FIRST-ORDER LOGIC: PROOFS

Course “Computational Logic”

Wolfgang Schreiner
Research Institute for Symbolic Computation (RISC)
Wolfgang.Schreiner@risc.jku.at

JOHANNES KEPLER UNIVERSITY LINZ

First-Order Logic Proofs

Our core goal is to show the validity of first-order formulas.

- Problem: how to show $\models F$?
  - Does $M \models F$ hold for every structure M (i.e., is every structure M a model of F)?
  - But there are infinitely many structures with different domains and interpretations!

Can we reduce first-order reasoning to reasoning in some “canonical structures”?

Herbrand Structures

A Herbrand structure $H := (D_H, I_H)$ for a formula (language) with symbols $\mathcal{C}, \mathcal{F}, \mathcal{P}$ consists of the Herbrand universe D_H and some Herbrand interpretation I_H.

- The Herbrand universe D_H is the set of all terms t formed as follows:

$$t ::= c \mid f(t_1, \ldots, t_n)$$

  - Every constant $c \in \mathcal{C}$ (if $\mathcal{C} = \{\ \}$, we extend $\mathcal{C}$ by a constant c).
  - Every n-ary function symbol $f \in \mathcal{F}$.
  - D_H is the set of ground terms (no variables) that includes all constants and is closed under the application of all function symbols (thus D_H is generally infinite).
- A Herbrand interpretation I_H must satisfy the following:

$$I(c) := c\ (\in D_H) \qquad I(f)(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)\ (\in D_H) \qquad I(p)(t_1, \ldots, t_n) \subseteq D_H^n$$

  - I_H interprets constant c as itself, n-ary function symbol f as a term constructor, and n-ary predicate p as an arbitrary n-ary relation over D_H.

A Herbrand structure is a (generalization of a) “term algebra”.

Herbrand Structures as Models of Formulas

- Theorem: Let F be a quantifier-free formula. Then there exists a structure M with $M \models F$ if and only if there exists a Herbrand structure H with $H \models F$.
  - Proof sketch: Since the implication from right to left clearly holds, only the implication from left to right has to be shown. For this, we assume $M \models F$ for arbitrary structure $M = (D, I)$ and show $H \models F$ for the Herbrand structure $H = (D_H, I_H)$ over F with

    $$I_H(p)(t_1, \ldots, t_n) :\Leftrightarrow M \models p(t_1, \ldots, t_n)$$

    We take arbitrary valuation v_H over D_H and show $[\![F]\!]^H_{v_H}$ = true. Let $x_1, \ldots, x_n$ be the free variables of F and consider the closed formula instance $F' := F[v_H(x_1)/x_1, \ldots, v_H(x_n)/x_n]$. From $M \models F$, we can show $M \models F'$. Furthermore, we can show $[\![F]\!]^H_{v_H} = [\![F']\!]^M_{v'}$ for arbitrary valuation v' over D. From $M \models F'$, we have $[\![F']\!]^M_{v'}$ = true and thus also $[\![F]\!]^H_{v_H}$ = true.

Herbrand structures are “canonical structures” for reasoning in first-order logic; all proof calculi use these structures in some way or another.

The Sequent Calculus

An extension of the propositional sequent calculus by two additional rules.

$$\frac{\Gamma, A[t/x], (\forall x.\ A), \Delta \vdash \Lambda}{\Gamma, (\forall x.\ A), \Delta \vdash \Lambda}\ (\forall\text{-L}) \qquad \frac{\Gamma \vdash \Delta, A[y/x], \Lambda}{\Gamma \vdash \Delta, (\forall x.\ A), \Lambda}\ (\forall\text{-R})$$

$$\frac{\Gamma, A[y/x], \Delta \vdash \Lambda}{\Gamma, (\exists x.\ A), \Delta \vdash \Lambda}\ (\exists\text{-L}) \qquad \frac{\Gamma \vdash \Delta, A[t/x], (\exists x.\ A), \Lambda}{\Gamma \vdash \Delta, (\exists x.\ A), \Lambda}\ (\exists\text{-R})$$

- Substitution $F[t/x]$:
  - Substitution of term t for every free occurrence of variable x in formula F.
- Eigenvariable (Skolem constant) y
  - y must not occur in the conclusion of the rule.
- Witness term t
  - Term t may contain arbitrary variables, constants, and function symbols; however, every variable in t different from x must not be not bound by any quantifier in A.

