

FIRST-ORDER LOGIC: SYNTAX AND SEMANTICS

Course “Computational Logic”



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Abstract Syntax

A first-order formula F is a “sentence” that talks about “objects”.

Sentence: “The successor of every natural number has a natural number as its predecessor.”

Formula: $\forall x. \text{isNat}(x) \Rightarrow \exists y. \text{isNat}(y) \wedge \text{isPred}(y, \text{succ}(x))$

Two kinds of syntactic phrases (“expressions”):

- **Terms** denoting **objects** (values).

$$t ::= x \mid c \mid f(t_1, \dots, t_n)$$

- **Formulas** denoting **properties** of objects (i.e., the truth values “true” or “false”).

$$F ::= \top \mid \perp \mid p(t_1, \dots, t_n) \mid \dots \mid (\forall x. F) \mid (\exists x. F)$$

- **Variables** $x \in \mathcal{X}$.
- **Constants** $c \in \mathcal{C}$, n -ary **function symbols** $f \in \mathcal{F}$ and **predicate symbols** $p \in \mathcal{P}$.
- **Quantifiers** \forall (“universal quantifier”) and \exists (“existential quantifier”).
 - $\forall x. F$: “for all (possible objects assigned to) x , F is true”.
 - $\exists x. F$: “there exists some (possible object assigned to) x , for which F is true”.

Other versions of concrete syntax can be transformed into above “standard form”.

Concrete Syntax

- We use the following abbreviations:

$$\forall x_1, \dots, x_n. F \rightsquigarrow \forall x_1. \dots \forall x_n. F$$

$$\exists x_1, \dots, x_n. F \rightsquigarrow \exists x_1. \dots \exists x_n. F$$

- We apply the following binding rules:

$$(\neg) > \dots > (\forall, \exists)$$

- Without parentheses, the scope of a quantifiers ranges till the *end* of the formula.

$$\forall x. \text{isNat}(x) \Rightarrow \exists y. \text{isNat}(y) \wedge \text{isPred}(y, \text{succ}(x))$$

$$\rightsquigarrow \forall x. (\text{isNat}(x) \Rightarrow \exists y. (\text{isNat}(y) \wedge \text{isPred}(y, \text{succ}(x))))$$

Be sure to (mentally) insert parentheses appropriately.

Abstract Syntax in OCaml

- OCaml Type:

```
type term = Var of string | Fn of string * term list;;
type ('a)formula = False | True | Atom of 'a
| Not of ('a)formula | And of ('a)formula * ('a)formula | Or of ('a)formula * ('a)formula
| Imp of ('a)formula * ('a)formula | Iff of ('a)formula * ('a)formula
| Forall of string * ('a)formula | Exists of string * ('a)formula;;
```

```
type fol = R of string * term list;;
type folformula = fol formula;;
```

- Execution:

```
# let f = << forall x. P(x) ==> exists y. R(x,F(x,y)) >> ;;
val f : fol formula = <<forall x. P(x) ==> (exists y. R(x,F(x,y)))>>
```

```
let g = Forall("x", Imp(Atom(R("P", [Var "x"])),
  Exists("y", Atom(R("R", [Var "x"; Fn("F", [Var "x"; Var "y"])])))))) ;;
val g : fol formula = <<forall x. P(x) ==> (exists y. R(x,F(x,y)))>>
```

First-order formulas are values of type `fol formula`.

Interpretation of Quantifiers

- Formula E : “everybody loves somebody”

$$\forall x. \exists y. \text{loves}(x, y)$$

- Formula S : “somebody is loved by everybody”

$$\exists y. \forall x. \text{loves}(x, y)$$

	$y = 0$	$y = 1$	$y = 2$	$y = 3$
$x = 0$		■		
$x = 1$			■	■
$x = 2$	■			
$x = 3$		■		

E : true, S : false

	$y = 0$	$y = 1$	$y = 2$	$y = 3$
$x = 0$		■		
$x = 1$		■	■	■
$x = 2$	■	■		
$x = 3$		■		

E : true, S : true

The nesting order of quantifier matters.

Free and Bound Variables

- **Non-closed** formula:

$\text{equal}(x, \text{zero})$

- Truth value depends on value assigned to x : “true” for $x = \text{zero}$, “false”, otherwise.
 - Variable x is **free** in the formula.
 - If some of its variables are free, a formula is **non-closed**.
- **Closed** formulas (**sentences**):

$\forall x. \text{equal}(x, \text{zero})$ $\exists x. \text{equal}(x, \text{zero})$

- Truth values do not depend on x : first formula is “false”, second one is “true”.
- Variable x is **bound** in both formulas (by the quantifier \forall respectively \exists).
- If all of its variables are bound, a formula is **closed**.

The truth value of a formula only depends on the values assigned to the formula's free variables; it is independent of the values of the bound variables.

