_1 \mapsto f(c), y_2 \mapsto f(c), y_3 \mapsto f(c)]$.

Paramodulation

An extension of first-order resolution (George Robinson and Lawrence Wos, 1969).

$$\frac{C \cup \{L[t]\} \in F \qquad D \cup \{s = u\} \in F \qquad \sigma \text{ is mgu of t and s} \qquad C \cup \{L[t]\} \text{ and } D \cup \{s = u\} \text{ have no common variables} \qquad F \cup \{C\sigma \cup D\sigma \cup \{L[u]\sigma\}\} \vdash}{F \vdash} \text{(PARA)}$$

- The paramodulation rule (PARA):
  - Literal $L[t]$ with an occurrence of term t that is replaced by term u in $L[u]$.
  - Clause $C\sigma \cup D\sigma \cup \{L[u]\sigma\}$ is the paramodulant of $C \cup \{L[t]\}$ and $D \cup \{s = u\}$.
- The paramodulation calculus consists of rules (AX), (RES), (REN), (FACT), (PARA).
  - Soundness: if $F \cup feq(F) \vdash$ can be derived, F is not satisfiable by a normal model.
  - Completeness: if F is not satisf. by a normal model, $F \cup feq(F) \vdash$ can be derived.
    - $feq(F)$ consists of the reflexivity axiom $x = x$ and one function reflexivity axiom $f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ for every n-ary function symbol f in F.
    - In most proofs, function reflexivity axioms are not needed; thus many implementations only use the reflexity axiom.

A much more restricted form of the application of equalities.

Example

We show the unsatisfiability of

$$\{\{q(c)\}, \{\neg q(c), f(x) = x\}, \{p(x), p(f(c))\}, \{\neg p(x), \neg p(f(x))\}\}$$

by the following refutation (here reflexivity is not needed):

$$\begin{tikzcd}
\{q(c)\} \arrow[ddr] & & \{\neg q(c), f(x) = x\} \arrow[ddl] & & & \{p(x), p(f(c))\} \arrow[d] & & \{\neg p(x), \neg p(f(x))\} \arrow[ddl] \\
 & & & & & \{p(f(c))\} \arrow[dr] \arrow[dddl] & & \\
 & \{f(x) = x\} \arrow[drr] & & & & & \{\neg p(f(f(c)))\} \arrow[dlll] & \\
 & & & \{\neg p(f(c))\} \arrow[dr] & & & & \\
 & & & & \{\,\} & & &
\end{tikzcd}$$

3 resolution steps, 1 paramodulation step, 1 factorization step.

Paramodulation in OCaml

```
let rec overlapl (l,r) fm rfn = (* Find paramodulations with l = r inside a literal fm. *)
  match fm with
    Atom(R(f,args)) -> listcases (overlaps (l,r))
                                (fun i a -> rfn i (Atom(R(f,a)))) args []
  | Not(p) -> overlapl (l,r) p (fun i p -> rfn i (Not(p)))
  | _ -> failwith "overlapl: not a literal";;

(* Now find paramodulations within a clause. *)
let overlapc (l,r) cl rfn acc = listcases (overlapl (l,r)) rfn cl acc;;

(* Overall paramodulation of ocl by equations in pcl. *)
let paramodulate pcl ocl =
  itlist (fun eq -> let pcl' = subtract pcl [eq] in
                    let (l,r) = dest_eq eq
                    and rfn i ocl' = image (subst i) (pcl' @ ocl') in
                    overlapc (l,r) ocl rfn ** overlapc (r,l) ocl rfn)
         (filter is_eq pcl) [];;
```

Paramodulation in OCaml

```
let para_clauses cls1 cls2 =
  let cls1' = rename "x" cls1 and cls2' = rename "y" cls2 in
  paramodulate cls1' cls2' @ paramodulate cls2' cls1';;

let rec paraloop (used,unused) = (* Incorporation into resolution loop. *)
  match unused with
    [] -> failwith "No proof found"
  | cls::ros ->
        print_string(string_of_int(length used) ^ " used; "^
                     string_of_int(length unused) ^ " unused.");
        print_newline();
        let used' = insert cls used in
        let news =
          itlist (@) (mapfilter (resolve_clauses cls) used')
            (itlist (@) (mapfilter (para_clauses cls) used') []) in
        if mem [] news then true else
        paraloop(used',itlist (incorporate cls) news ros);;
```

