Equational Weighted Tree Transformations[1]

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[1] Common work with Symeon Bozapalidis and Zoltán Fülöp

Motivation

- Weighted tree transducers are currently used in practical applications in *Computational Linguistics*
- Question: *How can we specify the weighted tree transformation computed by a weighted tree transducer?*
- Usual method in other frameworks: *Associate a system of equations to the model, and compute the behavior of the model by solving the system*
- Examples: Finite automata, context free grammars, tree automata, ...

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- Σ, Δ, Γ: ranked alphabets

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- $L \subseteq T_\Sigma(X_n)$: *tree language*

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- *relabeling* if $\forall k \geq 0$, $\sigma \in \Sigma_k$ we have $h_k(\sigma) = \delta(\xi_1, \ldots, \xi_k)$ for some $\delta \in \Delta_k$

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Tree transformations

Example

$\sigma \in \Sigma_3$, $\delta \in \Delta_2$, $(s,t) = (\sigma(x_1, x_1, x_3), \delta(x_3, x_1))$,
$R_1 = \{(s_1, t_1), (s_1', t_1')\}$, $R_2 = \emptyset$, $R_3 = \{(s_3, t_3)\}$

$$(s,t)\,[R_1, R_2, R_3]_{[IO]} = \{(\sigma(s_1, s_1, s_3), \delta(t_3, t_1)), (\sigma(s_1', s_1', s_3), \delta(t_3, t_1'))\}$$

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$$\mathcal{A} = (A, +, 0, \Sigma^{\mathcal{A}})$$

where $(A, +, 0)$ is a K-semimodule, and $\Sigma^{\mathcal{A}} = (\sigma^{\mathcal{A}} \mid \sigma \in \Sigma)$ a family of multilinear operations on A such that $\forall n \geq 0$, $\sigma \in \Sigma_n$, we have $\sigma^{\mathcal{A}} : A^n \to A$ satisfying
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- if $\varphi_0 \leq \varphi_1 \leq \ldots$ is an ω-chain in $K\langle\langle T_\Sigma(X_n)\rangle\rangle$, then

$$\varphi_k = \sum_{0\leq i\leq k} \rho_i,$$

  where $\rho_i \in K\langle\langle T_\Sigma(X_n)\rangle\rangle$, $(i \geq 0)$, and thus

$$\sup_{k\geq 0}(\varphi_k) = \sum_{i\geq 0}\rho_i.$$

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  where $\mathbf{r}^{(i)} = \left(\left(s_1^{(i)}, t_1^{(i)}\right), \ldots, \left(s_{m_i}^{(i)}, t_{m_i}^{(i)}\right)\right)$,
  $\left(\tau_i, \mathbf{r}^{(i)}\right) = \left(\tau_i, \left(s_1^{(i)}, t_1^{(i)}\right)\right) \cdot \ldots \cdot \left(\tau_i, \left(s_{m_i}^{(i)}, t_{m_i}^{(i)}\right)\right)$, and
  $\left(\tau_i, \mathbf{r}^{(i)}\right) = 1$ if $\mathbf{r}^{(i)} = ()$ $\forall 1 \le i \le n$

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Weighted tree transformations: systems of equations

Definition

A *system of equations of weighted tree transformations over* (Σ, Δ, K) is a system

$$\text{(E)} \qquad x_i = \rho_i,\ 1 \leq i \leq n,$$

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Equational weighted tree transformations

Definition

$\tau \in K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ *u-equational* (u=[IO], OI) if it is a component of the least *u*-solution of a system of equations of weighted tree transformations over (Σ, Δ, K).

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Weighted bimorphisms

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- $\varphi \in K\langle\langle T_\Gamma(X_n)\rangle\rangle$
- h, h', and φ define a weighted tree transformation over (Σ, Δ, K) by:

$$\langle h, h'\rangle(\varphi) = \sum_{u\in T_\Gamma(X_n)} (\varphi, u).(h(u), h'(u))$$

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- A *weighted bimorphism over* $(\Gamma, \Sigma, \Delta, K)$:

$$(h, \varphi, h')$$

where $\varphi \in K\langle\langle T_\Gamma \rangle\rangle$ is a recognizable tree series,
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- $\langle h, h' \rangle\, (\varphi)$: the *weighted tree transformation computed by* (h, φ, h')

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- $B(uc(l\text{-}H, l\text{-}H))$

Equational weighted tree transformations: results

Theorem

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Rule-like equational weighted tree transformations

Definition

A system

$$(\mathrm{E}) \qquad x_i = \rho_i,\ 1 \leq i \leq n,$$

of equations of weighted tree transformations is called *rule-like* if each pair $(s, t) \in \mathrm{supp}(\rho_i)$ $(1 \leq i \leq n)$ has the form $(\sigma(x_{i_1}, \ldots, x_{i_k}), t)$, where $k \geq 0$, $\sigma \in \Sigma_k$, $\sigma(x_{i_1}, \ldots, x_{i_k})$ is linear, and $t \in T_\Delta(\{x_{i_1}, \ldots, x_{i_k}\})$ or the form (x_j, x_j).

rl-vi-$EQUT_u$: the class of all weighted tree transformations obtained as components of the least *u*-solutions of rule-like variable identical systems of equations of weighted tree transformations

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- $\mathit{ln\text{-}BOT} = B(REL, lc\text{-}H)$
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Equational weighted tree transformations: results