The Set of Free Variables

$\text{fv}(F)$ and $\text{fv}(t)$ compute the set of free vars of formula F and term t .

$$\text{fv}(\top) = \emptyset$$

$$\text{fv}(\perp) = \emptyset$$

$$\text{fv}(x) = \{x\} \quad \text{fv}(c) = \emptyset$$

$$\text{fv}(f(t_1, \dots, t_n)) = \text{fv}(t_1) \cup \dots \cup \text{fv}(t_n)$$

$$\text{fv}(p(t_1, \dots, t_n)) = \text{fv}(t_1) \cup \dots \cup \text{fv}(t_n)$$

$$\text{fv}(\neg F) = \text{fv}(F)$$

$$\text{fv}(F_1 \wedge F_2) = \text{fv}(F_1) \cup \text{fv}(F_2)$$

$$\text{fv}(F_1 \vee F_2) = \text{fv}(F_1) \cup \text{fv}(F_2)$$

$$\text{fv}(F_1 \Rightarrow F_2) = \text{fv}(F_1) \cup \text{fv}(F_2)$$

$$\text{fv}(F_1 \Leftrightarrow F_2) = \text{fv}(F_1) \cup \text{fv}(F_2)$$

$$\text{fv}(\forall x. F) = \underline{\text{fv}(F)} \setminus \{x\}$$

$$\text{fv}(\exists x. F) = \underline{\text{fv}(F)} \setminus \{x\}$$

Quantifiers bind variables.

Example

$$\text{fv}(q(x, y, z)) = \{x, y, z\}$$

$$\begin{aligned} \text{fv}(\exists y. q(x, y, z)) &= \text{fv}(q(x, y, z)) \setminus \{y\} \\ &= \{x, y, z\} \setminus \{y\} = \{x, z\} \end{aligned}$$

$$\text{fv}(p(x, w)) = \{x, w\}$$

$$\begin{aligned} \text{fv}(p(x, w) \Rightarrow \exists y. q(x, y, z)) &= \text{fv}(p(x, w)) \cup \text{fv}(\exists y. q(x, y, z)) \\ &= \{x, w\} \cup \{x, z\} = \{x, w, z\} \end{aligned}$$

$$\begin{aligned} \text{fv}(\forall x. p(x, w) \Rightarrow \exists y. q(x, y, z)) &= \text{fv}(p(x, w) \Rightarrow \exists y. q(x, y, z)) \setminus \{x\} \\ &= \{x, w, z\} \setminus \{x\} = \{w, z\} \end{aligned}$$

The Set of Free Variables in OCaml

```
let rec fvt tm =
  match tm with
  | Var x -> [x]
  | Fn(f,args) -> unions (map fvt args);;

let rec fv fm =
  match fm with
  | False | True -> []
  | Atom(R(p,args)) -> unions (map fvt args)
  | Not(p) -> fv p
  | And(p,q) | Or(p,q) | Imp(p,q) | Iff(p,q) -> union (fv p) (fv q)
  | Forall(x,p) | Exists(x,p) -> subtract (fv p) [x];;

let generalize fm = itlist mk_forall (fv fm) fm;;

# fv <<forall x. p(x,w) ==> exists y. q(x,y,z) >>;
- : string list = ["w"; "z"]
# generalize <<forall x. p(x,w) ==> exists y. q(x,y,z) >>;
- : fol formula = <<forall w z x. p(x,w) ==> (exists y. q(x,y,z))>>
```


Formal Semantics: Structures and Valuations

- A **structure** (D, I) consists of a **domain** D and an **interpretation** I on D :
 - D is a non-empty set of objects ($D \neq \emptyset$).
 - The “universe” about which a first-order formula talks.
 - I maps every constant and function/predicate symbol to its meaning:
 - **Constant** $c \in \mathcal{C}$: $I(c)$ is an object in D ($I(c) \in D$).
 - **Function symbol** $f \in \mathcal{F}$ of **arity** n : $I(f)$ is an n -ary function on D ($I(f): D^n \rightarrow D$).
 - **Predicate symbol** $p \in \mathcal{P}$ of **arity** n : $I(p)$ is an n -ary predicate/relation on D ($I(p) \subseteq D^n$).
- A **valuation (assignment)** v maps every variable to its meaning:
 - **Variable** $x \in \mathcal{X}$: $v(x)$ is an object in D ($v(x) \in D$).

$$D = \mathbb{N}$$

$$I = [0 \mapsto \text{zero}, + \mapsto \text{add}, < \mapsto \text{less-than}, \dots]$$

$$v = [x \mapsto \text{one}, y \mapsto \text{zero}, z \mapsto \text{three}, \dots]$$

The Formal Semantics: Terms

- Term semantics $\llbracket t \rrbracket_v^{D,I} \in D$

- Given structure (D, I) and valuation v , the semantics of term t is an object in D .

$$t ::= x \mid c \mid f(t_1, \dots, t_n)$$

- The meaning of a **variable** is the value given by the valuation:

$$\llbracket x \rrbracket_v^{D,I} := v(x)$$

- The meaning of a **constant** is the value given by the interpretation:

$$\llbracket c \rrbracket_v^{D,I} := I(c)$$

- The meaning of a **function application** is the result of the interpretation of the function symbol applied to the values of the argument terms:

$$\llbracket f(t_1, \dots, t_n) \rrbracket_v^{D,I} := I(f)(\llbracket t_1 \rrbracket_v^{D,I}, \dots, \llbracket t_n \rrbracket_v^{D,I})$$

The recursive definition of a function evaluating a term.