Paramodulation in OCaml

```
let pure_paramodulation fm =
  paraloop([],[mk_eq (Var "x") (Var "x")]::simpcnf(specialize(pnf fm)));;

let paramodulation fm =
  let fm1 = askolemize(Not(generalize fm)) in
  map (pure_paramodulation ** list_conj) (simpdnf fm1);;

# paramodulation
 <<(forall x. f(f(x)) = f(x)) /\ (forall x. exists y. f(y) = x)
   ==> forall x. f(x) = x>>;;
0 used; 4 unused.
...
10 used; 108 unused.
11 used; 125 unused.
- : bool list = [true]
```

The naive application of paramodulation leads to huge proof search spaces; in practice, strong restrictions and sophisticated strategies are implemented.

Equational Logic

Let Δ be a set of equations of form $t = u$ which are implicitly universally quantified.

$$\frac{(s = t) \in \Delta}{\Delta \vdash s = t}\ \text{(AXIOM)} \qquad \frac{\Delta \vdash s = t}{\Delta \vdash (s = t)[u/x]}\ \text{(INST)}$$

$$\frac{}{\Delta \vdash t = t}\ \text{(REFL)} \qquad \frac{\Delta \vdash u = t}{\Delta \vdash t = u}\ \text{(SYM)} \qquad \frac{\Delta \vdash t = s \quad \Delta \vdash s = u}{\Delta \vdash t = u}\ \text{(TRANS)}$$

$$\frac{\Delta \vdash t_1 = u_1 \quad \dots \quad \Delta \vdash t_n = u_n}{\Delta \vdash f(t_1, \dots, t_n) = f(u_1, \dots, u_n)}\ \text{(CONG)}$$

- Judgement $\Delta \vdash t = u$
  - Interpreted as “every normal model of Δ satisfies $t = u$”.
  - Equivalent to: $\Delta \models t = u$ holds in first-order logic with equality.
- Birkhoff’s Theorem (Garrett Birkhoff, 1935):
  - If $\Delta \vdash s = t$ is derivable by above inference rules (the “Birkhoff rules”), then every normal model of Δ satisfies $t = u$, and vice versa.

Birkhoff’s rules denote a sound and complete inference calculus for equational logic; like first-order logic, however, equational logic is undecidable.

Equational Proving

- Let set Δ consist of the following equations:

$$g(x, c) = x \tag{1}$$

$$g(x, f(y)) = f(g(x, y)) \tag{2}$$

$$h(x, c) = c \tag{3}$$

$$h(x, f(y)) = g(x, h(x, y)) \tag{4}$$

- How to prove $\Delta \models h(f(f(c)), f(f(c))) = g(h(f(c), f(c)), f(f(f(c))))$?

$$\begin{aligned}
\underline{h(f(f(c)), f(f(c)))} &\overset{(4)}{=} g(f(f(c)), h(f(f(c)), f(c))) \overset{(4)}{=} g(f(f(c)), g(f(f(c)), h(f(f(c)), c))) \\
&\overset{(3)}{=} g(f(f(c)), g(f(f(c)), c)) \overset{(1)}{=} g(f(f(c)), f(f(c))) \overset{(2)}{=} f(g(f(f(c)), f(c))) \\
&\overset{(2)}{=} f(f(g(f(f(c)), c))) \overset{(1)}{=} \underline{f(f(f(f(c))))} \\
\underline{g(h(f(c), f(c)), f(f(f(c))))} &\overset{(4)}{=} g(g(f(c), h(f(c), c)), f(f(f(c)))) \overset{(3)}{=} g(g(f(c), c), f(f(f(c)))) \\
&\overset{(1)}{=} g(f(c), f(f(f(c)))) \overset{(2)}{=} f(g(f(c), f(f(c)))) \overset{(2)}{=} f(f(g(f(c), f(c)))) \\
&\overset{(2)}{=} f(f(f(g(f(c), c)))) \overset{(1)}{=} \underline{f(f(f(f(c))))}
\end{aligned}$$

By a sequence of equality substitutions in the left term and a sequence of equality substitutions in the right term the same term can be derived; thus the left term and the right term are equal.

Equational Proving

We have just performed a strategy of “simplifying calculations”.