Corollary

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  - if $s = \sigma(s_1, \ldots, s_k)$ $k \geq 0$, $\sigma \in \Sigma_k$ and $s_1, \ldots, s_k \in T_\Sigma(X_n)$, then $s[a_1, \ldots, a_n]_{\mathcal{A}} = \sigma^{\mathcal{A}}(s_1[a_1, \ldots, a_n]_{\mathcal{A}}, \ldots, s_k[a_1, \ldots, a_n]_{\mathcal{A}})$

Mezei-Wright result

Algebras

- $|s|_{x_i} = \lambda_i$, $\mathbf{a}^{(i)} = \left(a_1^{(i)}, \ldots, a_{\lambda_i}^{(i)}\right) \in A^{\lambda_i}$ $\forall 1 \leq i \leq n$

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  - $s\left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\right]_{\mathcal{A}} = \sigma^{\mathcal{A}}\left(s_1\left[\mathbf{a}^{(1,1)}, \ldots, \mathbf{a}^{(1,n)}\right]_{\mathcal{A}}, \ldots, s_k\left[\mathbf{a}^{(k,1)}, \ldots, \mathbf{a}^{(k,n)}\right]_{\mathcal{A}}\right)$

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- there is a unique morphism $H_{\mathcal{A}} : K\langle\langle T_\Sigma \rangle\rangle \to K\langle\langle A \rangle\rangle$ given by

$$H_{\mathcal{A}}(\varphi) = \sum_{s \in T_\Sigma} (\varphi, s).H_{\mathcal{A}}(s)$$

- $\mathcal{A} = (A, +, 0, \Sigma^{\mathcal{A}})$ K-Σ-algebra

Mezei-Wright result

Algebras

- $\mathcal{A} = (A, +, 0, \Sigma^{\mathcal{A}})$ K-Σ-algebra
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Mezei-Wright result

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- $(s, t) \in T_\Sigma(X_n) \times T_\Delta(X_n)$,
  $|s|_{x_i} = \lambda_i$, $|t|_{x_i} = \mu_i$, $m_i = \max\{\lambda_i, \mu_i\}$ $\forall 1 \leq i \leq n$
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  $$\left(s\left[\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}\right]_{\mathcal{A}}, t\left[\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(n)}\right]_{\mathcal{B}}\right)$$
  $\mathbf{v}^{(i)} = \left(\left(a_1^{(i)}, b_1^{(i)}\right), \ldots, \left(a_{m_i}^{(i)}, b_{m_i}^{(i)}\right)\right)$,
  $\mathbf{a}^{(i)} = \left(a_1^{(i)}, \ldots, a_{\lambda_i}^{(i)}\right)$, $\mathbf{b}^{(i)} = \left(b_1^{(i)}, \ldots, b_{\mu_i}^{(i)}\right)$ $\forall 1 \leq i \leq n$

Mezei-Wright result

Algebras

- $(s,t) \in T_\Sigma(X_n) \times T_\Delta(X_n)\,,\;\; var(s) \cup var(t) = \{x_{i_1}, \ldots, x_{i_k}\}$

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Mezei-Wright result

Systems of equations

A system of equations of weighted tree transformations over (Σ, Δ, K)

$$(\mathrm{E}) \qquad x_i = \rho_i,\ 1 \leq i \leq n$$

- $(\theta_1, \ldots, \theta_n) \in (K\langle\langle A \times B \rangle\rangle)^n$ *u-solution* of (E) in $(\mathcal{A}, \mathcal{B}, K)$ if $\theta_i = \rho_i[\theta_1, \ldots, \theta_n]_u \quad \forall 1 \leq i \leq n$

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- $(\theta_1, \ldots, \theta_n) \in (K\langle\langle A \times B \rangle\rangle)^n$ *least u-solution* of (E) in $(\mathcal{A}, \mathcal{B}, K)$ if $\theta_i \leq \theta'_i$ $(1 \leq i \leq n)$ for every other u-solution $(\theta'_1, \ldots, \theta'_n)$ of (E) in $(\mathcal{A}, \mathcal{B}, K)$

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- $(\theta_1, \ldots, \theta_n) \in (K\langle\langle A \times B \rangle\rangle)^n$ *least u-solution* of (E) in $(\mathcal{A}, \mathcal{B}, K)$ if $\theta_i \leq \theta_i'$ $(1 \leq i \leq n)$ for every other u-solution $(\theta_1', \ldots, \theta_n')$ of (E) in $(\mathcal{A}, \mathcal{B}, K)$
- Existence of the least u-solution of (E) in $(\mathcal{A}, \mathcal{B}, K)$: as in tree transformations case
- $\theta \in K\langle\langle A \times B \rangle\rangle$ *u-equational* if it a component of the least u-solution in $(\mathcal{A}, \mathcal{B}, K)$ of a system of equations of weighted tree transformations

Mezei-Wright result

Theorem (Mezei-Wright)

A weighted transformation $\theta \in K\langle\langle A \times B \rangle\rangle$ is u-equational iff there exists a u-equational weighted tree transformation $\tau \in K\langle\langle T_\Sigma \times T_\Delta \rangle\rangle$ such that $H_{(\mathcal{A},\mathcal{B})}(\tau) = \theta$, where $H_{(\mathcal{A},\mathcal{B})}(\tau) = \sum_{(s,t) \in T_\Sigma \times T_\Delta} (\tau, (s,t)).(H_\mathcal{A}(s), H_\mathcal{B}(t))$.

Thank you