Example

$$D = \mathbb{N} = \{\text{zero, one, two, three, } \dots\}$$

$$I = [0 \mapsto \text{zero}, + \mapsto \text{add}, \dots]$$

$$v = [x \mapsto \text{one}, y \mapsto \text{two}, \dots]$$

$$\begin{aligned} \llbracket x + (y + 0) \rrbracket_v^{D,I} &= \text{add}(\llbracket x \rrbracket_v^{D,I}, \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{add}(v(x), \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{add}(\text{one}, \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{add}(\text{one}, \text{add}(\llbracket y \rrbracket_v^{D,I}, \llbracket 0 \rrbracket_v^{D,I})) \\ &= \text{add}(\text{one}, \text{add}(v(y), I(0))) \\ &= \text{add}(\text{one}, \text{add}(\text{two}, \text{zero})) \\ &= \text{add}(\text{one}, \text{two}) \\ &= \text{three}. \end{aligned}$$

The meaning of the term with the “usual” interpretation.

Example

$$D = \mathcal{P}(\mathbb{N}) = \{\emptyset, \{\text{zero}\}, \{\text{one}\}, \{\text{two}\}, \dots, \{\text{zero, one}\}, \dots\}$$

$$I = [0 \mapsto \emptyset, + \mapsto \text{union}, \dots]$$

$$a = [x \mapsto \{\text{one}\}, y \mapsto \{\text{two}\}, \dots]$$

$$\begin{aligned} \llbracket x + (y + 0) \rrbracket_v^{D,I} &= \text{union}(\llbracket x \rrbracket_v^{D,I}, \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{union}(v(x), \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{union}(\{\text{one}\}, \llbracket y + 0 \rrbracket_v^{D,I}) \\ &= \text{union}(\{\text{one}\}, \text{union}(\llbracket y \rrbracket_v^{D,I}, \llbracket 0 \rrbracket_v^{D,I})) \\ &= \text{union}(\{\text{one}\}, \text{union}(v(y), I(0))) \\ &= \text{union}(\{\text{one}\}, \text{union}(\{\text{two}\}, \emptyset)) \\ &= \text{union}(\{\text{one}\}, \{\text{two}\}) \\ &= \{\text{one, two}\} \end{aligned}$$

The meaning of the term with another interpretation.

Formal Semantics: Basic Formulas

- Formula semantics $\llbracket F \rrbracket_v^{D,I} \in \mathbb{B}$
 - Given structure (D, I) and valuation v , the semantics of formula F is a truth value.

$$F ::= \top \mid \perp \mid p(t_1, \dots, t_n) \mid \dots \mid (\forall x. F) \mid (\exists x. F)$$

- The meaning of the **logical constants** is a fixed truth value:

$$\llbracket \top \rrbracket_v^{D,I} := \text{true} \quad \llbracket \perp \rrbracket_v^{D,I} := \text{false}$$

- The meaning of an **atomic formula** is the result of the interpretation of the predicate symbol applied to the values of the argument terms.

$$\llbracket p(t_1, \dots, t_n) \rrbracket_v^{D,I} := I(p)(\llbracket t_1 \rrbracket_v^{D,I}, \dots, \llbracket t_n \rrbracket_v^{D,I})$$

The meaning of the basic formulas.

Formal Semantics: Logical Connectives

- The meaning of the **logical connectives**:

$$\llbracket \neg F \rrbracket_v^{D,I} := \begin{cases} \text{true} & \text{if } \llbracket F \rrbracket_v^{D,I} = \text{false} \\ \text{false} & \text{else} \end{cases}$$

$$\llbracket F_1 \wedge F_2 \rrbracket_v^{D,I} := \begin{cases} \text{true} & \text{if } \llbracket F_1 \rrbracket_v^{D,I} = \llbracket F_2 \rrbracket_v^{D,I} = \text{true} \\ \text{false} & \text{else} \end{cases}$$

$$\llbracket F_1 \vee F_2 \rrbracket_v^{D,I} := \begin{cases} \text{false} & \text{if } \llbracket F_1 \rrbracket_v^{D,I} = \llbracket F_2 \rrbracket_v^{D,I} = \text{false} \\ \text{true} & \text{else} \end{cases}$$

$$\llbracket F_1 \Rightarrow F_2 \rrbracket_v^{D,I} := \begin{cases} \text{false} & \text{if } \llbracket F_1 \rrbracket_v^{D,I} = \text{true} \text{ and } \llbracket F_2 \rrbracket_v^{D,I} = \text{false} \\ \text{true} & \text{else} \end{cases}$$

$$\llbracket F_1 \Leftrightarrow F_2 \rrbracket_v^{D,I} := \begin{cases} \text{true} & \text{if } \llbracket F_1 \rrbracket_v^{D,I} = \llbracket F_2 \rrbracket_v^{D,I} \\ \text{false} & \text{else} \end{cases}$$

An embedding of the semantics of propositional logic into first-order logic.

Formal Semantics: Quantifiers

- $(\forall x. F)$ is true, if F is true for every possible object d assigned to variable x :

$$\llbracket \forall x. F \rrbracket_v^{D,I} := \begin{cases} \text{true} & \text{if } \llbracket F \rrbracket_{v[x \mapsto d]}^{D,I} = \text{true for all } d \text{ in } D \\ \text{false} & \text{else} \end{cases}$$

- $(\exists x. F)$ is true, if F is true for at least one possible object d assigned to x :

$$\llbracket \exists x. F \rrbracket_v^{D,I} := \begin{cases} \text{true} & \text{if } \llbracket F \rrbracket_{v[x \mapsto d]}^{D,I} = \text{true for some } d \text{ in } D \\ \text{false} & \text{else} \end{cases}$$

- Valuation v updated by the assignment of object d to variable x :

$$v[x \mapsto d](y) = \begin{cases} d & \text{if } x = y \\ v(y) & \text{else} \end{cases}$$

The core of the semantics of first-order logic.