Example Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(\overline{x},\overline{y}), \forall y.\ p(\overline{x},y) \vdash p(\overline{x},\overline{y}), \exists x.\ p(x,\overline{y})}\,\text{(AX)}}{p(\overline{x},\overline{y}), \forall y.\ p(\overline{x},y) \vdash \exists x.\ p(x,\overline{y})}\,\text{($\exists$-R)}}{\forall y.\ p(\overline{x},y) \vdash \exists x.\ p(x,\overline{y})}\,\text{($\forall$-L)}}{\exists x.\ \forall y.\ p(x,y) \vdash \exists x.\ p(x,\overline{y})}\,\text{($\exists$-L)}}{\exists x.\ \forall y.\ p(x,y) \vdash \forall y.\ \exists x.\ p(x,y)}\,\text{($\forall$-R)}}{\vdash (\exists x.\ \forall y.\ p(x,y)) \Rightarrow (\forall y.\ \exists x.\ p(x,y))}\,\text{($\Rightarrow$-R)}
$$

A simple proof that applies all quantifier rules.

Another Proof

- We may apply some additional "convenience" rules:

$$\frac{\Gamma, \Delta \vdash \Lambda}{\Gamma, A, \Delta \vdash \Lambda}\ \text{(DROP)} \qquad\qquad \frac{\Gamma \vdash \Delta, \Lambda}{\Gamma \vdash \Delta, A, \Lambda}\ \text{(DROP)}$$

  - Reduce size of sequent; soundness can be easily derived.

$$\dfrac{\dfrac{}{p(\overline{x}) \vdash p(\overline{x})}\ \text{(AX)} \qquad \dfrac{\dfrac{\dfrac{}{p(\overline{x}), q(\overline{x}, \overline{y}) \vdash q(\overline{x}, \overline{y})}\ \text{(AX)}}{p(\overline{x}), q(\overline{x}, \overline{y}) \vdash \exists x, y.\ q(x, y)}\ (\exists\text{-R,DROP})}{p(\overline{x}), \exists y.\ q(\overline{x}, y) \vdash \exists x, y.\ q(x, y)}\ (\exists\text{-L})}{\dfrac{p(\overline{x}), p(\overline{x}) \Rightarrow \exists y.\ q(\overline{x}, y) \vdash \exists x, y.\ q(x, y)}{\dfrac{p(\overline{x}), \forall x.\ p(x) \Rightarrow \exists y.\ q(x, y) \vdash \exists x, y.\ q(x, y)}{\dfrac{\exists x.\ p(x), \forall x.\ p(x) \Rightarrow \exists y.\ q(x, y) \vdash \exists x, y.\ q(x, y)}{\dfrac{(\exists x.\ p(x)) \wedge (\forall x.\ p(x) \Rightarrow \exists y.\ q(x, y)) \vdash \exists x, y.\ q(x, y)}{\vdash ((\exists x.\ p(x)) \wedge (\forall x.\ p(x) \Rightarrow \exists y.\ q(x, y))) \Rightarrow \exists x, y.\ q(x, y)}\ (\Rightarrow\text{-R})}\ (\wedge\text{-L})}\ (\exists\text{-L})}\ (\forall\text{-L,DROP})}\ (\Rightarrow\text{-L})$$

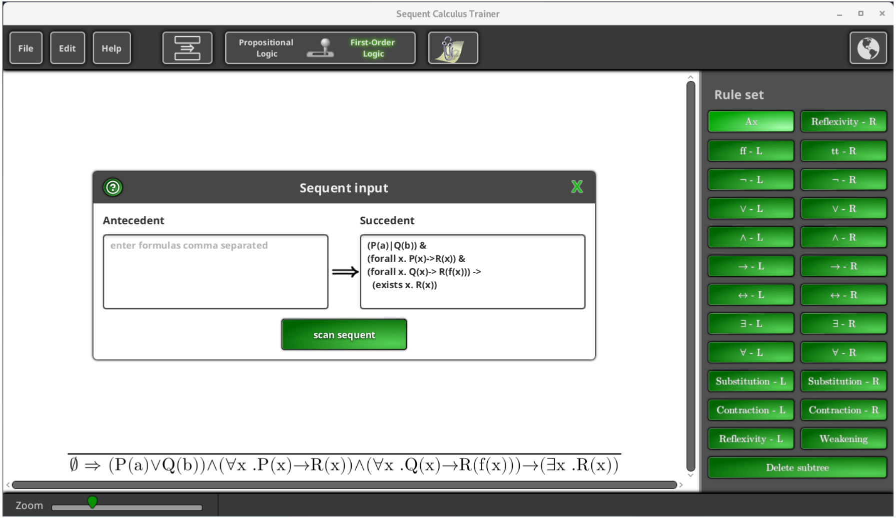
We may drop formulas that have served their purpose.