- Set Δ described some arithmetic axioms:

$$x + 0 = x \tag{1}$$
$$x + (y') = (x + y)' \tag{2}$$
$$x \cdot 0 = 0 \tag{3}$$
$$x \cdot (y') = x + (x \cdot y) \tag{4}$$

- We have proved $\Delta \models (0'') \cdot (0'') = ((0') \cdot (0')) + (0''')$ (i.e., $2 \cdot 2 = 1 + 3$):

$$\begin{aligned}
\underline{(0'') \cdot (0'')} &\overset{(4)}{=} (0'') + ((0'') \cdot (0')) \overset{(4)}{=} (0'') + ((0'') + ((0'') \cdot 0)) \\
&\overset{(3)}{=} (0'') + ((0'') + 0) \overset{(1)}{=} (0'') + (0'') \overset{(2)}{=} ((0'') + (0'))' \\
&\overset{(2)}{=} ((0'') + 0)'' \overset{(1)}{=} \underline{0''''} \\
\underline{((0') \cdot (0')) + (0''')} &\overset{(4)}{=} ((0') + ((0') \cdot 0)) + (0''') \overset{(3)}{=} ((0') + 0) + (0''') \\
&\overset{(1)}{=} (0') + (0''') \overset{(2)}{=} ((0') + (0''))' \overset{(2)}{=} ((0') + (0'))'' \\
&\overset{(2)}{=} ((0') + 0)''' \overset{(1)}{=} \underline{0''''}
\end{aligned}$$

When can this strategy be performed?

Term Rewriting

Consider the elements of Δ not as equations but as (left-to-right) rewrite rules.

- Abstract reduction system $(S, \rightarrow)$: a set S and a binary relation $\rightarrow$ on S.
  - $x \leftrightarrow y$: $x \rightarrow y$ or $y \rightarrow x$.
  - $x \rightarrow^* y$ and $x \leftrightarrow^* y$: the reflexive transitive closure of $\rightarrow$ and $\leftrightarrow$.
- Term rewriting system: an abstract reduction system induced by Δ.
  - S is the set of terms and $\rightarrow$ is the "term rewriting relation" generated by Δ when considering every equation $t = u$ as a (left-to-right) rewrite rule.
- Theorem: Let $\rightarrow$ be the term rewriting relation induced by Δ. Then we have $\Delta \models t = u$ if and only if $t \leftrightarrow^* u$.
  - Proof sketch: If $\Delta \models t = u$, by Birkhoff's theorem $\Delta \vdash t = u$ is derivable. One can show by induction on the Birkhoff rules that this implies $t \leftrightarrow^* u$. Conversely, by the semantics of substitution $t \rightarrow u$ implies $\Delta \models t = u$; from this one can show by induction that also $t \leftrightarrow^* u$ implies $\Delta \models t = u$.

To show $\Delta \models t = u$ it suffices to show $t \leftrightarrow^* u$.

Term Rewriting as a Decision Strategy

Some fundamental notions and properties of an abstract reduction system $(S, \rightarrow)$.

- Element $x \in S$ is a normal form: there is no $y \in S$ such that $x \rightarrow y$.
- $\rightarrow$ is terminating (Noetherian): there are no infinite reduction sequences $x_0 \rightarrow x_1 \rightarrow \cdots$, i.e., every reduction sequence ends with a normal form $x_n \in S$.
- $\rightarrow$ has the Church-Rosser property: if $x \leftrightarrow^* y$, then $x \rightarrow^* z$ and $y \rightarrow^* z$ for some $z \in S$.
  - Lemma: If $\rightarrow$ has the Church-Rosser property, then for every $x \in S$ there exists *at most* one normal form $x' \in S$ such that $x \rightarrow^* x'$.
- $\rightarrow$ is canonical: $\rightarrow$ is terminating and also has the Church rosser property.
  - Lemma: If $\rightarrow$ is canonical, then for every $x \in S$ there exists *exactly one* normal form $x' \in S$ such that $x \rightarrow^* x'$.
- Theorem (Trevor Evans, 1951): If $\rightarrow$ is canonical, then $x \leftrightarrow^* y$ if and only if $x \rightarrow^* x'$ and $y \rightarrow^* y'$ with $x' = y'$ for normal forms $x' \in S$ and $y' \in S$.

If Δ induces a canonical term rewriting system, we can decide $\Delta \models t = u$ by rewriting terms t and u to normal forms t' and u' and comparing t' with u'.