Example

$D = \mathbb{N}_3 = \{\text{zero}, \text{one}, \text{two}\}$ $I = [0 \mapsto \text{zero}, + \mapsto \text{add}, \dots]$ $v = [x \mapsto \text{one}, y \mapsto \text{two}, z \mapsto \text{two}, \dots]$

$\llbracket \forall x. \exists y. x + y = z \rrbracket_v^{D,I} = ?$

- $\llbracket \exists y. x + y = z \rrbracket_{v[x \mapsto \text{zero}]}^{D,I} = \text{true}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{zero}, y \mapsto \text{zero}]}^{D,I} = \text{false}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{zero}, y \mapsto \text{one}]}^{D,I} = \text{false}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{zero}, y \mapsto \text{two}]}^{D,I} = \underline{\text{true}}$
- $\llbracket \exists y. x + y = z \rrbracket_{v[x \mapsto \text{one}]}^{D,I} = \text{true}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{one}, y \mapsto \text{zero}]}^{D,I} = \text{false}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{one}, y \mapsto \text{one}]}^{D,I} = \underline{\text{true}}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{one}, y \mapsto \text{two}]}^{D,I} = \text{false}$
- $\llbracket \exists y. x + y = z \rrbracket_{v[x \mapsto \text{two}]}^{D,I} = \text{true}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{two}, y \mapsto \text{zero}]}^{D,I} = \underline{\text{true}}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{two}, y \mapsto \text{one}]}^{D,I} = \text{false}$
 - $\llbracket x + y = z \rrbracket_{v[x \mapsto \text{two}, y \mapsto \text{two}]}^{D,I} = \text{false}$

$\llbracket \forall x. \exists y. x + y = z \rrbracket_v^{D,I} = \text{true.}$

Term and Formula Semantics in OCaml

```
let rec termval (domain,func,pred as m) v tm =  
  match tm with  
    Var(x) -> apply v x  
  | Fn(f,args) -> func f (map (termval m v) args);;
```

```
let rec holds (domain,func,pred as m) v fm =  
  match fm with  
    False -> false  
  | True -> true  
  | Atom(R(r,args)) -> pred r (map (termval m v) args)  
  | Not(p) -> not(holds m v p)  
  | And(p,q) -> (holds m v p) & (holds m v q)  
  | Or(p,q) -> (holds m v p) or (holds m v q)  
  | Imp(p,q) -> not(holds m v p) or (holds m v q)  
  | Iff(p,q) -> (holds m v p = holds m v q)  
  | Forall(x,p) -> forall (fun a -> holds m ((x |-> a) v) p) domain  
  | Exists(x,p) -> exists (fun a -> holds m ((x |-> a) v) p) domain;;
```

The structure is represented by a triple $m = (domain, func, pred)$.

Term and Formula Semantics in OCaml

```
let bool_interp =
  let func f args =
    match (f,args) with
      ("0",[ ]) -> false
    | ("1",[ ]) -> true
    | ("+",[x;y]) -> not(x = y)
    | ("*",[x;y]) -> x & y
    | _ -> failwith "uninterpreted function"
  and pred p args =
    match (p,args) with
      ("=", [x;y]) -> x = y
    | _ -> failwith "uninterpreted predicate" in
  ([false; true],func,pred);;

# holds bool_interp undefined <<forall x. (x = 0) \ / (x = 1)>>;
- : bool = true
# holds bool_interp undefined <<forall x. (x + 1) + 1 = x>>;
- : bool = true
```

Term and Formula Semantics in OCaml

```
let mod_interp n =
  let func f args =
    match (f,args) with
      ("0",[ ]) -> 0 | ("1",[ ]) -> 1 mod n
    | ("+",[x;y]) -> (x + y) mod n | ("*",[x;y]) -> (x * y) mod n
    | _ -> failwith "uninterpreted function"
  and pred p args =
    match (p,args) with
      ("=", [x;y]) -> x = y | _ -> failwith "uninterpreted predicate" in
    (0--(n-1),func,pred));;

# holds (mod_interp 2) undefined <<forall x. (x = 0) \\/ (x = 1)>>;
- : bool = true
# holds (mod_interp 3) undefined <<forall x. (x = 0) \\/ (x = 1)>>;
- : bool = false
# let fm = <<forall x. ~(x = 0) ==> exists y. x * y = 1>>;
val fm : fol formula = <<forall x. ~x = 0 ==> (exists y. x * y = 1)>>
# filter (fun n -> holds (mod_interp n) undefined fm) (1--45);;
- : int list = [1; 2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43]
```