A Proof with More Branches

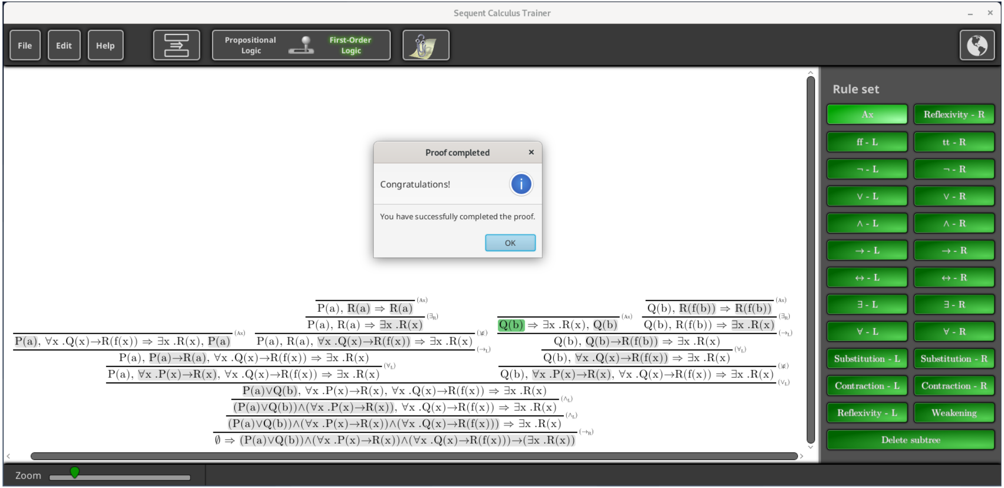
$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{p(a) \vdash p(a)}\;(\text{AX})
\qquad
\dfrac{\dfrac{}{p(a), r(a) \vdash r(a)}\;(\text{AX})}{p(a), r(a) \vdash \exists x.\, r(x)}\;(\exists\text{-R})
}{p(a), p(a) \Rightarrow r(a) \vdash \exists x.\, r(x)}\;(\Rightarrow\text{-L})
}{p(a), (\forall x.\, p(x) \Rightarrow r(x)) \vdash \exists x.\, r(x)}\;(\forall\text{-L,DROP})
\qquad
\dfrac{
\dfrac{
\dfrac{}{q(b), \vdash q(b)}\;(\text{AX})
\qquad
\dfrac{\dfrac{}{q(b), r(f(b)) \vdash r(f(b))}\;(\text{AX})}{q(b), r(f(b)) \vdash \exists x.\, r(x)}\;(\exists\text{-R})
}{q(b), q(b) \Rightarrow r(f(b)) \vdash \exists x.\, r(x)}\;(\Rightarrow\text{-L})
}{q(b), (\forall x.\, q(x) \Rightarrow r(f(x))) \vdash \exists x.\, r(x)}\;(\forall\text{-L,DROP})
}{
\dfrac{p(a) \lor q(b), (\forall x.\, p(x) \Rightarrow r(x)), (\forall x.\, q(x) \Rightarrow r(f(x))) \vdash \exists x.\, r(x)}{
\dfrac{(p(a) \lor q(b)), (\forall x.\, p(x) \Rightarrow r(x)) \land (\forall x.\, q(x) \Rightarrow r(f(x))) \vdash \exists x.\, r(x)}{
\dfrac{(p(a) \lor q(b)) \land (\forall x.\, p(x) \Rightarrow r(x)) \land (\forall x.\, q(x) \Rightarrow r(f(x))) \vdash \exists x.\, r(x)}{\vdash \big((p(a) \lor q(b)) \land (\forall x.\, p(x) \Rightarrow r(x)) \land (\forall x.\, q(x) \Rightarrow r(f(x)))\big) \Rightarrow \exists x.\, r(x)}\;(\Rightarrow\text{-R})
}\;(\land\text{-L})
}\;(\land\text{-L})
}\;(\lor\text{-L,DROP})
$$

A proof by “case distinction”.