Term Rewriting in OCaml

```
let rec rewrite1 eqs t = (* Rewriting at the top level with first of list of equations. *)
  match eqs with
    Atom(R("=",[l;r]))::oeqs ->
     (try tsubst (term_match undefined [l,t]) r
      with Failure _ -> rewrite1 oeqs t)
  | _ -> failwith "rewrite1";;

let rec rewrite eqs tm = (* Rewriting repeatedly and at depth (top-down). *)
  try rewrite eqs (rewrite1 eqs tm) with Failure _ ->
  match tm with
    Var x -> tm
  | Fn(f,args) -> let tm' = Fn(f,map (rewrite eqs) args) in
                  if tm' = tm then tm else rewrite eqs tm';;

# rewrite [<<0 + x = x>>; <<S(x) + y = S(x + y)>>;
         <<0 * x = 0>>; <<S(x) * y = y + x * y>>]
        <<|S(S(S(0))) * S(S(0)) + S(S(S(S(0))))|>>;;
- : term = <<|S(S(S(S(S(S(S(S(S(S(0))))))))))|>>
```

Non-Canonical Term Rewriting

- Not Terminating:

$$x + y = y + x \tag{1}$$

$$c + d \longrightarrow d + c \longrightarrow c + d \longrightarrow \cdots$$

- No Church-Rosser Property:

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{1}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z \tag{2}$$

$$\begin{aligned}
(a+b)\cdot(c+d) &\overset{(2)}{\longrightarrow} a\cdot(c+d) + b\cdot(c+d) \\
&\overset{(1)}{\longrightarrow} (a\cdot c + a\cdot d) + b\cdot(c+d) \overset{(1)}{\longrightarrow} (a\cdot c + a\cdot d) + (b\cdot c + b\cdot d) \\
(a+b)\cdot(c+d) &\overset{(1)}{\longrightarrow} (a+b)\cdot c + (a+b)\cdot d \\
&\overset{(2)}{\longrightarrow} (a\cdot c + b\cdot c) + (a+b)\cdot d \overset{(2)}{\longrightarrow} (a\cdot c + b\cdot c) + (a\cdot d + b\cdot d)
\end{aligned}$$

If a term rewriting system is not canonical, rewriting fails as a decision strategy.

Ensuring Termination

- It is generally undecidable whether a term rewriting system is terminating.
  - Term rewriting systems can perform arbitrary computations.
  - The problem whether computing machines halt is undecidable (Alan Turing, 1937).
- But we can prove that a particular term rewriting system is terminating.
  - Determine a suitable termination ordering, i.e., a well-founded relation on terms that is decreased by the application of every rewrite rule.
  - One such termination ordering is the lexicographic path order $t > u$ defined as follows:
    - $t > u$, if u is a proper subterm of t.
    - $f(t_1, \ldots, t_n) > t$, if $t_i > t$ for some i.
    - $f(t_1, \ldots, t_n) > f(u_1, \ldots, u_n)$ if $t_i > u_i$ for some i and $t_j = u_j$ for all $j < i$.
    - $f(t_1, \ldots, t_n) > g(u_1, \ldots, u_m)$, if $f > g$ for some ordering of function/constant symbols.

    In the last two rules we additionally require $f(t_1, \ldots, t_n) > u_i$ for every i.
- Example: consider the lexicographic path order for '$\cdot$' > '+' > '$'$' > '0'.
  - $x + 0 > x$ because x is a proper subterm of $x + 0$.
  - $x + (y') > (x + y)'$ because '+' > '$'$' and $x + (y') > x + y$ (why?).
  - $x \cdot 0 > 0$ because 0 is a proper subterm of $x \cdot 0$.
  - $x \cdot (y') > x + (x \cdot y)$ because '$\cdot$' > '+' and $x \cdot (y') > x$ and $x \cdot (y') > x \cdot y$ (why?).

Thus the previously stated arithmetic term rewriting system is terminating.

Ensuring the Church-Rosser Property

- Does the following term rewriting system have the Church-Rosser Property?

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{1}$$

$$1 \cdot x = x \tag{2}$$

$$i(x) \cdot x = 1 \tag{3}$$

- We can rewrite term $(1 \cdot x) \cdot y$ in two different ways:

$$(1 \cdot x) \cdot y \overset{(1)}{\rightarrow} 1 \cdot (x \cdot y)$$

$$(1 \cdot x) \cdot y \overset{(2)}{\rightarrow} x \cdot y$$

- This does not violate the property, because both results have the same normal form:

$$1 \cdot (x \cdot y) \overset{(2)}{\rightarrow} x \cdot y$$

- But we can also rewrite term $(i(x) \cdot x) \cdot y$ in two different ways:

$$(i(x) \cdot x) \cdot y \overset{(1)}{\rightarrow} i(x) \cdot (x \cdot y)$$

$$(i(x) \cdot x) \cdot y \overset{(3)}{\rightarrow} 1 \cdot y \overset{(2)}{\rightarrow} y$$

- Thus we have derived two different normal forms which violates the Church-Rosser property.