The Model Checker RISCAL

<https://www.risc.jku.at/research/formal/software/RISCAL/>

The screenshot shows the RISCAL IDE interface. The main window displays a program for computing the greatest common divisor (gcd) using the Euclidean algorithm. The program is written in RISCAL and includes several theorems and a procedure. The analysis panel on the right shows the execution settings and the results of the analysis.

```
1 // -----
2 // Computing the greatest common divisor by the Euclidean Algorithm
3 // -----
4
5 val N: N;
6 type nat = N[N];
7
8 pred divides(m:nat,n:nat) =  $\exists p:nat. m = p \cdot n$ ;
9
10 fun gcd(m:nat,n:nat): nat
11   requires m  $\neq$  0  $\vee$  n  $\neq$  0;
12 = choose result:nat with
13   divides(result,m)  $\wedge$  divides(result,n)  $\wedge$ 
14    $\neg \exists r:nat. divides(r,m) \wedge divides(r,n) \wedge r >$  result;
15
16 theorem gcd0(m:nat) = m=0  $\rightarrow$  gcd(m,0) = m;
17 theorem gcd1(m:nat,n:nat) = m  $\neq$  0  $\vee$  n  $\neq$  0  $\rightarrow$  gcd(m,n) = gcd(n,m);
18 theorem gcd2(m:nat,n:nat) = 1  $\leq$  n  $\wedge$  n  $\leq$  m  $\rightarrow$  gcd(m,n) = gcd(m/n,n);
19
20 proc gcdp(m:nat,n:nat): nat
21   requires m $\neq$ 0  $\vee$  n $\neq$ 0;
22   ensures result = gcd(m,n);
23 {
24   var a:nat = m;
25   var b:nat = n;
26   while a > 0  $\wedge$  b > 0 do
27     invariant a  $\neq$  0  $\vee$  b  $\neq$  0;
28     invariant gcd(a,b) = gcd(old_a,old_b);
29     decreases a+b;
30   {
31     if a > b then
32       a = a/b;
33     else
34       b = b/a;
35   }
36   return if a = 0 then b else a;
37 }
38
39 fun gcdf(m:nat,n:nat): nat
40   requires m $\neq$ 0  $\vee$  n $\neq$ 0;
41   ensures result = gcd(m,n);
42   decreases m+n;
```

Analysis

Translation: Nondeterminism Default Value: 0 Other Values: [...]

Execution: Silent Inputs: [] Per Mille: [] Branches: [] Depth: []

Visualization: Trace Tree Width: 1500 Height: 800

Parallelism: Multi-Threaded Threads: 4 Distributed Servers: [...]

Operation: [gcdp(Z,Z)]

RISC Algorithm Language 4.0 (July 7, 2022)
<https://www.risc.jku.at/research/formal/software/RISCAL>
(C) 2016-. Research Institute for Symbolic Computation (RISC)
This is free software distributed under the terms of the GNU GPL.
Execute "RISCAL -h" to see the available command line options.

Reading file /home/schreine/software/RISCAL/trunk/spec/gcd.txt
Using N=10.
Type checking and translation completed.

The Model Checker RISCAL

```
val N:N; type Num = N[N];

pred divides(m:Num,n:Num) ⇔
  ∃p:Num. m·p = n;
pred isgcd(g:Num,m:Num,n:Num) ⇔
  divides(g,m) ∧ divides(g,n) ∧
  ∀g0:Num. divides(g0,m) ∧ divides(g0,n) ⇒
    g0 ≤ g;

theorem t1() ⇔ isgcd(1,3,5);
theorem gcdbound(m:Num,n:Num) ⇔
  ∀g:Num. isgcd(g,m,n) ⇒ g ≤ m ∧ g ≤ n;
theorem gcdbound0(m:Num,n:Num) ⇔
  m ≠ 0 ∧ n ≠ 0 ⇒ ∀g:Num. isgcd(g,m,n) ⇒
    g ≤ m ∧ g ≤ n;
```

```
Using N=5.
Type checking and translation completed.
Executing t1().
Execution completed (24 ms).
Executing gcdbound( $\mathbb{Z}$ , $\mathbb{Z}$ ) with all 36 inputs.
ERROR in execution of gcdbound(0,0): evaluation of
gcdbound
at line 23 in file algebra.txt:
theorem is not true
ERROR encountered in execution (8 ms).
Executing gcdbound0( $\mathbb{Z}$ , $\mathbb{Z}$ ) with all 36 inputs.
Execution completed for ALL inputs (46 ms,
36 checked, 0 inadmissible).
```

First-order logic over finite domains with fixed interpretations.

Satisfiability and Validity

Let F denote a first-order formula, $M = (D, I)$ a structure, ν a valuation.

- Formula F is **satisfiable**, if there exists some structure M and valuation ν such that $\llbracket F \rrbracket_{\nu}^M = \text{true}$.
 - Example: $p(0, x)$ is satisfiable; $q(x) \wedge \neg q(x)$ is not.
- Structure M is a **model** of formula F , written as $M \models F$, if for every valuation ν , we have $\llbracket F \rrbracket_{\nu}^M = \text{true}$.
 - Example: $(\mathbb{N}, [0 \mapsto \text{zero}, p \mapsto \text{less-equal}]) \models p(0, x)$
- Formula F is **valid**, written as $\models F$, if every structure M is a model of F , i.e., for every structure M we have $M \models F$.
 - Example: $\models p(x) \wedge (p(x) \Rightarrow q(x)) \Rightarrow q(x)$