Sequent Calculus Trainer

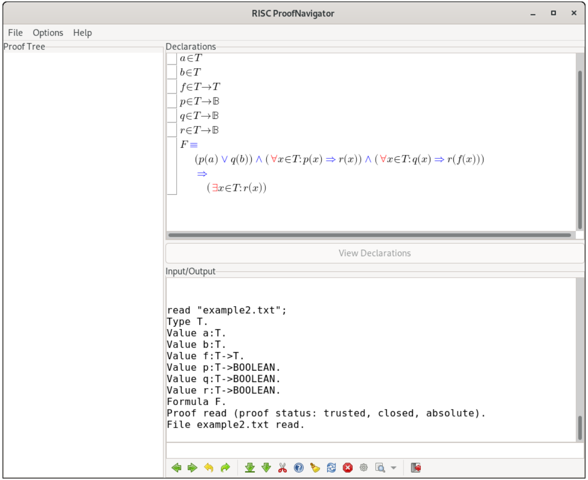


Sequent Calculus Trainer



RISC ProofNavigator

ProofNavigator &

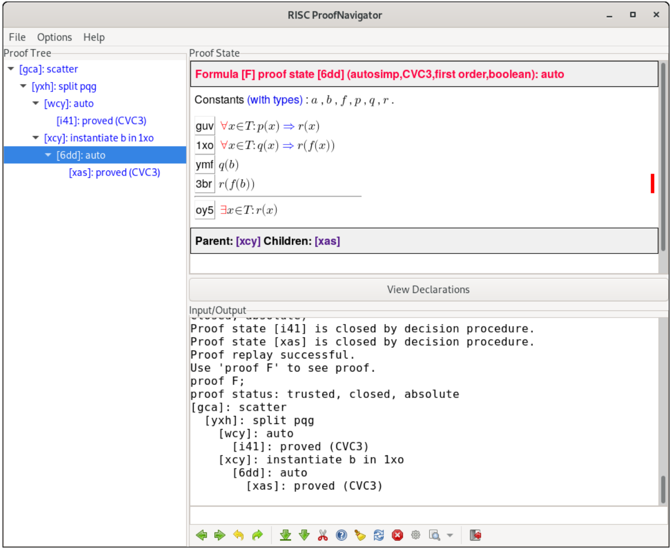



```
% example2.txt
newcontext "example2";

T:TYPE;
a:T;
b:T;
f:T->T;
p:(T)->BOOLEAN;
q:(T)->BOOLEAN;
r:(T)->BOOLEAN;

F: FORMULA
  (p(a) OR q(b)) AND
  (FORALL(x:T): p(x) => r(x)) AND
  (FORALL(x:T): q(x) => r(f(x))) =>
    (EXISTS(x:T): r(x));
```

RISC ProofNavigator



Soundness of the Sequent Calculus

Theorem: Every derivable sequent is valid.

Proof Sketch: It suffices to show that, if the conclusion of a rule is not valid, also some premise is not valid.

$$\frac{\Gamma, A[t/x], (\forall x.\ A), \Delta \vdash \Lambda}{\Gamma, (\forall x.\ A), \Delta \vdash \Lambda}\ (\forall\text{-L}) \qquad\qquad \frac{\Gamma \vdash \Delta, A[y/x], \Lambda}{\Gamma \vdash \Delta, (\forall x.\ A), \Lambda}\ (\forall\text{-R})$$