This may spark the idea of how to decide the Church-Rosser property.

Ensuring the Church-Rosser Property

- Reduction relation $\rightarrow$ is locally confluent if the following property holds: if $x \rightarrow y_1$ and $x \rightarrow y_2$, then $y_1 \rightarrow^* z$ and $y_2 \rightarrow^* z$ for some $z \in S$.
  - Newman's Lemma: If a reduction relation $\rightarrow$ is both terminating and locally confluent, it is confluent (and thus has the Church-Rosser property).
- Thus, given a set Δ of rewrite rules whose reduction relation $\rightarrow$ is terminating, the following algorithm decides whether $\rightarrow$ has the Church-Rosser property:
  - Consider every pair $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ of rewrite rules (both rules may be the same).
    - Rename the variables in these rules such that variables in l_1 and l_2 are disjoint.
  - Determine every critical pair of these rules, i.e., terms $\underline{r_1\sigma}$ and $\underline{l_1[r_2]\sigma}$ such that
    - l_1 contains an occurrence of a non-variable term l_2' where
    - l_2' can be unified with l_2 by a most general unifier σ and
    - $l_1[r_2]$ denotes l_1 with l_2' replaced by r_2.

$$\begin{tikzcd}
\boxed{l_1 \boxed{l_2'}}\,\sigma \arrow[r] \arrow[d, equal] & \boxed{r_1}\,\sigma \\
\boxed{l_1 \boxed{l_2}}\,\sigma \arrow[r] & \boxed{l_1 \boxed{r_2}}\,\sigma
\end{tikzcd}$$

  and we thus have the potentially divergent reductions $l_1\sigma \rightarrow \underline{r1\sigma}$ and $l_1\sigma \rightarrow \underline{l_1[r_2]\sigma}$.
  - The reduction reduction system has the Church-Rosser property if and only if every critical pair y_1 and y_2 can be rewritten by $\rightarrow$ to a common normal form z.

The Church-Rosser property can be decided via critical pair computation.

Computing Critical Pairs

- Example: equations $x_1 + 0 = x_1$ and $x_2 + 0 = x_2$ (the first equation renamed).
  - $x_1 + 0$ and $x_2 + 0$ have mgu $[x_1 \mapsto x_2]$ which yields the trivial critical pair x_2 and x_2.
  - The arithmetic system has only trivial critical pairs and thus is Church-Rosser.
  - We only need to consider the overlap of a rule with itself at a proper subterm of the left side.
- Example: $\Delta := \{f(g(f(x))) \xrightarrow{r} g(x)\}$
  - Rule instances $f(g(f(x_1))) \xrightarrow{r_1} g(x_1), f(g(f(x_2))) \xrightarrow{r_2} g(x_2)$
  - Unify $f(x_1)$ and $f(g(f(x_2)))$ with mgu $\sigma = [x_1 \mapsto g(f(x_2))]$.
  - Reduction $f(g(f(g(f(x_2))))) \xrightarrow{r_1} g(g(f(x_2)))$ with normal form $g(g(f(x_2)))$.
  - Reduction $f(g(f(g(f(x_2))))) \xrightarrow{r_2} f(g(g(x_2)))$ with normal form $f(g(g(x_2)))$.
  - Critical pair $g(g(f(x_2)))$ and $f(g(g(x_2)))$ with different normal forms.
  - Δ does not have the Church-Rosser property.

Critical Pairs in OCaml

```
let renamepair (fm1,fm2) = ... ;;
let rec listcases fn rfn lis acc = (* Rewrite with l = r inside tm to give a critical pair. *)
  match lis with
    [] -> acc
  | h::t -> fn h (fun i h' -> rfn i (h'::t)) @ listcases fn (fun i t' -> rfn i (h::t')) t acc;;
let rec overlaps (l,r) tm rfn =
  match tm with
    Fn(f,args) -> listcases (overlaps (l,r)) (fun i a -> rfn i (Fn(f,a))) args
                        (try [rfn (fullunify [l,tm]) r] with Failure _ -> [])
  | Var x -> [];;

let crit1 (Atom(R("=",[l1;r1]))) (Atom(R("=",[l2;r2]))) =
  overlaps (l1,r1) l2 (fun i t -> subst i (mk_eq t r2));;
let critical_pairs fma fmb = (* Generate all critical pairs between two equations. *)
  let fm1,fm2 = renamepair (fma,fmb) in
  if fma = fmb then crit1 fm1 fm2
  else union (crit1 fm1 fm2) (crit1 fm2 fm1);;

# let eq = <<f(f(x)) = g(x)>> in critical_pairs eq eq;;
- : fol formula list = [<<f(g(x0)) = g(f(x0))>>; <<g(x1) = g(x1)>>]
```