Logical Consequence and Equivalence

- Formula F_2 is a **logical consequence** of formula F_1 , written as $F_1 \models F_2$, if for every structure M and valuation ν , the following is true:
 - If $\llbracket F_1 \rrbracket_{\nu}^M = \text{true}$, then also $\llbracket F_2 \rrbracket_{\nu}^M = \text{true}$.
 - Example: $p(x) \wedge (p(x) \Rightarrow q(x)) \models q(x)$
- Formula F is a **logical consequence** of formulas F_1, \dots, F_n , written $F_1, \dots, F_n \models F$, if for every M and ν the following is true:
 - If for every formula F_i we have $\llbracket F_i \rrbracket_{\nu}^M = \text{true}$, then $\llbracket F \rrbracket_{\nu}^M = \text{true}$.
 - Example: $p(x), q(x) \models p(x) \wedge q(x)$
- Formulas F_1 and F_2 are **logically equivalent**, written as $F_1 \equiv F_2$, if and only if F_1 is a logical consequence of F_2 and vice versa, i.e., $F_1 \models F_2$ and $F_2 \models F_1$.
 - Example: $p(x) \Rightarrow q(x) \equiv \neg p(x) \vee q(x)$

Semantic Relationships

- Satisfiability and Validity:
 - F is satisfiable, if $\neg F$ is not valid.
 - F is valid, if $\neg F$ is not satisfiable.
- Logical Consequence and Equivalence
 - Formula F_2 is a logical consequence of formula F_1 (i.e., $F_1 \models F_2$) if and only if the formula $(F_1 \Rightarrow F_2)$ is valid.
 - Formula F is a logical consequence of formulas F_1, \dots, F_n (i.e., $F_1, \dots, F_n \models F$) if and only if the formula $(F_1 \wedge \dots \wedge F_n \Rightarrow F)$ is valid.
 - Formula F_1 and formula F_2 are logically equivalent (i.e., $F_1 \equiv F_2$) if and only if the formula $(F_1 \Leftrightarrow F_2)$ is valid.

Logical consequence/equivalence reduced to validity of an implication/equivalence.

Logical Equivalence: Formula Substitutions

Assume $F \equiv F'$ and $G \equiv G'$. Then we have the following equivalences:

$$\neg F \equiv \neg F'$$

$$F \wedge G \equiv F' \wedge G'$$

$$F \vee G \equiv F' \vee G'$$

$$F \Rightarrow G \equiv F' \Rightarrow G'$$

$$F \Leftrightarrow G \equiv F' \Leftrightarrow G'$$

$$\forall x. F \equiv \forall x. F'$$

$$\exists x. F \equiv \exists x. F'$$

Logically equivalent formulas can be substituted in any context.

Logical Equivalence: Rules

In addition to the logical equivalences for connectives in propositional logic:

$$\neg\forall x. F \equiv \exists x. \neg F \quad (\text{De Morgan's Law})$$

$$\neg\exists x. F \equiv \forall x. \neg F \quad (\text{De Morgan's Law})$$

$$\forall x. (F_1 \wedge F_2) \equiv (\forall x. F_1) \wedge (\forall x. F_2)$$

$$\exists x. (F_1 \vee F_2) \equiv (\exists x. F_1) \vee (\exists x. F_2)$$

$$\forall x. (F_1 \vee F_2) \equiv F_1 \vee (\forall x. F_2) \quad \text{if } x \notin \text{fv}(F_1)$$

$$\exists x. (F_1 \wedge F_2) \equiv F_1 \wedge (\exists x. F_2) \quad \text{if } x \notin \text{fv}(F_1)$$

For a finite domain whose values are denoted by constants $\{c_1, \dots, c_n\}$:

$$\forall x. F \equiv F[c_1/x] \wedge \dots \wedge F[c_n/x]$$

$$\exists x. F \equiv F[c_1/x] \vee \dots \vee F[c_n/x]$$

Logical Equivalence: Examples

- Push negations from the outside to the inside:

$$\begin{aligned}\neg(\forall x. p(x) \Rightarrow \exists y. q(x, y)) &\equiv \exists x. \neg(p(x) \Rightarrow \exists y. q(x, y)) \\ &\equiv \exists x. \neg((\neg p(x)) \vee \exists y. q(x, y)) \\ &\equiv \exists x. ((\neg\neg p(x)) \wedge \neg\exists y. q(x, y)) \\ &\equiv \exists x. (p(x) \wedge \neg\exists y. q(x, y)) \\ &\equiv \exists x. (p(x) \wedge \forall y. \neg q(x, y))\end{aligned}$$

- Reduce the scope of quantifiers:

$$\begin{aligned}\forall x, y. (p(x) \Rightarrow q(x, y)) &\equiv \forall x, y. (\neg p(x) \vee q(x, y)) \\ &\equiv \forall x. (\neg p(x) \vee \forall y. q(x, y)) \\ &\equiv \forall x. (p(x) \Rightarrow \forall y. q(x, y))\end{aligned}$$

- Replace quantification in a finite domain $\{0, 1, 2\}$:

$$\forall x. p(x) \equiv p(0) \wedge p(1) \wedge p(2)$$

Prenex Normal Form

- A formula F is in **prenex normal form (PNF)** if it is of the following form:

$$Q_1x_1. \dots Q_nx_n. M$$

- Quantifiers Q_1, \dots, Q_n .
- Formula M (the **matrix**) without quantifiers.
- **Example:**

$$\forall x. \exists y. \forall z. P(x) \wedge P(y) \Rightarrow P(z)$$

- We can compute PNF by applying logical equivalences:
 - Perform the simplifications of propositional logic.
 - Remove quantifiers whose variable does not occur freely in body.
 - Compute negation normal form (“push down negations”).
 - Pull out quantifiers (renaming bound variables if necessary).