- Rule ($\forall$-L): Since the conclusion is not valid, we have some structure M and valuation v with $[\![\Gamma]\!]_v^M$ = true, $[\![\forall x.\ A]\!]_v^M$ = true, $[\![\Delta]\!]_v^M$ = true, and $[\![\Lambda]\!]_v^M$ = false. From above, to show that the premise is not valid, it suffices to show $[\![A[t/x]]\!]_v^M$ = true. Let $d := [\![t]\!]_v^M$. From the side condition on t, we can show $[\![A[t/x]]\!]_v^M = [\![A]\!]_{v[x \mapsto d]}^M$. From $[\![\forall x.\ A]\!]_v^M$ = true, we know $[\![A]\!]_{v[x \mapsto d]}^M$ = true and are done.
- Rule ($\forall$-R): Since the conclusion is not valid, we have some structure M and valuation v with $[\![\Gamma]\!]_v^M$ = true, $[\![\Delta]\!]_v^M$ = false, $[\![\forall x.\ A]\!]_v^M$ = false, and $[\![\Lambda]\!]_v^M$ = false. From $[\![\forall x.\ A]\!]_v^M$ = false, there is some $d \in D$ such that $[\![A]\!]_{v[x \mapsto d]}^M$ = false. Let $v' := v[y \mapsto d]$. Since y does not occur in the conclusion, we have $[\![\Gamma]\!]_{v'}^M$ = true, $[\![\Delta]\!]_{v'}^M$ = false, and $[\![\Lambda]\!]_{v'}^M$ = false. Thus, to show that the premise is not valid, it suffices to show $[\![A[y/x]]\!]_{v'}^M$ = false, i.e., $[\![A[y/x]]\!]_{v[y \mapsto d]}^M$ = false. Since y does not occur in A, we can show $[\![A[y/x]]\!]_{v[y \mapsto d]}^M = [\![A]\!]_{v[x \mapsto d]}^M$ = false and are done.
- Rules ($\exists$-L) and ($\exists$-R): analogously.

Proof Tree Construction: Data

To construct a proof tree for sequent $\Gamma \vdash \Delta$, we use the following data:

- $y = [y_0, y_1, \ldots]$: an infinite sequence of variables that do *not* occur in $\Gamma \vdash \Delta$.
  - These variables can be used as eigenvariables in rules ($\forall$-R) and ($\exists$-L).
- $a = [a_0, a_1, \ldots]$: an infinite sequence of term sequences:
  - The terms in these sequences are available as witnesses in rules ($\forall$-L) and ($\exists$-R).
    - If some function symbols occur in $\Gamma \vdash \Delta$, all sequences $a_0, a_1, \ldots$ are infinite.
  - $[t_0] \circ a_0 = [t_0, \ldots]$: an enumeration of all terms constructed from the free variables, constants, and function symbols in $\Gamma \vdash \Delta$.
    - If $\Gamma \vdash \Delta$ does not contain any free variable or constant, we use $t_0 := y_0$.
  - $a_{i \geq 1}$: an enumeration $[y_i, \ldots]$ of all terms that contain y_i and are constructed from $y_1, \ldots, y_i$ and the free variables, constants, and function symbols in $\Gamma \vdash \Delta$.

During the proof tree construction, the value of program variable n indicates that $y_1, \ldots, y_n$ have been used as eigenvariables in rules ($\forall$-R) or ($\exists$-L); the sequences $a_0, a_1, \ldots, a_n$ contain all terms in which these variables may occur.

Proof Tree Construction: Algorithm

procedure SEARCH($\Gamma \vdash \Delta$)
  INITIALIZE(y, a, t_0)
  $T, ts, n \leftarrow \langle \Gamma \vdash \Delta \rangle, [t_0], 0$
  while T has some open leaf node **do**
    for every open leaf node N in T **do**
      EXPAND(N, T, ts, y, n)
    end for
    for i **from** 0 **do** n **do**
      if $\neg$empty(a_i) **then**
        $ts, a_i \leftarrow ts \circ [\text{head}(a_i)], \text{tail}(a_i)$
      end if
    end for
  end while
  if T is complete **then**
    WRITE("T proves $\Gamma \vdash \Delta$")
  else
    WRITE("T refutes $\Gamma \vdash \Delta$")
  end if
end procedure

procedure EXPAND($N, T, ts, y, \updownarrow n$)
  Let S be the subtree of T with root N
  Apply the propositional rules until the formulas
   in all leaf nodes of S are atomic or quantified
  for every leaf formula in S to which ($\forall$-L) or ($\exists$-R) applies **do**
    repeatedly apply the rule for every $t \in ts$
  end for
  for every leaf formula in S to which ($\forall$-R) or ($\exists$-L) applies **do**
    $n \leftarrow n + 1$
    apply the rule for $x \leftarrow y_n$
  end for
end procedure

A leaf node is open if it does not match any axiom and there is a non-atomic node formula whose outermost symbol is

- either a connective
- or a quantifier to which ($\forall$-L) or ($\exists$-R) has not yet been applied for every term in ts.
  - This has to be recorded in EXPAND.