Knuth-Bendix Completion

A semi-algorithm to derive a canonical term rewriting system (Donald Knuth and Peter Bendix, 1970).

```
procedure COMPLETE($\Delta$)        ▹ if the procedure terminates, it returns a canonical system equivalent to $\Delta$
    $\Delta_1 \leftarrow \Delta$
    repeat                                                                          ▹ may not terminate
        $\Delta_0 \leftarrow \Delta_1$
        for every critical pair $(t, u)$ in $\Delta_0$ do
            reduce $t$ and $u$ to normal forms $t'$ and $u'$ according to $\Delta_0$          ▹ may not terminate
            if $t' \neq u'$ then
                choose $l = r \in \{t = u, u = t\}$
                $\Delta_1 \leftarrow \Delta_1 \cup \{l = r\}$
            end if
        end for
    until $\Delta_1 = \Delta_0$
    return $\Delta_1$
end procedure
```

There are numerous improvements to increase the practical applicability.

Knuth Bendix Completion

- Example: $\Delta := \{f(g(f(x))) \xrightarrow{r} g(x)\}$
  - Rule instances $f(g(f(x_1))) \xrightarrow{r_1} g(x_1), f(g(f(x_2))) \xrightarrow{r_2} g(x_2)$
    - Unify $f(x_1)$ and $f(g(f(x_2)))$ with mgu $\sigma = [x_1 \mapsto g(f(x_2))]$.
    - Reduction $f(g(f(g(f(x_2))))) \xrightarrow{r_1} g(g(f(x_2)))$ with normal form $g(g(f(x_2)))$.
    - Reduction $f(g(f(g(f(x_2))))) \xrightarrow{r_2} f(g(g(x_2)))$ with normal form $f(g(g(x_2)))$.
    - Critical pair $g(g(f(x_2)))$ and $f(g(g(x_2)))$ with different normal forms.
    - $\Delta' := \{f(g(f(x))) \xrightarrow{r} g(x), g(g(f(x))) \xrightarrow{s} f(g(g(x)))\}$
  - Rule instances $g(g(f(x_1))) \xrightarrow{s_1} f(g(g(x_1)))$ and $g(g(f(x_2))) \xrightarrow{s_2} f(g(g(x_2)))$
    - Only trivial mgu $[x_1 \rightarrow x_2]$ and trivial critical pair.
  - Rule instances $f(g(f(x_1))) \xrightarrow{r_1} g(x_1)$ and $g(g(f(x_2))) \xrightarrow{s_1} f(g(g(x_2)))$
    - Unify $f(x_2)$ and $f(g(f(x_1)))$ with mgu $[x_2 \mapsto g(f(x_1))]$.
    - $g(g(f(g(f(x_1))))) \xrightarrow{r_1} g(g(g(x_1)))$ with normal form $g(g(g(x_1)))$.
    - $g(g(f(g(f(x_1))))) \xrightarrow{s_1} f(g(g(g(f(x_1))))) \xrightarrow{s_1} f(g(f(g(g(x_1))))) \xrightarrow{r_1} g(g(g(x_1)))$.
    - Critical pair $g(g(g(x_1)))$ and $f(g(f(g(g(x_1)))))$ has common normal form.
  - No more non-trivial rule overlaps.

Δ' has the Church-Rosser property.

The Case of Variable-Free Equations

Our goal is to derive $\Delta \vdash (t = u)$.

- Consider the special case of only variable-free equations in $\Delta \vdash (t = u)$.
  - Any occurrence of a symbol x in $t = u$ does not denote any more a "variable" (that is universally quantified in the equation) but a "constant" (whose value is the same in all equations in which x occurs).
- Then proofs need not apply the Birkhoff rule (INST).
- This makes the theory decidable.

We will next consider decision procedures for variable-free equational logic and other decidable theories.