The steps can be best described by actual code.

Prenex Normal Form in OCaml

We have $\forall x. F \equiv F$ if $x \notin fv(F)$.

```
let simplify1 fm =
  match fm with
  | Forall(x,p) -> if mem x (fv p) then fm else p
  | Exists(x,p) -> if mem x (fv p) then fm else p
  | _ -> psimplify1 fm;;

let rec simplify fm =
  match fm with
  | Not p -> simplify1 (Not(simplify p))
  | And(p,q) -> simplify1 (And(simplify p,simplify q))
  | ...
  | Forall(x,p) -> simplify1(Forall(x,simplify p))
  | Exists(x,p) -> simplify1(Exists(x,simplify p))
  | _ -> fm;;

# simplify <<(forall x y. P(x) /\ (P(y) /\ false)) ==> exists z. Q>>;
- : fol formula = <<(forall x. P(x)) ==> Q>>
```

Prenex Normal Form in OCaml

We have $\neg\forall x. F \equiv \exists x. \neg F$ and $\neg\exists x. F \equiv \forall x. \neg F$.

```
let rec nnf fm =
  match fm with
  | And(p,q) -> And(nnf p,nnf q) | Or(p,q) -> Or(nnf p,nnf q)
  | Imp(p,q) -> Or(nnf(Not p),nnf q)
  | Iff(p,q) -> Or(And(nnf p,nnf q),And(nnf(Not p),nnf(Not q)))
  | Not(Not p) -> nnf p
  | Not(And(p,q)) -> Or(nnf(Not p),nnf(Not q)) | Not(Or(p,q)) -> And(nnf(Not p),nnf(Not q))
  | Not(Imp(p,q)) -> And(nnf p,nnf(Not q))
  | Not(Iff(p,q)) -> Or(And(nnf p,nnf(Not q)),And(nnf(Not p),nnf q))
  | Forall(x,p) -> Forall(x,nnf p) | Exists(x,p) -> Exists(x,nnf p)
  | Not(Forall(x,p)) -> Exists(x,nnf(Not p)) | Not(Exists(x,p)) -> Forall(x,nnf(Not p))
  | _ -> fm;;

# nnf <<(forall x. P(x)) ==> ((exists y. Q(y)) <=> exists z. P(z) /\ Q(z))>>;
- : fol formula =
<<(exists x. ~P(x)) \/ (exists y. Q(y)) /\ (exists z. P(z) /\ Q(z)) \/
  (forall y. ~Q(y)) /\ (forall z. ~P(z) \/ ~Q(z))>>
```

Prenex Normal Form in OCaml

We have (for instance) $F_1 \vee (\forall x. F_2) \equiv \forall x. (F_1 \vee F_2)$ if $x \notin fv(F_1)$.

- Thus $F_1 \vee (\forall x. F_2) \equiv \forall y. (F_1 \vee F_2[y/x])$ if $y \notin fv(F_1) \cup fv(F_2)$.

```
let rec pullquants fm =
  match fm with
  | And(Forall(x,p),Forall(y,q)) -> pullq(true,true) fm mk_forall mk_and x y p q
  | Or(Exists(x,p),Exists(y,q))  -> pullq(true,true) fm mk_exists mk_or x y p q
  | And(Forall(x,p),q)           -> pullq(true,false) fm mk_forall mk_and x x p q
  | And(p,Forall(y,q))           -> pullq(false,true) fm mk_forall mk_and y y p q
  | Or(Forall(x,p),q)            -> pullq(true,false) fm mk_forall mk_or x x p q
  | Or(p,Forall(y,q))            -> pullq(false,true) fm mk_forall mk_or y y p q
  | And(Exists(x,p),q)           -> pullq(true,false) fm mk_exists mk_and x x p q
  | And(p,Exists(y,q))           -> pullq(false,true) fm mk_exists mk_and y y p q
  | Or(Exists(x,p),q)            -> pullq(true,false) fm mk_exists mk_or x x p q
  | Or(p,Exists(y,q))            -> pullq(false,true) fm mk_exists mk_or y y p q
  | _ -> fm
and ...
```


Prenex Normal Form in OCaml

```
...
and pullq(l,r) fm quant op x y p q =
  let z = variant x (fv fm) in
  let p' = if l then subst (x | => Var z) p else p
  and q' = if r then subst (y | => Var z) q else q in
  quant z (pullquants(op p' q'));;

let rec prenex fm =
  match fm with
  | Forall(x,p) -> Forall(x,prenex p)
  | Exists(x,p) -> Exists(x,prenex p)
  | And(p,q) -> pullquants(And(prenex p,prenex q))
  | Or(p,q) -> pullquants(Or(prenex p,prenex q))
  | _ -> fm;;

let pnf fm = prenex(nnf(simplify fm));;

# pnf <<(forall x. P(x) \ / R(y)) ==> exists y z. Q(y) \ / ~(exists z. P(z) /\ Q(z))>>;
- : fol formula = <<exists x. forall z. ~P(x) /\ ~R(y) \ / Q(x) \ / ~P(z) \ / ~Q(z)>>
```