Correctness Properties of the Algorithm

By the soundness of the calculus, if SEARCH terminates with a complete proof tree, $\Gamma \vdash \Delta$ is valid.

- Theorem: if $\Gamma \vdash \Delta$ is valid, SEARCH terminates with a complete proof tree.
  - Proof Sketch: we assume that $\Gamma \vdash \Delta$ is valid but SEARCH does not terminate with a complete proof tree; from this, we derive a contradiction. There are two cases:

    First, SEARCH may terminate with an incomplete tree T, i.e., there is a leaf node $\Gamma_k \vdash \Delta_k$ at some depth k that does not match any axiom. But, from the loop condition, no leaf node of T is open. Thus, $\Gamma_k \vdash \Delta_k$ only contains atoms and quantified formulas to which (∀-L) and (∃-R) have been applied for every term in ts. Consider every node $\Gamma_i \vdash \Delta_i$ along the path $\Gamma \vdash \Delta \to \ldots \to \Gamma_k \vdash \Delta_k$ from the root $\Gamma \vdash \Delta$ to the leaf $\Gamma_k \vdash \Delta_k$. Let $S := \bigcup\{\Gamma_i \cup \neg\Delta_i \mid 0 \le i \le k\}$ where $\neg\Delta := \{\neg A \mid A \in \Delta\}$. Now it is possible to prove that every formula in S is satisfied by the Herbrand structure $H_S = (D_S, I_S)$ where (considering all free variables as constants) $D_S := \bigcup\{a_i \mid 0 \le i \le n\} \cup ts$ (for the final values of ts, a, n) and $I_S(p)(t_1, \ldots, t_n) :\Leftrightarrow p(t_1, \ldots, t_n) \in \bigcup\{\Gamma_i \mid 0 \le i \le k\}$. Since $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$, this structure H_S refutes $\Gamma \vdash \Delta$, which contradicts the assumption that $\Gamma \vdash \Delta$ is valid.

    Second, SEARCH may not terminate. Then its execution describes the construction of an infinite tree T (even if only a finite part of T is ever computed). Since T is infinite but finitely branching, by König's lemma it contains some infinite path $\Gamma \vdash \Delta \to \ldots$. Analogously to the first case, we can construct from this path a satisfiable set S and structure H_S that refutes $\Gamma \vdash \Delta$ (to show this, it is essential that for every universal formula in some Γ_i respectively existential formula in some Δ_i, every instance of that formula appears in the branch in some $\Gamma_{j \ge i}$ respectively $\Delta_{j \ge i}$).

Fundamental Properties of First-Order Logic

- Completeness: every valid first-order formula is provable.
  - Kurt Gödel, 1929 (for another proof calculus of first-order logic).
  - A corollary of the previous theorem: given a valid formula F, procedure SEARCH finds a complete proof tree for the sequent $\vdash F$.
  - However, if F is invalid, SEARCH may run forever.
- Undecidability: there cannot exist any procedure that, when given an arbitrary first-order formula F, always halts and correctly states whether F is valid.
  - Alonzo Church/Alan Turing, 1936/1937.
    - The halting problem for computing machines is undecidable.
    - The halting problem can be reduced to the decision problem of first-order logic.

The power and the limit of reasoning in first-order logic.

The Problem of the Sequent Calculus

Procedure SEARCH looks a bit difficult to implement.

- Complex traversal of proof tree to make sure that all quantified formulas in all leafs to which the rules (∀-L) and (∃-R) are applicable are indeed instantiated by all possible terms.

Is there no “easier” way to achieve the same result?

Herbrand's Theorem

Actually, the Gödel-Herbrand-Skolem theorem (≈1930).

- Theorem: Let F be a quantifier-free first-order formula. Then F is first-order satisfiable if the set of all its ground instances $\{F_1, F_2, \ldots\}$ is propositionally satisfiable.
  - F is first-order satisfiable: there exists some structure M such that $M \models F$.
  - F' is a ground instance of F if F' is identical to F except that every variable has been replaced by a term in which only constants and function symbols appear.
  - F is propositionally satisfiable: F is satisfied by some valuation v, considering every atom as a propositional variable. A set $\{F_1, F_2, \ldots\}$ is propositionally satisfiable if there exists some valuation v that satisfies every formula F_i in the set.
- Example: formula $p(x) \wedge \neg q(x, y)$.
  - Ground instances: $\{p(c) \wedge \neg q(c, c), p(c) \wedge \neg q(c, f(c)), p(f(c)) \wedge \neg q(f(c), c), \ldots\}$
  - Valuation: $[p(c) \mapsto \text{true}, q(c, c) \mapsto \text{false}, q(c, f(c)) \mapsto \text{false}, p(f(c)) \mapsto \text{true}, q(f(c), c) \mapsto \text{false}, \ldots]$

The previously stated theorem abound Herbrand structures as models is actually a consequence of Herbrand's theorem.

Corollaries of Herbrand's Theorem

- Theorem: Quantifier-free F is first-order satisfiable if every conjunction $F_1 \wedge \ldots \wedge F_n$ of a finite subset of its instances is propositionally satisfiable.
    - Proof sketch: a corollary of the "compactness theorem" of propositional logic: a set of propositional formulas is satisfiable, if each finite subset is satisfiable.
- Theorem: Quantifier-free F is first-order unsatisfiable if some conjunction $F_1 \wedge \ldots \wedge F_n$ of a finite subset of its instances is propositionally unsatisfiable.
    - Proof sketch: the contraposition of the previous theorem.
- Theorem: Formula $\forall x_1, \ldots, x_n.\ F$ in Skolem normal form is unsatisfiable if some conjunction $F_1 \wedge \ldots \wedge F_n$ of a finite number of instances of its matrix F is propositionally unsatisfiable.
    - Proof sketch: by induction on n, using the previous theorem as the induction base.

The basis of various "Herbrand procedures" for first-order proving.

The Gilmore Algorithm

Paul C. Gilmore, 1960.

procedure GILMORE(G)
  $F \leftarrow$ SKOLEMNORMALFORMMATRIX($\neg G$)
  $Fs \leftarrow \top$
  $i \leftarrow 1$
  loop
    $Fs \leftarrow Fs \wedge F(i)$ ▹ Add instance i of F
    if Fs is propositionally unsatisfiable **then**
      WRITE("G is first-order valid")
      return
    end if
    $i \leftarrow i + 1$
  end loop
end procedure

A systematic enumeration of all instances of the matrix.

The Gilmore Algorithm in OCaml

```
(* Get the constants for Herbrand base, adding nullary one if necessary. *)
let herbfuns fm =
  let cns,fns = partition (fun (_,ar) -> ar = 0) (functions fm) in
  if cns = [] then ["c",0],fns else cns,fns;;

(* Enumeration of ground terms and m-tuples, ordered by total fns. *)
let rec groundterms cntms funcs n =
  if n = 0 then cntms else
  itlist (fun (f,m) l -> map (fun args -> Fn(f,args))
                             (groundtuples cntms funcs (n - 1) m) @ l)
          funcs []

and groundtuples cntms funcs n m =
  if m = 0 then if n = 0 then [[]] else [] else
  itlist (fun k l -> allpairs (fun h t -> h::t)
                       (groundterms cntms funcs k)
                       (groundtuples cntms funcs (n - k) (m - 1)) @ l)
         (0 -- n) [];;
```

The Gilmore Algorithm in OCaml

```
let rec herbloop mfn tfn fl0 cntms funcs fvs n fl tried tuples =
  print_string(string_of_int(length tried)^" ground instances tried; "^
               string_of_int(length fl)^" items in list"); print_newline();
  match tuples with
    [] -> let newtups = groundtuples cntms funcs n (length fvs) in
          herbloop mfn tfn fl0 cntms funcs fvs (n + 1) fl tried newtups
  | tup::tups -> let fl' = mfn fl0 (subst(fpf fvs tup)) fl in
                 if not(tfn fl') then tup::tried else
                 herbloop mfn tfn fl0 cntms funcs fvs n fl' (tup::tried) tups;;
let gilmore_loop fl0 cntms funcs fvs n fl tried tuples =
  let mfn djs0 ifn djs = filter (non trivial) (distrib (image (image ifn) djs0) djs) in
  herbloop mfn (fun djs -> djs <> []) fl0 cntms funcs fvs n fl tried tuples;;
let gilmore fm =
  let sfm = skolemize(Not(generalize fm)) in
  let fvs = fv sfm and consts,funcs = herbfuns sfm in
  let cntms = image (fun (c,_) -> Fn(c,[])) consts in
  length(gilmore_loop (simpdnf sfm) cntms funcs fvs 0 [[]] [] []);;
```

Verify propositional unsatisfiability of a formula in DNF by finding a pair of complimentary literals in each disjunct.