Prenex Normal Form (Manual Computation)

$$\begin{aligned} & (\forall x. P(x) \vee R(y)) \Rightarrow (\exists y, z. Q(y) \vee \neg(\exists z. P(z) \wedge Q(z))) \\ \equiv & \neg(\forall x. P(x) \vee R(y)) \vee (\exists y, z. Q(y) \vee \neg(\exists z. P(z) \wedge Q(z))) \\ \equiv & (\exists x. \neg(P(x) \vee R(y))) \vee (\exists y, z. Q(y) \vee (\forall z. \neg(P(z) \wedge Q(z)))) \\ \equiv & (\exists x. \neg P(x) \wedge \neg R(y)) \vee (\exists y, z. Q(y) \vee (\forall z. \neg P(z) \vee \neg Q(z))) \\ \equiv & (\exists x. \neg P(x) \wedge \neg R(y)) \vee (\exists y. Q(y) \vee (\forall z. \neg P(z) \vee \neg Q(z))) \\ \equiv & (\exists x. \neg P(x) \wedge \neg R(y)) \vee (\exists y. \forall z. Q(y) \vee \neg P(z) \vee \neg Q(z)) \\ \equiv & (\exists x. \neg P(x) \wedge \neg R(y)) \vee (\exists x. \forall z. Q(x) \vee \neg P(z) \vee \neg Q(z)) \\ \equiv & \exists x. (\neg P(x) \wedge \neg R(y)) \vee (\forall z. Q(x) \vee \neg P(z) \vee \neg Q(z)) \\ \equiv & \exists x. \forall z. (\neg P(x) \wedge \neg R(y)) \vee (Q(x) \vee \neg P(z) \vee \neg Q(z)) \end{aligned}$$

Skolem Normal Form

- A formula is in **Skolem normal form (SNF)** if it is in prenex normal form and only contains universal quantifiers.
 - But how to remove the existential quantifiers?
- **Theorem (Skolemization):** Let F be a formula with free variables x_1, \dots, x_n, y . Let f be an n -ary function symbol that does not occur in F . Then $\forall x_1, \dots, x_n. \exists y. F$ is satisfiable if and only if $\forall x_1, \dots, x_n. F[f(x_1, \dots, x_n)/y]$ is.
 - **Skolem function** f ($n = 0$: Skolem constant c), **substitution** $F[t/x]$ of t for x in F .
 - **Proof sketch:** First, let (D, I) and v satisfy $\forall x_1, \dots, x_n. \exists y. F$. Then for all $d_1, \dots, d_n \in D$ there exists $d \in D$ such that $v[x_1 \mapsto d_1, \dots, x_n \mapsto d_n, y \mapsto d]$ satisfies F . Thus there exists $f_D(d_1, \dots, d_n): D^n \rightarrow D$ such that for all $d_1, \dots, d_n \in D$ structure (D, I) and valuation $v[x_1 \mapsto d_1, \dots, x_n \mapsto d_n, y \mapsto f_D(d_1, \dots, d_n)]$ satisfy F . Thus $\forall x_1, \dots, x_n. F[f(x_1, \dots, x_n)/y]$ is satisfied by structure (D, I') and valuation v where I' is identical to I except that $I'(f) := f_D$. Second, let (D, I) and v satisfy $\forall x_1, \dots, x_n. F[f(x_1, \dots, x_n)/y]$. Then, for all $d_1, \dots, d_n \in D$, (D, I) and $v[x_1 \mapsto d_1, \dots, x_n \mapsto d_n, y \mapsto I(f)(d_1, \dots, d_n)]$ satisfy F . Thus (D, I) and v satisfy $\forall x_1, \dots, x_n. \exists y. F$.

We can construct an *equisatisfiable* formula without existential quantifiers.

Skolem Normal Form in OCaml

```
let rec skolem fm fns =
  match fm with
  | Exists(y,p) -> let xs = fv(fm) in
    let f = variant (if xs = [] then "c_"^y else "f_"^y) fns in
    let fx = Fn(f,map (fun x -> Var x) xs) in
    skolem (subst (y | => fx) p) (f::fns)
  | Forall(x,p) -> let p',fns' = skolem p fns in Forall(x,p'),fns'
  | And(p,q) -> skolem2 (fun (p,q) -> And(p,q)) (p,q) fns
  | Or(p,q) -> skolem2 (fun (p,q) -> Or(p,q)) (p,q) fns
  | _ -> fm,fns
and skolem2 cons (p,q) fns =
  let p',fns' = skolem p fns in let q',fns'' = skolem q fns' in
  cons(p',q'),fns'';;
let askolemize fm = fst(skolem (nnf(simplify fm)) (map fst (functions fm)));;
let rec specialize fm = match fm with Forall(x,p) -> specialize p | _ -> fm;;
let skolemize fm = specialize(pnf(askolemize fm));;

# pnf(askolemize <<forall x. P(x) ==> (exists y z. Q(x,y,z)) /\ (exists y z. R(y,z)) >>);;
- : fol formula = <<forall x. ~P(x) \/ Q(x,f_y(x),f_z(x)) /\ R(c_y,c_z)>>
```