The Gilmore Algorithm in OCaml

```
# gilmore << (P(a) \/ Q(b)) /\ (forall x. P(x) ==> R(x)) /\ (forall x. Q(x) ==> R(f(x)))
    ==> (exists x. R(x)) >>;;
0 ground instances tried; 1 items in list
1 ground instances tried; 2 items in list
2 ground instances tried; 2 items in list
2 ground instances tried; 2 items in list
3 ground instances tried; 2 items in list
- : int = 4

# skolemize << ~((P(a) \/ Q(b)) /\ (forall x. P(x) ==> R(x)) /\ (forall x. Q(x) ==> R(f(x)))
    ==> (exists x. R(x)) >>;;
<<((P(a) \/ Q(b)) /\ (~P(x) \/ R(x)) /\ (~Q(x) \/ R(f(x)))) /\ ~R(x)) >>
# satisfiable <<
  ((P(a) \/ Q(b)) /\ (~P(a) \/ R(a)) /\ (~Q(a) \/ R(f(a))) /\ ~R(a)) /\
  ((P(a) \/ Q(b)) /\ (~P(b) \/ R(b)) /\ (~Q(b) \/ R(f(b))) /\ ~R(b)) /\
  ((P(a) \/ Q(b)) /\ (~P(f(b)) \/ R(f(b))) /\ (~Q(f(b)) \/ R(f(f(b)))) /\ ~R(f(b))) >> ;;
- : bool = false
```

Our example formula can be proved with 3 ground instances: $x = a, x = b, x = f(b)$.

The Gilmore Algorithm in OCaml

```
# val p45 = gilmore <<
  (forall x. P(x) /\ (forall y. G(y) /\ H(x,y) ==> J(x,y))
     ==> (forall y. G(y) /\ H(x,y) ==> R(y))) /\
  ~(exists y. L(y) /\ R(y)) /\
  (exists x. P(x) /\ (forall y. H(x,y) ==> L(y)) /\ (forall y. G(y) /\ H(x,y) ==> J(x,y)))
  ==> (exists x. P(x) /\ ~(exists y. G(y) /\ H(x,y))) >>;;
0 ground instances tried; 1 items in list
1 ground instances tried; 13 items in list
1 ground instances tried; 13 items in list
2 ground instances tried; 57 items in list
3 ground instances tried; 84 items in list
4 ground instances tried; 405 items in list
val p45 : int = 5
```

DNF representations explode, problems soon become intractable.

The Davis Putnam Algorithm

An optimization of the Gilmore algorithm where the formula is represented in CNF and propositional satisfiability is tested by DPLL.

```
let dp_mfn cjs0 ifn cjs = union (image (image ifn) cjs0) cjs;;
let dp_loop = herbloop dp_mfn dpll;;
let davisputnam fm =
  let sfm = skolemize(Not(generalize fm)) in
  let fvs = fv sfm and consts,funcs = herbfuns sfm in
  let cntms = image (fun (c,_) -> Fn(c,[])) consts in
  length(dp_loop (simpcnf sfm) cntms funcs fvs 0 [] [] []);;
```

```
# let p20 = gilmore <<(forall x y. exists z. forall w. P(x) /\ Q(y) ==> R(z) /\ U(w))
   ==> (exists x y. P(x) /\ Q(y)) ==> (exists z. R(z))>>;;
...
18 ground instances tried; 15060 items in list
val p20 : int = 19
# let p20 = davisputnam <<(forall x y. exists z. forall w. P(x) /\ Q(y) ==> R(z) /\ U(w))
   ==> (exists x y. P(x) /\ Q(y)) ==> (exists z. R(z))>>;;
...
18 ground instances tried; 37 items in list
val p20 : int = 19
```

However, the number of ground instances does not change.

The Problem with Herbrand Procedures

Optimizing satisfiability checking does not eliminate the core problem.

> Davis, 1983: *...effectively eliminating the truth-functional satisfiability obstacle only uncovered the deeper problem of the combinatorial explosion inherent in unstructured search through the Herbrand universe ...*

A more intelligent way of choosing instances is required rather than blindingly trying out all possibilities.